Lecture 13

Mortality profit and Multiple life insurance

Reading: Gerber Chapter 8, CT5 Core Reading Units 3 and 6

13.1 Reserves for life assurances – mortality profit

Let us look more specifically at the risk of an insurer who has underwritten a portfolio of identical whole life assurances of independent lives aged x at the start, with annual net premium P and sum assured S. Remember that the reserve $V(k)$ for each policy satisfies

$$(V(k)+P)(1+i) = q_{x+k}S + p_{x+k}V(k+1) = V(k+1) + q_{x+k}(S - V(k+1)).$$

Actuaries consider

- the *death strain* $\Delta_k = 0$ if the life survives, $\Delta_k = S - V(k+1)$ otherwise; the *death strain at risk* or *net amount at risk* is $S - V(k+1)$;
- the *expected death strain* $\mathbb{E}(\Delta_k) = q_{x+k}(S - V(k+1))$, or *risk premium* $v\mathbb{E}(\Delta_k)$; the remaining premium is called the *savings premium* $P_s = P - P_r = vV(k+1) - V(k)$;
- the *mortality profit* $\mathbb{E}(\Delta_k) - \Delta_k$.

Note that the mortality profit is positive if the life survives, and then just accumulated from the part P_r of the premium P which is not needed to save for $V(k+1)$ – it is hence a profit for the insurer on that policy in that year. On the other hand, the mortality "profit" is negative if the life dies. Specifically, note that the payment of the sum assured

$$S = (V(k)+P)(1+i) + (\Delta_k - \mathbb{E}(\Delta_k))$$

is in two parts, first the funds held from that policy and then topped up to S by the amount $\Delta_k - \mathbb{E}(\Delta_k)$ causing the insurer's loss on that policy in that year.

Suppose there are n survivors to age $x+k$ and N survivors to age $x+k+1$. Then we have

- a *total death strain at risk* of $n(S - V(k+1))$,
- a total expected death strain at risk of $nq_{x+k}(S - V(k+1))$,
- a total actual death strain at risk of $(n-N)(S - V(k+1))$,
- a total (net) mortality profit of $M = (nq_{x+k} - (n-N))(S - V(k+1))$.

From the mortality profit discussion of individual policies, we see how the premium payments of survivors are now naturally used to help top up to the sum assured the benefits payable to the deceased, if we offset profits and losses within the portfolio.

If the insurer charges a premium $Y > P$ exceeding the net premium P, this increases the mortality profit by $(Y - P)(1 + i)$. We can relate the insurer's loss probability ε and premium Y using the Central Limit Theorem for the binomially distributed random variable $n - N$, as follows:

$$\begin{aligned}\varepsilon &\geq \mathbb{P}(M + n(Y - P)(1 + i) \leq 0)\\ &= \mathbb{P}\left(\frac{(n - N) - nq_{x+k}}{\sqrt{nq_{x+k}(1 - q_{x+k})}} \geq \frac{n(Y - P)(1 + i)}{(S - V(k + 1))\sqrt{nq_{x+k}(1 - q_{x+k})}}\right)\\ &\approx \mathbb{P}\left(Z \geq \frac{n(Y - P)(1 + i)}{(S - V(k + 1))\sqrt{nq_{x+k}(1 - q_{x+k})}}\right)\end{aligned}$$

where Z is a standard normally distributed random variable. For $\varepsilon = 1\%$, this gives

$$Y - P \geq 2.33\, v\, (S - V(k + 1))\sqrt{q_{x+k}(1 - q_{x+k})/n}.$$

13.2 Insurance products contingent on multiple lives

Many life insurance products such as pensions are contingent on several lives, often a couple, but sometimes other dependent persons such as disabled children or former spouses. Splitting cash flows of benefits between several recipients is purely an accounting problem and we only deal with the aggregate cash flow. Our focus will be on the interaction of multiple lives, modelled by a corresponding number of lifetime random variables.

Consider m lives aged $x_1, \dots, x_m$.

- Life annuity paying S per annum until the last of the m lives dies.
- (Whole-life/term/endowment) life assurance paying S when the last of the m lives dies.
- Life assurance paying S when the first of the m lives dies.
- Life assurance paying S_i when the ith of the m lives dies, $i = 1, \dots, m$.
- Life annuity paying S_i per annum while i of the lives are alive, $i = m, \dots, 1$.
- Life assurances paying S_i on death of the first life if the first life to die is x_i, $i = 1, \dots, m$.

13.3 Joint lifetime distributions

Consider m independent lifetime random variables $T^{(1)}, \dots, T^{(m)}$ with respective forces of mortality $\mu^{(1)}, \dots, \mu^{(m)}$. Then

$$\mathbb{P}(\min\{T^{(1)}, \dots, T^{(m)}\} > t) = \mathbb{P}(T^{(1)} > t, \dots, T^{(m)} > t) = \exp\left(-\int_0^t \sum_{j=1}^m \mu^{(j)}(s)ds\right).$$

Hence, the time $T_{\min} = \min\{T^{(1)}, \dots, T^{(m)}\}$ of the first life to die has force of mortality

$$\mu_{\min} = \mu^{(1)} + \cdots + \mu^{(m)}.$$

We can view this as the *force of failure* of the *joint life status.*

Suppose now that we are given the force of mortality μ of a full lifetime T, and that $T^{(j)}$ is distributed as T_{x_j}, i.e. according to the conditional distribution of $T - x_j$ given $T > x_j$, for all $j = 1, \dots, m$. Then actuarial notation is $T_{x_1 \cdots x_m} = T_{\min}$ and we obtain

$$f_{T_{x_1 \cdots x_m}}(t) = \overline{F}_{T_{\min}}(t)\mu_{\min}(t) = {}_tp_{x_1} \cdots {}_tp_{x_m}(\mu_{x_1+t} + \cdots + \mu_{x_m+t}).$$

For $T_{\max} = \max\{T^{(1)}, \dots, T^{(m)}\}$, we obtain

$$\mathbb{P}(\max\{T^{(1)}, \dots, T^{(m)}\} \le t) = \mathbb{P}(T^{(1)} \le t, \dots, T^{(m)} \le t) = \prod_{j=1}^{m} \mathbb{P}(T^{(j)} \le t),$$

so that the product rule yields the density. We say that the *last survivor status* ends at $T_{\max}$. Specifically, when $T^{(j)}$ is distributed as T_{x_j}, actuarial notation is $T_{\overline{x_1 \cdots x_m}} = T_{\max}$ and this yields

$$f_{T_{\max}}(t) = \left(\prod_{j=1}^{m} {}_tq_{x_j}\right) \sum_{j=1}^{m} \frac{f_{x_j}(t)}{{}_tq_{x_j}} = \left(\prod_{j=1}^{m} (1 - {}_tp_{x_j})\right) \sum_{j=1}^{m} \frac{\mu_{x_j+t}\, {}_tp_{x_j}}{1 - {}_tp_{x_j}}.$$

In the special case $m = 2$ this can be simplified to

$$f_{T_{\overline{xy}}}(t) = f_{T_x}(t) + f_{T_y}(t) - f_{T_{xy}}(t).$$

The formulas get gradually more involved as we consider the other order statistics of the m lifetime variables. We can also consider a more general *survival status* such as when precisely individuals $j_1, \dots, j_k$ are alive, or when precisely k out of m are alive (any k individuals rather than a specific set of k individuals). Generic notation for a survival status is u, e.g. $u = xy$ or $u = \overline{xy}$ for the joint life status and the last survivor status of a pair of lives aged x and y.

We naturally define curtate lifetimes $K^{(1)} = [T^{(1)}], \dots, K^{(m)} = [T^{(m)}]$ and obtain similar relationships between probability mass functions (exercise).

In practice, the independence assumption for, say, the pair of lifetime variables of a couple, may be questioned.

13.4 Pricing of multiple life insurance products

Consider a survival status u. Denote by f_{T_u} the density of the residual lifetime of status u. Then, as for single-life products, the single net premium of an assurance paying 1 at time T_u is

$$\overline{A}_u = \mathbb{E}(v^{T_u}) = \int_0^\infty v^t f_{T_u}(t)dt$$

and

$$\mathrm{Var}(v^{T_u}) = {}^2\overline{A}_u - (\overline{A}_u)^2,$$

where ${}^2\overline{A}_u$ is calculated using the modified interest rate $i' = 2i + i^2$.

Note that $f_{T_{\overline{xy}}}(t) = f_{T_x}(t) + f_{T_y}(t) - f_{T_{xy}}(t)$ yields

$$\overline{A}_{\overline{xy}} = \overline{A}_x + \overline{A}_y - \overline{A}_{xy}.$$

This can also be argued directly by consideration of the underlying cash-flows.

Pricing of annuities is again done via conversion formulas

$$\overline{a}_u = \mathbb{E}(\overline{a}_{\overline{T_u}|}) = \frac{1-\overline{A}_u}{\delta}, \qquad \mathrm{Var}(\overline{a}_{\overline{T_u}|}) = \frac{1}{\delta^2}\left({}^2\overline{A}_u - (\overline{A}_u)^2\right).$$

The same reasoning allows us to transfer formulas for term and endowment assurances, pure endowments, term annuities etc. from the single-life framework to any survival status, as long as the product only depends on T_u. Products that depend on several survival statuses can often be evaluated by linearity.

Example 82 A pension is payable to a couple aged x and y. Payments are continuous at rate r_1 for as long as x survives and then at rate r_2 for as long as y survives x (if y survives x; if x survives y, the full pension is payable for as long as x survives y). The expected net present value of this pension is

$$\mathbb{E}\left(\int_0^{T_x} r_1 v^t dt + \int_{T_{xy}}^{T_y} r_2 v^t dt\right) = \overline{a}_x + r_2(\overline{a}_y - \overline{a}_{xy}).$$

Example 83 A couple aged x and y have a life assurance policy under which a sum of S_1 is payable when the first partner dies and a further sum of S_2 when the second partner dies. The expected net present value of this policy is

$$\mathbb{E}\left(S_1 v^{T_{xy}} + S_2 v^{T_{\overline{xy}}}\right) = S_1\overline{A}_{xy} + S_2\overline{A}_{\overline{xy}}.$$

In some cases, a direct argument is needed.

Example 84 A life assurance policy pays S when x dies, if x dies before y, and pays nothing otherwise. The expected net present value of this policy is

$$\mathbb{E}\left(Sv^{T_x}1_{\{T_y>T_x\}}\right) = \int_0^\infty \int_t^\infty Sv^t f_{T_y}(s) f_{T_x}(t) ds dt = \int_0^\infty Sv^t {}_tp_y f_{T_x}(t) dt = \int_0^\infty Sv^t {}_tp_{xy}\mu_{x+t} dt.$$

Lecture 14

Multiple decrement model and investment funds

Reading: Gerber Chapter 7, CT5 Core Reading Units 4 and 8

14.1 The multiple decrement model

The model of a single lifetime with hazard rate (force of mortality) μ can be viewed as the *single decrement model*: an individual moves from an "alive" state to a "dead" state. The transition is referred to as a "decrement". A multiple decrement model includes several decrements, each leading from the "active" state to one of two or more "absorbing states". For example, the single decrement model could be extended by a "sick" state or a "retired" state (ignoring further transitions away from "sick" or "retired"), or it could distinguish different causes of death (road accident, domestic accident, cancer, heart attack, etc.).

In the *multiple decrement model*, we assume that each decrement has an (age-dependent) force of decrement $\mu^{(j)}$, and that the decrements constitute competing risks. We associate with each decrement an independent random variable $T^{(j)}$ with hazard rate $\mu^{(j)}$. We declare that the decrement j takes place at time $T^{(j)}$ if $T^{(j)} = T_{\text{min}}$, where we note that the framework is the same as in the multiple life model, we use the same notation, but the interpretation is different. Specifically, after the decrement at time T_{min}, the variables $T^{(k)} > T_{\text{min}}$ have no further relevance (as an individual who died from one cause does not go on to die from another).

Now consider a life assurance issued to a life aged x that pays 1 at the time of death if death is by cause j, and pays nothing if death is by a different cause. First recall that $T^* = \min\{T^{(k)}, k \neq j\}$ has hazard rate $\mu^* = \sum_{k\neq j} \mu^{(k)}$. Denote by $T_x^{(j)}$, T_x^* and T_x random variables distributed as $T^{(j)} - x$, $T^* - x$ and $T_{\text{min}} - x$ conditioned to exceed x. Then the net discounted value of this assurance is calculated as in Example 84 and given by

$$\mathbb{E}\left(v^{T_x^{(j)}} 1_{\{T_x^* > T_x^{(j)}\}}\right) = \int_0^\infty v^t \mathbb{P}(T_x^* > t) f_{T_x^{(j)}}(t) dt = \int_0^\infty {}_tp_x \mu_{x+t}^{(j)} dt.$$

By linearity, an assurance that pays S_j at the time of death if death is by cause j, for all $j = 1, \ldots, m$, has expected net present value

$$\sum_{j=1}^m S_j \int_0^\infty {}_tp_x \mu_{x+t}^{(j)} dt.$$

Note that this simplifies to A_x, as expected, if $S_j = 1$ for all j since $\sum_j \mu_{x+t}^{(j)} = \mu_{x+t}$ and ${}_tp_x\mu_{x+t} = f_{T_x}(t)$.

14.2 More general Markov models

Example 85 Consider the three-state healthy-sick-dead model. Transitions are possible in both directions between the healthy state and the sick state and from both these states into the dead state. An income protection insurance charges premiums while in the healthy state and pays benefits while in the sick state. What are reasonable model assumptions, and how can they be used to relate the premium rate $\overline{P}$ to the benefit rate $\overline{B}$?

Based on the discussions of the multiple decrement model, starting from the healthy state, there are two decrements. We model age-dependent rates μ_x into the dead state and σ_x into the sick state. Similarly, from the sick state we consider rates ρ_x into the healthy state and ν_x into the dead state.

If we are given the (random) state $X_t \in \{H, S, D\}$ for all times $t \geq 0$, the situation is captured by the continuous cash-flow $\overline{P}1_{\{X_t=H\}} - \overline{B}1_{\{X_t=S\}}$. For $\overline{P}$ to correspond to a net premium rate in a constant-δ model, we require

$$\mathbb{E}\left(\int_0^\infty e^{-\delta t}\overline{P}1_{\{X_t=H\}}dt\right) = \mathbb{E}\left(\int_0^\infty e^{-\delta t}\overline{B}1_{\{X_t=S\}}dt\right).$$

Interchanging integration and expectation, and using $\mathbb{E}(1_A) = \mathbb{P}(A)$, this can be written in terms of the transition probabilities ${}_tp_x^{H,H} = \mathbb{P}(X_t = H)$ and ${}_tp_x^{H,S} = \mathbb{P}(X_t = S)$ as

$$\overline{P} = \overline{B}\frac{\displaystyle\int_0^\infty e^{-\delta t}{}_tp_x^{H,S}dt}{\displaystyle\int_0^\infty e^{-\delta t}{}_tp_x^{H,H}dt}.$$

Methods to calculate the transition probabilities are developed in SB3a Applied Probability (for age-independent μ, σ, ν, ρ) and in SB3b Statistical Lifetime-Models.

14.3 Discounted dividend model

A share which has just paid a dividend of d_0. Suppose that each year (or half-year), the dividend increases by an independent random factor F_m, $m \geq 1$, with $\mathbb{E}(F_m) = 1+g$ (could also decrease for $F_m < 1$). Then the mth dividend will be

$$D_m = d_0 \times F_1 \times \cdots \times F_m, \qquad \text{with } \mathbb{E}(D_m) = d_0 \times \mathbb{E}(F_1) \times \cdots \times \mathbb{E}(F_m) = d_0(1+g)^m.$$

What is the fair price for this share? We assume that annual dividends continue indefinitely, so the random cash-flow is $C = ((1, D_1), (2, D_2), \ldots) = ((m, D_m), m \geq 1)$ with

$$\begin{aligned}\mathbb{E}(\mathrm{NPV}(i)) = \sum_{m\geq 1}\mathbb{E}(D_m)(1+i)^{-m} &= \sum_{m\geq 1} d_0(1+g)^m(1+i)^{-m} \\ &= \frac{d_0(1+g)}{1+i}\frac{1}{1-(1+g)(1+i)^{-1}} = \frac{d_0(1+g)}{i-g},\end{aligned}$$

provided that $(1+g)(1+i)^{-1} < 1$, i.e. $g < i$, for the geometric series to converge.

With this reasoning, today's value of the share is $P = d_0(1+g)/(i-g)$. In a year's time the dividend of D_1 will have been paid, and the value of the share will be $D_1(1+g)/(i-g)$, with expected value $d_0(1+g)^2/(i-g) = P(1+g)$. Similarly, the expected share value at time m is $P(1+g)^m$ for all $m \geq 0$. In words, we expect the share value to increase at constant rate g.

14.4 Units in investment funds

When the investments of two or more investors are merged into one investment portfolio, we need to keep track of the value of each investor's money in the fund.

Example 86 Suppose a fund is composed of holdings of two investors, as follows.

- Investor A invests £100 at time 0 and withdraws his holdings of £130 at time 3.
- Investor B invests £290 at time 2 and withdraws his holdings of £270 at time 4.

The yields of the two investors are $y_A = 9.14\%$ and $y_B = -3.51\%$, based respectively on cash-flows $((0, 100), (3, -130))$ and $((2, 290), (4, -270))$.

The cash-flow of the fund is

$$c = ((0, 100), (2, 290), (3, -130), (4, -270)).$$

Its yield is 1.16%.

A convenient way to keep track of the value of each investor's money is to assign units to investors and to track prices $P(t)$ per unit. There is always a normalisation choice, typically but not necessarily fixed by setting *£*1 per unit at time 0.

Cash-flows of investors can now be conveniently described in terms of units. The yield achieved by an investor I making a single investment buying N_I units for $N_I P(s)$ and selling N_I units for $N_I P(t)$ is the rate of growth in the unit price, because the number of units cancels in the yield equation $N_I P(s)(1+i)^{t-s} = N_I P(t)$.

Example 86 (continued) With $P(0) = 1$, Investor A buys 100 units, we obtain $P(2) = 1.45$ (from $F(2-) = 145$), $P(3) = 1.30$ (from the sale proceeds of 130 for Investor A's 100 units). With $P(2) = 1.45$, Investor B receives 200 units for £290, and so $P(4) = 1.35$ (from the sale proceeds of 270 for Investor B's 200 units).

Pension schemes invest pensions savings in a variety of ways, sometimes giving members a choice that sets some constraints on the risk class or the spread over different investment types or industries. Rather than managing separate investment portfolios for each scheme member, they generally set up investment funds that merge investments from many members, recording the numbers of units for each scheme member.

As with the discounted dividend model for a single share, investment funds are usually expected to increase at a constant rate g.

14.5 Charges

Usually, investment funds charge investors by a bid-ask spread. The quoted price process $P(s)$, $s \geq 0$, may then either be the price per unit at which units can be sold, while the purchase price is higher, often by a certain percentage $\lambda_{\text{purchase}}$, i.e. $(1 + \lambda_{\text{purchase}})P(s)$. Or, the price process is the price at which units can be purchased, with a fee being retained at sale, which effectively reduces the sales proceeds to $(1 - \lambda_{\text{sale}})P(s)$. A third possibility is that the quoted price is the mid-point of the spread, the purchase price being $(1 + \lambda)P(s)$ and the sales price being $(1 - \lambda)P(s)$.

Alternatively, a fixed rate f may be payable on the fund value, e.g. if unit prices would be $\widetilde{P}(t)$ without the fee deducted, then the actual unit price is reduced to $P(t) = \widetilde{P}(t)/(1+f)^t$. Or, for an associated force $\varphi = \log(1 + f)$, with the portfolio accumulating according to $\widetilde{A}(s,t) = \exp(\int_s^t \delta(r)dr)$, i.e. $\widetilde{P}(t) = P(0)\widetilde{A}(0,t)$ satisfies $\widetilde{P}'(t) = \delta(t)P(t)$, then $P'(t) = (\delta(t) - \varphi)P(t)$, i.e.

$$P(t) = \exp(-\int_0^t (\delta(s) - \varphi)ds)P(0) = e^{-\varphi t}\widetilde{P}(t).$$

This fee is then incorporated in the unit price.

Lecture 15

With-profits contracts

Reading: CT5 Core Reading Unit 4

So far, we have studied life insurance contracts where benefits and premiums are predetermined and guaranteed. Such products are sometimes referred to as without-profits contracts. Some pensions and life assurances have benefits that depend on investment performance, or where premiums may be lowered or a bonus payment made when investment performance has exceeded expectations – these are ways in which "profits" resulting from investment performance are paid back to the policy holder. Naturally, such extra benefits come at the cost of a higher premium ("bonus loading").

15.1 Conventional with-profits contracts

With-profits contracts are variants of whole-life or endowment assurances. They have an initial sum assured that is payable upon death (or at maturity). In addition to the initial sum assured, they may have annual bonuses or a terminal bonus or both. Annual bonuses increase the initial sum assured as a guaranteed benefit. A terminal bonus is only added at maturity or at the time of the claim. As for without-profit contracts, the insurer pools the risk associated with such contracts in policy portfolios. The size of a bonus depends on the investment performance across the portfolio.

A simple method to add bonuses in a portfolio of policies is to distribute a surplus in proportion to the initial sum assured. Another method would be to distribute in proportion to the sum assured including previous years' bonuses. A third method combines the two by applying two separate rates to the initial sum assured and any previous years' bonuses.

In all three methods, these rates will depend on the size of the surplus, so they are variable rates. Typically, only part of the surplus is distributed each year, the remainder being retained for a terminal bonus (but is not guaranteed year on year so that it may serve to fund investment losses that may arise from riskier investment strategies with a higher expected rate of return, or for a higher than expected total death strain in the portfolio).

Example 87 Consider a life x and a sum assured S. As derived previously, a without-profits whole-life contract has net annual premium $P_x = SA_x/\ddot{a}_x$. This is based on a constant-i interest rate model. The insurance company not only carries the mortality risk, but also investment risk, since reserves will take the form of an investment fund.

Now consider a with-profits whole-life contract with net annual premium $P > P_x$ and an underlying investment fund that is expected to grow at constant rate i. Then we expect the premium excess to lead to an investment surplus. Suppose that the actual investment surplus each year is added as a bonus to the sum assured. Denote by B_k the aggregate investment surplus up to time k. Given survival, the surplus B_k has expected value $(P - P_x)\ddot{s}_{\overline{k}|}$. The actual surplus will depend on the investment performance.

The insurer should hold a reserve at time k of $(S + B_k)A_{x+k} - P\ddot{a}_{x+k}$.

15.2 Unit-linked assurances

Unit-linked assurances are variants of whole-life or endowment assurances, in which the sum assured is not pre-determined, but dependent on investment performance.

In the most basic product, the benefit payment is simply the value at the time of death (or survival in the case of an endowment assurance) of the total units recorded when investing the premium payments. In this case, there is no pooling of risk, as each contract separately accumulates the benefit payment from the premium payments. The risk is entirely with the policy holder. If the premium is P, then the fund value F_k relates to the growth rate i_k in year k as $F_k = (F_{k-1} + P)(1 + i_k)$.

Example 88 A unitised accumulating with-profits contract is based on a unitised fund whose unit price includes any bonus. Then, each premium payment is used to buy units of the fund (the premium payments are level premium payments, the number of units this fixed premium will buy each year will vary).

Other assurance products may set a minimum sum assured. This is a hybrid between the basic product and an assurance product with fixed sum assured as studied previously. Qualitatively speaking, as the fixed sum assured may exceed the investment value (particularly when death occurs early), some pooling of risk is again required. The premium will typically exceed the premium of the product that only provides the fixed sum assured, and in each year part of the premium is used for a pooled reserve to guarantee the difference between the sum assured and the investment value, and the remainder is invested in fund units. We will discuss this further in the context of profit testing.

15.3 Expected cash flows under the policy basis

We take the insurer's perspective. Recall that we defined a policy basis as a set of assumptions about interest rates, mortality and charges that include expenses and may also include the insurer's profit margin. This was in a context of a without-profits contract, where no profits are returned to the policy holder. In the context of a with-profits contract or a contract that allows the policy holder to surrender, this may include more contributions to the cash flow or to the growth assumptions. To set up the insurer's cash flow, we distinguish two entities:

1. the net reserve or unit fund, which we may think of as belonging to the policy holder,

2. the insurer's revenue account (cash account, non-unit fund), showing the insurer's profit.

We focus on the latter.

The cash flow affecting the revenue account will contain some or all of the following:

- premium receipts,

- interest receipts, accrued on reserves and cash balances,
- other receipts such as management charges made on unit funds,
- expense payments,
- expected benefit payments (or difference of the benefit payment and the value of a unit fund or reserve),
- expected surrender payments (or difference of the surrender payment and the value of a unit fund or reserve),
- expected tax payments,
- expected payment into a unit fund or reserve.

Note in particular, that "charges" or "fees" are inflows into the insurer's revenue account (at the cost of the policy holder, possibly as part of the premium payments), whereas "expenses" are outflows (at the cost of the insurer, passed on to the policy holder as part of the charges). The cash flow is usually worked out separately for consecutive time periods, e.g. for periods of one year.

Example 89 For a whole-life assurance of sum assured S issued to a life $[x]$, the expected income cash flow for the year from k to $k+1$ is

$$C^I = ((k, P), (k+1, (S\ {}_kV_{[x]} + P - e_{k+1})i), (k+1, q_{[x]+k} S\ {}_kV_{[x]}))$$

consisting of

- the premium receipt of P in advance,
- interest in arrear at rate i on the reserve ${}_{k-1}V_{[x]}$ and on the premium minus the expenses of e_{k+1}, which are payable in advance,
- and of the transfer from the reserve in the case of death.

The expected expenditure cash flow for the year from k to $k+1$ is

$$C^E = ((k, e_{k+1}), (k+1, q_{[x]+k} S), (k+1, p_{[x]+k}(S\ {}_{k+1}V_{[x]} - S\ {}_kV_{[x]}))),$$

consisting of

- the expense payment in advance,
- the payment of the death benefit in the case of death,
- and the transfer into the reserve in the case of survival.

The balancing item completing this set of accounts is the profit of the insurer for the year from k to $k+1$

$$(\mathrm{PRO})_{k+1} = (P - e_{k+1})(1+i) + S\ {}_kV_{[x]}(1+i) - q_{[x]+k} S - p_{[x]+k} S\ {}_{k+1}V_{[x]}.$$

In the following we consider a unit fund building up values F_k at time k, where F_k is the cash-in value at time k. Suppose that the fund has two types of charges. The first is that when K is invested, a percentage charge at rate λ is levied, so that the fund value only increases by $(1-\lambda)K$. The second is an annual management percentage charge of κ on each end-of-year fund value. Here, the convention is that F_{k+1} already takes into account the management charge, so that the fund value just before the management charge was applied was $(1-\kappa)^{-1}F_{k+1}$

Example 90 For a whole-life unit-linked assurance with guaranteed sum assured S issued to a life $[x]$, the expected income cash flow for the year from k to $k+1$ is

$$C^I = ((k, P), (k+1, ((1-(1-\lambda)\alpha_k)P - e_{k+1})i), (k+1, (1-\kappa_k)^{-1}F_{k+1}\kappa), (k+1, q_{[x]+k}F_{k+1}))$$

consisting of

- the premium receipt of P in advance,
- interest in arrear at rate i on the premium minus two quantities that are diverted elsewhere at the beginning of the year: the first is the proportion $(1-\lambda)\alpha_k$ of the premium paid into the unit fund for investment, where α_k is the allocation percentage for time k and λ is the percentage charge on investment retained by the insurer; the second is the expenses e_{k+1} that the insurer has managing the policy,
- of management charges $(1-\kappa)^{-1}F_{k+1}\kappa$,
- and of the transfer from the unit fund in the case of death.

The expected expenditure cash flow for the year from k to $k+1$ is

$$C^E = ((k, e_{k+1}), (k, (1-\lambda)\alpha P), (k+1, q_{[x]+k}\max\{S, F_{k+1}\}))$$

consisting of

- the expense payment in advance,
- the transfer to the unit fund in advance,
- and the payment of the death benefit in the case of death.

The balancing item completing this set of accounts is the profit of the insurer for the year from k to $k+1$

$$(\mathrm{PRO})_{k+1} = ((1-(1-\lambda)\alpha_k)P - e_{k+1})(1+i) + (1-\kappa_k)^{-1}F_{k+1}\kappa - q_{[x]+k}(S-F_{k+1})^+.$$

As this profit calculation is done in advance, the fund value is unknown and therefore substitute by projected fund values, using growth rates as specified in the policy basis.

Lecture 16

Profit testing

Reading: CT5 Core Reading Unit 9

In the first instance, the notion of "profit testing" refers to a check that an insurance product yields a profit for the insurance company for the underlying policy basis. A profit test is usually not only carried out under the assumptions of the policy basis, but also under modified assumptions ("sensitivity testing") in order to reveal and quantify scenarios under which profits may be diminished or turned into losses. Profit tests are also run during the design of a product to tune model parameters such as premiums and charges. Our main aim will be to use profit tests in order to price with-profits contracts.

16.1 Profit margins and premium calculation

In general, setting up annual revenue accounts as in Examples 89 and 90 yields a vector of profits. Each of these quantities $(\mathrm{PRO})_k$, $k \geq 1$, is meaningful for policies where the policy holder was alive at the beginning of the year, i.e. at time $k-1$. The weighted vector

$$((\mathrm{PS})_k, k \geq 1) = ({}_{k-1}p_{[x]}(\mathrm{PRO})_k, k \geq 1)$$

is known as the *profit signature.*

To summarise the profit, we can calculate a net present value under a given interest rate i_d (which may include a risk premium over and above the market interest rate i that was used in the calculation of the profit signature):

$$\mathrm{NPV} = \sum_{k\geq 1}(1+i_d)^{-k}(\mathrm{PS})_k.$$

When set relative to the present value of expected premium income, this yields the *profit margin*

$$\frac{\sum_{k\geq 1}(1+i_d)^{-k}(\mathrm{PS})_k}{\sum_{k\geq 1}(1+i_d)^{-(k-1)}{}_{k-1}p_{[x]}P_k}.$$

Using the same equations of expected values as we used to calculate the profit signature and profit margin, we can instead fix the profit margin and solve the equations for the annual level premium. For example, if we want to achieve a profit margin of 3%, with level premiums $P_k = P$ for all $k \geq 1$, then we need to solve the linear equation

$$\sum_{k\geq 1}(1+i_d)^{-k}(\mathrm{PS})_k = 3\% \sum_{k\geq 1}(1+i_d)^{-(k-1)}{}_{k-1}p_{[x]}P$$

for P, where we point out that the LHS also depends on P, in a way that depends on the policy contract, see e.g. Examples 89 and 90.

16.2 Reserve calculation using profit tests

Let us look at an example and then state general principles.

Example 91 Consider a 5-year endowment assurance with sum assured $S = £1,000.00$ and annual premiums of $P = £215.00$, issued to a life aged $x = 55$ on the following policy basis:

- interest rates constant $i = 3\%$,
- mortality A1967/70 Ultimate (i.e. life unchecked, assumed to follow ultimate mortality)
- expenses payable in advance: $e_1 = £40.00$ for year 1 and $e_2 = \cdots = e_5 = £10.00$ for subsequent years.
- profit margin: to be determined along with reserve requirements.

The profit vector can be calculated for years 1-4 as in Example 89, and for year 5 by adjusting the benefit payment so that it applies also on the event of survival. Let us here ignore the as yet undetermined reserve payments so that the "profit vector ignoring reserves" yields an expected cash flow with entries

$$c_k = (P - e_k)(1 + i) - Sq_{x+k-1}, \quad 1 \leq k \leq 4, \qquad c_5 = (P - e_5)(1 + i) - S.$$

The policy basis is such that this cash flow is

$$c = ((1, 171.81), (2, 201.73), (3, 200.65), (4, 199.46), (5, -788.85)).$$

The reason for setting up a reserve is the negative cash flow at time 5, when the endowment becomes payable even on the event of survival, so that the premium payment falls well short of providing the required sum. A reserve is required to meet this shortfall. It is best practice to build the reserve that changes the profit vector $((k, c_k), 1 \leq k \leq 5)$ to a vector $((k, c''_k), 1 \leq k \leq 5)$ with positive values followed by 0 values. We calculate the reserves R_4, R_3, R_2, R_1 required to achieve zero profit for the later years, and stop when the reserve can be set up by the "profit ignoring reserves" at the time:

- as $c_5 < 0$, we need $c''_5 = c_5 + R_4(1 + i) = 0$, i.e. $R_4 = -(1 + i)^{-1}c_5 = 765.87$,
- as $c'_4 := c_4 - p_{x+3}R_4 < 0$, we need $c''_4 = c'_4 + R_3(1+i) = 0$, i.e. $R_3 = -(1+i)^{-1}c'_4 = 552.18$,
- as $c'_3 := c_3 - p_{x+2}R_3 < 0$, we need $c''_3 = c'_3 + R_2(1+i) = 0$, i.e. $R_2 = -(1+i)^{-1}c'_3 = 343.33$,
- as $c'_2 := c_2 - p_{x+1}R_2 < 0$, we need $c''_2 = c'_2 + R_1(1+i) = 0$, i.e. $R_1 = -(1+i)^{-1}c'_2 = 139.32$,
- then $c'_1 := c_1 - p_x R_1 > 0$, and we find the revised profit vector by setting $c''_1 = c'_1 = 33.67$.

To summarise, we find

$$(R_1, R_2, R_3, R_4) = (139.32, 343, 33, 552.18, 765.87) \quad \text{and} \quad (c''_1, c''_2, c''_3, c''_4, c''_5) = (33.67, 0, 0, 0, 0),$$

We can now also calculate the expected discounted value of premium payments and obtain 995.75, hence a profit margin of

$$(1.03)^{-1}33.67/995.75 = 3.28\%.$$

In Examples 89, 90 and 91, we have seen profit vector and/or reserve calculations for three standard types of life assurance. In the first two of these, we separated inflows and outflows as is often required for accounting purposes. For the same reason, we recorded interest payments as inflows. In Example 91, we demonstrated that in- and outflows may combine naturally to net cash flows that can be described directly (since outflows may be tied to inflows, e.g. expense payments here, but also investments and benefit payments more generally), and we may use our notion of accumulated values of cash flows that take care of interest payments as accumulations of payments made in advance.

Now consider a general life insurance product (assurance or annuity or other) contingent on one or more lives, with a given policy basis (which might even specify assumptions of more general multi-state Markov transitions, but it may be best to ignore this at first reading). The expected "profit ignoring reserves" for each year k of the contract is calculated as the accumulated value at the end of the year of the net cash flow on the insurer's revenue account that is attributed to year k including, as appropriate

- expected premium income in advance minus any payments due in advance such as expenses payable in advance and/or investments into unit funds etc.,
- expected income from charges levied on investments (if applicable), payable in advance, in arrear or continuously, as specified in the policy contract,
- expected benefit outgo in the event of maintained survival status or in the events of changes of survival status, or at maturity, possibly partly or entirely offset by an inflow from an investment fund.

The *profit vector* $((\mathrm{PRO})_k, k \geq 1)$ as defined previously, records these accumulated values as conditional expectations given survival to the beginning of year k, for each entry. In this case, there is only one relevant survival status. For multiple life products (or other multi-state Markov models), we can similarly set up a profit vector $((\mathrm{PRO})_k^u, k \geq 1)$ for each relevant survival status u (or other state of the model, also denoted by u to simplify the presentation).

In general, the insurance product will be underwritten in a specific survival status u_0, and there will be a set $\mathcal{U}$ of survival statuses that can be reached while the policy is in force. We will naturally define the *profit signature* $((\mathrm{PS})_k, k \geq 1)$ for the policy starting in this initial survival status u_0. At time k, the survival status will be $u \in \mathcal{U}$ with a probability that we denote by ${}_kp^{u_0,u}$. We may also write ${}_kq^{u_0} = 1 - \sum_{u\in\mathcal{U}} {}_kp^{u_0,u}$ for the probability that the policy end was reached before time k. In this notation, we define

$$(\mathrm{PS})_k = \sum_{u\in\mathcal{U}} {}_kp^{u_0,u}(\mathrm{PRO})_k^u, \qquad k \geq 1.$$

This is again usefully summarised by

$$\mathrm{NPV} = \sum_{k\geq 1}(1+i_d)^{-k}(\mathrm{PS})_k$$

and the notion of *profit margin* generalises to

$$\frac{\sum_{k\geq 1}(1+i_d)^{-k}(\mathrm{PS})_k}{\sum_{k\geq 1}(1+i_d)^{-(k-1)}\sum_{u\in\mathcal{U}} {}_{k-1}p^{u_0,u}P_k^u},$$

where P_k^u is the premium payable in advance of the kth year if the survival status in advance of the kth year is u.