BS4a
Actuarial Science I

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MT 2017

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16 lectures MT 2017

Prerequisites

A8 Probability is useful, but not essential. If you have not done A8 Probability, make sure that you are familiar with Prelims work on Probability.

Synopsis

Fundamental nature of actuarial work. Use of generalised cash-flow model to describe financial transactions. Time value of money using the concepts of compound interest and discounting.

Interest rate models. Present values and the accumulated values of a stream of equal or unequal payments using specified rates of interest. Interest rates in terms of different time periods. Equation of value, rate of return of a cash-flow, existence criteria.

Loan repayment schemes.

Single decrement model. Present values and accumulated values of a stream of payments taking into account the probability of the payments being made according to a signle decrement model. Annuity functions and assurance functions for a single decrement model. Risk and premium calculation.

Liabilities under a simple assurance contract or annuity contract. Premium reserves, death strain. Expenses and office premiums. Conventional and accumulation with-profits and unit-linked contracts, assurance and annuity functions involving two lives, valuing expected cash-flows in multiple decrement models, profit testing.

Reading

All of the following are available from the Faculty and Institute of Actuaries, 1st Floor, Park Central, 40/41 Park End Street, Oxford, OX1 1JD and https://www.actuaries.org.uk/shop.

- Subject CT1: *Financial Mathematics Core reading.* Institute & Faculty of Actuaries
- Subject CT5: *Contingencies Core Reading.* Institute & Faculty of Actuaries.
- J. J. McCutcheon and W. F. Scott: *An Introduction to the Mathematics of Finance.* Heinemann (1986)
- P. Zima and R. P. Brown: *Mathematics of Finance.* McGraw-Hill Ryerson (1993)
- N. L. Bowers et al, Actuarial mathematics, 2nd edition, Society of Actuaries (1997)
- J. Danthine and J. Donaldson: *Intermediate Financial Theory.* 2nd edition, Academic Press Advanced Finance (2005)

Lecture 1

Introduction

Reading: CT1 Core Reading Unit 1
Further reading: http://www.actuaries.org.uk

After some general information about relevant history and about the work of an actuary, we introduce cash-flow models as the basis of this course and as a suitable framework to describe and look beyond the contents of this course.

1.1 The actuarial profession

Actuarial Science is a discipline with its own history. The Institute of Actuaries was formed in 1848, (the Faculty of Actuaries in Scotland in 1856, the two merged in 2010), but the roots go back further. An important event was the construction of the first life table by Sir Edmund Halley in 1693. However, Actuarial Science is not old-fashioned. The language of probability theory was gradually adopted as it developed in the 20th century; computing power and new communication technologies have changed the work of actuaries. The growing importance and complexity of financial markets continues to fuel actuarial work; current debates on changes in life expectancy, retirement age, viability of pension schemes are core actuarial topics that the profession vigorously embraces.

Essentially, the job of an actuary is risk assessment. Traditionally, this was insurance risk, life insurance, later general insurance (health, home, property etc). As typically large amounts of money, reserves, have to be maintained, this naturally extended to investment strategies including the assessment of risk in financial markets. Today, the Actuarial Profession sees its role yet more broadly in helping "make financial sense of an uncertain future".

To become a Fellow of the Institute/Faculty of Actuaries in the UK, an actuarial trainee currently has to pass nine mathematics, statistics, economics and finance examinations (core technical series – CT), examinations on risk management, reporting and communication skills (core applications – CA), and three specialist examinations in the chosen areas of specialisation (specialist technical and specialist applications series – ST and SA) and for a UK fellowship an examination on UK specifics. This programme takes normally at least three or four years after a mathematical university degree and while working for an insurance company under the guidance of a Fellow of the Institute/Faculty of Actuaries.

This lecture course is an introductory course where important foundations are laid and an overview of further actuarial education and practice is given. An upper second mark in the examination of the SB4a and SB4b units normally entitles to an exemption from the CT1 and

CT5 papers. The CT3 paper is covered by the Part A Probability and Statistics courses. The units SB3a and SB3b include the material of the CT4 paper.

The Actuarial Profession is preparing a new module structure for 2019. The exemptions from CT1 and CT5 will translate into an exemption from the new module CM1, while the exemption from CT3 will translate into an exemption from the new module CS1. An exemption from CT4 would only translate into an exemption from CS2 in conjunction with a further exemption from CT6.

1.2 The generalised cash-flow model

The cash-flow model systematically captures payments either between different parties or, as we shall focus on, in an inflow/outflow way from the perspective of one party. This can be done at different levels of detail, depending on the purpose of an investigation, the complexity of the situation, the availability of reliable data etc.

Example 1 Look at the transactions on a bank statement for September 2011.

Date	Description	Money out	Money in
01-09-11	Gas-Elec-Bill	£21.37	
04-09-11	Withdrawal	£100.00	
15-09-11	Telephone-Bill	£14.72	
16-09-11	Mortgage Payment	£396.12	
28-09-11	Withdrawal	£150.00	
30-09-11	Salary		£1,022.54

Extracting the mathematical structure of this example we define elementary cash-flows.

Definition 2 A *cash-flow* is a vector $(t_j, c_j)_{1\le j\le m}$ of *times* $t_j \in \mathbb{R}$ and *amounts* $c_j \in \mathbb{R}$. Positive amounts $c_j > 0$ are called *inflows.* If $c_j < 0$, then $|c_j|$ is called an *outflow.*

Example 3 The cash-flow of Example 1 is mathematically given by

j	t_j	c_j
1	1	−21.37
2	4	−100.00
3	15	−14.72

j	t_j	c_j
4	16	−396.12
5	28	−150.00
6	30	1,022.54

Often, the situation is not as clear as this, and there may be uncertainty about the time/amount of a payment. This can be modelled stochastically.

Definition 4 A *random cash-flow* is a *random* vector $(T_j, C_j)_{1\le j\le M}$ of *times* $T_j \in \mathbb{R}$ and *amounts* $C_j \in \mathbb{R}$ with a possibly random length $M \in \mathbb{N}$.

Sometimes, in fact always in this course, the random structure is simple and the times or the amounts are deterministic, or even the only randomness is that a well specified payment may fail to happen with a certain probability.

Example 5 Future transactions on a bank account (say for November 2011)

j	T_j	C_j	Description
1	1	−21.37	Gas-Elec-Bill
2	T_2	C_2	Withdrawal?
3	15	C_3	Telephone-Bill

j	T_j	C_j	Description
4	16	−396.12	Mortgage payment
5	T_5	C_5	Withdrawal?
6	30	1,022.54	Salary

Here we assume a fixed Gas-Elec-Bill but a varying telephone bill. Mortgage payment and salary are certain. Any withdrawals may take place. For a full specification of the random cash-flow we would have to give the (joint!) laws of the random variables.

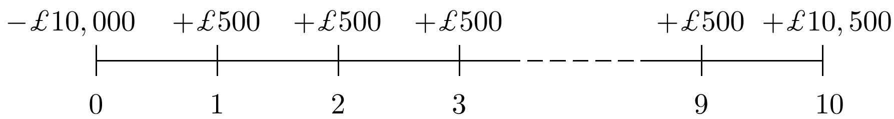
This example shows that simple situations are not always easy to model. It is an important part of an actuary's work to simplify reality into tractable models. Sometimes, it is worth dropping or generalising the time specification and just list approximate or qualitative ('big', 'small', etc.) amounts of income and outgo. cash-flows can be represented in various ways as the following important examples illustrate.

1.3 Examples and course overview

Example 6 (Zero-coupon bond) Usually short-term investments with interest paid at the end of the term, e.g. invest £99 for ninety days for a payoff of £100.

j	t_j	c_j
1	0	−99
2	90	100

Example 7 (Government bonds, fixed-interest securities) Usually long-term investments with annual or semi-annual *coupon payments* (interest), e.g. invest £10,000 for ten years at 5% per annum. [The government borrows money from investors.]

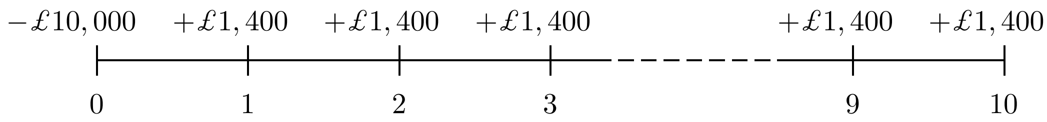


Example 8 (Corporate bonds) The underlying cash-flow looks the same as for government bonds, but they are not as secure. *Credit rating agencies* assess the insolvency risk. If a company goes bankrupt, invested money is often lost. One may therefore wish to add probabilities to the cash-flow in the above figure. Typically, the interest rate in corporate bonds is higher to allow for this extra *risk of default* that the investor takes.

Example 9 (Equities) Shares in the ownership of a company that entitle to regular *dividend payments* of amounts depending on the profit and strategy of the company. Equities can be bought and sold on *stock markets* (via a stock broker) at fluctuating market prices. In the above diagram (including payment probabilities) the inflow amounts are not fixed, the term at the discretion of the shareholder and sales proceeds are not fixed. There are *advanced stochastic models* for stock price evolution. A wealth of derivative products is also available, e.g. forward contracts, options to sell or buy shares. We will discuss forward contracts, but otherwise refer to B10b Mathematical Models for Financial Derivatives.

Example 10 (Index-linked securities) Inflation-adjusted securities: coupons and redemption payment increase in line with inflation, by tracking an *inflation index.*

Example 11 (Annuity-certain) Long term investments that provide a series of regular annual (semi-annual or monthly) payments for an initial lump sum, e.g.



Here the term is $n = 10$ years. *Perpetuities* provide regular payment forever ($n = \infty$).

Example 12 (Loans) Formally the negative of a bond cash-flow (interest-only loan) or annuity-certain (repayment loan), but the rights of the parties are not exactly opposite. Whereas the bond investor may be able to redeem or sell the bond early, the lender of a loan often has to obey stricter rules, to protect the borrower.

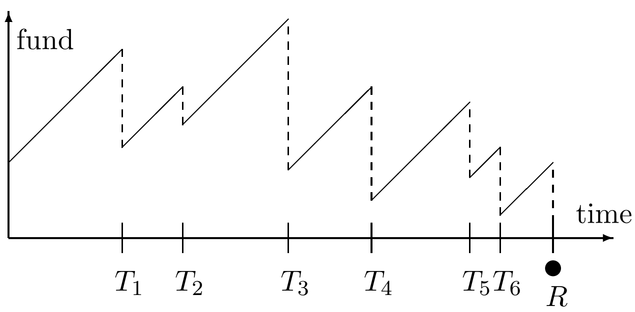
Example 13 (Appraisal of investment projects) Consider a building project. An initial construction period requires certain payments, the following exploitation (e.g. letting) yields income in return for the investment, but maintenance has to be taken into accout as well. Under what circumstances is the project profitable? How reliable are the estimated figures?

These "qualitative" questions, that can be answered qualitatively using specifications as stable, predictable, variable, increasing etc. are just as important as precise estimates.

Example 14 (Life annuity) Life annuities are like annuities-certain, but do not terminate at a fixed time but when the beneficiary dies. Risks due to age, health, profession etc. when entering the annuity contract determine the payment level. They are a basic form of a pension. Several modifications exist (minimal term, maximal term, etc.).

Example 15 (Life assurance) Pays a lump sum on death for monthly or annual premiums that depend on age and health of the policy holder when the policy is underwritten. The sum assured may be decreasing in accordance with an outstanding mortgage.

Example 16 (Property insurance) A class of general insurance (others are health, building, motor etc.). In return for regular premium payments, an insurance company replaces or refunds any stolen or damaged items included on the policy. From the insurer's point of view, all policy holders pay into a common fund to provide for those who claim. The claim history of policy holders affects their premium.



A branch of an insurance company is said to suffer *technical ruin* if the fund runs empty.

Lecture 2

The theory of compound interest

Reading: CT1 Core Reading Units 2-3, McCutcheon-Scott Chapter 1, Sections 2.1-2.4

Quite a few problems that we deal with in this course can be approached in an intuitive way. However, the mathematical and more powerful approach to problem solving is to set up a mathematical model in which the problem can be formalised and generalised. The concept of cash-flows seen in the last lecture is one part of such a model. In this lecture, we shall construct another part, the compound interest model in which interest on capital investments, loans etc. can be computed. This model will play a crucial role throughout the course.

In any mathematical model, reality is only partially represented. An important part of mathematical modelling is the discussion of model assumptions and the interpretation of the results of the model.

2.1 Simple versus compound interest

We are familiar with the concept of interest in everyday banking: the bank pays interest on positive balances on current accounts and savings accounts (not much, but some), and it charges interest on loans and overdrawn current accounts. Reasons for this include that

- people/institutions borrowing money are willing to pay a fee (in the future) for the use of this money now,
- there is price inflation in that £100 lose purchasing power between the beginning and the end of a loan as prices increase,
- there is often a risk that the borrower may not be able to repay the loan.

To develop a mathematical framework, consider an "interest rate h" per unit time, under which an investment of C at time 0 will receive interest Ch by time 1, giving total value $C(1+h)$:

$$C \quad \longrightarrow \quad C(1+h),$$

e.g. for $h = 4\%$ we get $C \longrightarrow 1.04C$.

There are two natural ways to extend this to general times t:

Definition 17 (Simple interest) *Invest C, receive $C(1+th)$ after t years. The* simple *interest on C at rate h for time t is Cth.*

Definition 18 (Compound interest) *Invest C, receive $C(1+i)^t$ after t years. The* compound *interest on C at rate i for time t is $C((1+i)^t-1)$.*

For integer $t=n$, this is as if a bank balance was updated at the end of each year

$$C \longrightarrow C(1+i) \longrightarrow (C(1+i))(1+i) = C(1+i)^2 \longrightarrow (C(1+i)^{n-1})(1+i) = C(1+i)^n.$$

Example 19 Given an interest rate of $i=6\%$ per annum (p.a.), investing $C=£1,000$ for $t=2$ years yields

$$I_{\text{simp}} = C2i = £120.00 \quad \text{and} \quad I_{\text{comp}} = C\left((1+i)^2-1\right) = C(2i+i^2) = £123.60,$$

where we can interpret Ci^2 as interest on interest, i.e. interest for the second year paid at rate i on the interet Ci for the first year.

Compound interest behaves well under term-splitting: for $t=s+r$

$$C \longrightarrow C(1+i)^s \longrightarrow (C(1+i)^s)\,(1+i)^r = C(1+i)^t,$$

i.e. investing C at rate i first for s years and then the resulting $C(1+i)^s$ for a further r years gives the same as directly investing C for $t=s+r$ years. Under simple interest

$$C \longrightarrow C(1+hs) \longrightarrow C(1+hs)(1+hr) = C(1+ht+srh^2) > C(1+ht),$$

(in the case $C>0$, $r>0$, $s>0$). The difference $Csrh^2=(Chs)hr$ is interest on the interest Chs that was already paid at time s for the first s years.

What is the best we can achieve by term-splitting under simple interest?

Denote by $S_t(C)=C(1+th)$ the accumulated value under simple interest at rate h for time t. We have seen that $S_r \circ S_s(C) > S_{r+s}(C)$.

Proposition 20 *Fix $t>0$ and h. Then*

$$\sup_{n\in\mathbb{N}, r_1,\ldots,r_n\in\mathbb{R}_+ : r_1+\cdots+r_n=t} S_{r_n}\circ S_{r_{n-1}}\circ\cdots\circ S_{r_1}(C) = \lim_{n\to\infty} S_{t/n}\circ\cdots\circ S_{t/n}(C) = e^{th}C.$$

Proof: For the second equality we first note that

$$S_{t/n}\circ\cdots\circ S_{t/n}(C) = \left(1+\frac{t}{n}h\right)^n C \to e^{th}C,$$

because

$$\log((1+th/n)^n) = n\log(1+th/n) = n(th/n+O(1/n^2)) \to th.$$

For the first equality,

$$e^{rh} = 1+rh+\frac{r^2h^2}{2}+\cdots \geq 1+rh$$

so if $r_1+\cdots+r_n=t$, then

$$e^{th}C = e^{r_1h}e^{r_2h}\cdots e^{r_nh}C \geq (1+r_1h)(1+r_2h)\cdots(1+r_nh)C = S_{r_n}\circ S_{r_{n-1}}\circ\cdots\circ S_{r_1}(C).$$

□

So, the optimal achievable is $C \longrightarrow Ce^{th}$. If $e^{th} = (1+i)^t$, i.e. $e^h = 1+i$ or $h = \log(1+i)$, we recover the compound interest case.

From now on, we will always consider compound interest.

Definition 21 Given an *effective interest rate i per unit time* and an *initial capital C* at time 0, the *accumulated value at time t* under the compound interest model (with constant rate) is given by

$$C(1+i)^t = Ce^{\delta t},$$

where

$$\delta = \log(1+i) = \frac{\partial}{\partial t}(1+i)^t \bigg|_{t=0},$$

is called the *force of interest.*

The second expression for the force of interest means that it is the "instantaneous rate of growth per unit capital per unit time".

2.2 Nominal and effective rates

The effective annual rate is i such that $C \longrightarrow C(1+i)$ after one year. We have already seen the force of interest $\delta = \log(1+i)$ as a way to describe the same interest rate model. In practice, rates are often quoted in other ways still.

Definition 22 A *nominal rate h convertible pthly* (or compounded p times per year) means that an accumulated value $C(1+h/p)$ is achieved after time $1/p$.

By compounding, the accumulated value at time 1 is $C(1+h/p)^p$, and at time t is $C(1+h/p)^{pt}$. This again describes the same model of accumulated values if $(1+h/p)^p = 1+i$, i.e. if $h = p((1+i)^{1/p} - 1$. Actuarial notation for the nominal rate convertible $p$thly associated with effective rate i is $i^{(p)} = p((1+i)^{1/p} - 1)$.

Example 23 An annual rate of 8% convertible quarterly, i.e. $i^{(4)} = 8\%$ means that $i^{(4)}/4 = 2\%$ is credited each 3 months (and compounded) giving an annual effecive rate $i = (1+i^{(4)}/4)^4 - 1 \approx 8.24\%$.

The most common frequencies are for $p = 2$ (half-yearly, semi-annually), $p = 4$ (quarterly), $p = 12$ (monthly), $p = 52$ (weekly), although the latter used to be approximated using

$$\lim_{p\to\infty} i^{(p)} = \lim_{p\to\infty} \frac{(1+i)^{1/p} - 1}{1/p} = \frac{\partial}{\partial t}(1+i)^t \bigg|_{t=0} = \log(1+i) = \delta;$$

the force of interest δ can be called the "nominal rate of interest convertible continuously".

Example 24 Here are two genuine and one artificial options for a savings account.

(1) 3.25% p.a. effective ($i_1 = 3.25\%$)

(2) 3.20% p.a. nominal convertible monthly ($i_2^{(12)} = 3.20\%$)

(3) 3.20% p.a. nominal "convertible continuously" ($\delta_3 = 3.20\%$)

After one year, an initial capital of £10,000 accumulates to

(1) $10,000 \times (1 + 3.25\%) = 10,325.00 = R_1$,

(2) $10,000 \times (1 + 3.20\%/12)^{12} = 10,324.74 = R_2$,

(3) $10,000 \times e^{3.20\%} = 10,325.18 = R_3$.

Although interest may be credited to the account differently, an investment into (j) just consists of deposit and withdrawal, so the associated cash-flow is $((0, -10000), (1, R_j))$, and we can use R_j to decide between the options. We can also compare $i_2 \approx 3.2474\%$ and $i_3 \approx 3.2518\%$ or calculate δ_1 and δ_2 to compare with δ_3 etc.

Interest rates always refer to some time unit. The standard choice is one year, but it sometimes eases calculations to choose six months, one month or one day. All definitions we have made reflect the assumption that the interest rate does not vary with the initial capital C nor with the term t. We refer to this model of accumulated values as the constant-i model, or the constant-δ model.

2.3 Discount factors and discount rates

Before we more fully apply the constant-i model to cash-flows in Lecture 3, let us discuss the notion of discount. We are used to discounts when shopping, usually a percentage reduction in price, time being implicit. Actuaries use the notion of an *effective rate of discount d per time unit* to represent a reduction of C to $C(1-d)$ if payment takes place a time unit early.

This is consistent with the constant-i model, if the payment of $C(1-d)$ accumulates to $C = (C(1-d))(1+i)$ after one time unit, i.e. if

$$(1-d)(1+i) = 1 \iff d = 1 - \frac{1}{1+i}.$$

A more prominent role will be played by the *discount factor* $v = 1 - d$, which answers the question

How much will we have to invest now to have 1 at time 1?

Definition 25 In the constant-i model, we refer to $v = 1/(1+i)$ as the associated *discount factor* and to $d = 1 - v$ as the associated effective annual rate of discount.

Example 26 How much do we have to invest now to have 1 at time t? If we invest C, this accumulates to $C(1+i)^t$ after t years, hence we have to invest $C = 1/(1+i)^t = v^t$.

Lecture 3

Valuing cash-flows

Reading: CT1 Core Reading Units 3 and 5, McCutcheon-Scott Chapter 2

In Lecture 2 we set up the constant-i interest rate model and saw how a past deposit accumulates and a future payment can be discounted. In this lecture, we combine these concepts with the cash-flow model of Lecture 1 by assigning time-t values to cash-flows. We also introduce general (deterministic) time-dependent interest models, and continuous cash-flows that model many small payments as infinitesimal payment streams.

3.1 Accumulating and discounting in the constant-i model

Given a cash-flow $c = (c_j, t_j)_{1\le j\le m}$ of payments c_j at time t_j and a time t with $t \ge t_j$ for all j, we can write the joint accumulated value of all payments by time t according to the constant-i model as

$$\text{AVal}_t(c) = \sum_{j=1}^{m} c_j(1+i)^{t-t_j} = \sum_{j=1}^{m} c_j e^{\delta(t-t_j)},$$

because each payment c_j at time t_j earns compound interest for $t-t_j$ time units. Note that some c_j may be negative, so the accumulated value could become negative. We assume implicitly that the same interest rate applies to positive and negative balances.

Similarly, given a cash-flow $c = (c_j, t_j)_{1\le j\le m}$ of payments c_j at time t_j and a time $t > t_j$ for all j, we can write the joint *discounted value* at time t of all payments as

$$\text{DVal}_t(c) = \sum_{j=1}^{m} c_j v^{t_j-t} = \sum_{j=1}^{m} c_j(1+i)^{-(t_j-t)} = \sum_{j=1}^{m} c_j e^{-\delta(t_j-t)}.$$

This discounted value is the amount we invest at time t to be able to spend c_j at time t_j for all j.

3.2 Time-dependent interest rates

So far, we have assumed that interest rates are constant over time. Suppose, we now let $i = i(k)$ vary with time $k \in \mathbb{N}$. We define the accumulated value at time n for an investment of C at time 0 as

$$C(1+i(1))(1+i(2))\cdots(1+i(n-1))\cdots(1+i(n)).$$

Example 27 A savings account pays interest at $i(1) = 2\%$ in the first year and $i(2) = 5\%$ in the second year, with interest from the first year reinvested. Then the account balance evolves as $1,000 \longrightarrow 1,000(1 + i(1)) = 1,020 \longrightarrow 1,000(1 + i(1))(1 + i(2)) = 1,071$.

When varying interest rates between non-integer times, it is often nicer to specify the force of interest $\delta(t)$ which we saw to have a local meaning as the infinitesimal rate of capital growth under compound interest:

$$C \longrightarrow C \exp\left(\int_0^t \delta(s)ds\right) = R(t).$$

Note that now (under some right-continuity assumptions)

$$\frac{\partial}{\partial t}R(t)\bigg|_{t=0} = \delta(0) \qquad \text{and more generally} \qquad \frac{\partial}{\partial t}R(t) = \delta(t)R(t),$$

so that the interpretation of $\delta(t)$ as local rate of capital growth at time t still applies.

Example 28 If $\delta(\cdot)$ is piecewise constant, say constant δ_j on $(t_{j-1}, t_j]$, $j = 1, \dots, n$, then

$$C \longrightarrow Ce^{\delta_1 r_1}e^{\delta_2 r_2} \cdots e^{\delta_n r_n}, \qquad \text{where } r_j = t_j - t_{j-1}.$$

Definition 29 Given a time-dependent force of interest $\delta(t)$, $t \in \mathbb{R}_+$, we define the *accumulated value* at time $t \geq 0$ of an initial capital $C \in \mathbb{R}$ *under a force of interest* $\delta(\cdot)$ as

$$R(t) = C \exp\left(\int_0^t \delta(s)ds\right).$$

Also, we may refer to $I(t) = R(t) - C$ as the *interest from time* 0 *to time* t *under* $\delta(\cdot)$.

3.3 Accumulation factors

Given a time-dependent interest model $\delta(\cdot)$, let us define *accumulation factors from* s *to* t

$$A(s,t) = \exp\left(\int_s^t \delta(r)dr\right), \qquad s < t. \tag{1}$$

Just as $C \longrightarrow R(t) = CA(0,t)$ for an investment of C at time 0 for a term t, we use $A(s,t)$ as a factor to turn an investment of C at time s into its accumulated value $CA(s,t)$ at time t. This behaves well under term-splitting, since

$$C \longrightarrow CA(0,s) \longrightarrow (CA(0,s))A(s,t) = C \exp\left(\int_0^s \delta(r)dr\right) \exp\left(\int_s^t \delta(r)dr\right) = CA(0,t).$$

More generally, note the *consistency property* $A(r,s)A(s,t) = A(r,t)$, and conversely:

Proposition 30 *Suppose,* $A\colon \{(s,t) : s \leq t\} \to (0,\infty)$ *satisfies the consistency property and* $t \mapsto A(s,t)$ *is differentiable for all* s*, then there is a function* $\delta(\cdot)$ *such that (1) holds.*

Proof: Since consistency for $r = s = t$ implies $A(t,t) = 1$, we can define as (right-hand) derivative

$$\delta(t) = \lim_{h\downarrow 0} \frac{A(t,t+h) - A(t,t)}{h} = \lim_{h\downarrow 0} \frac{A(0,t+h) - A(0,t)}{hA(0,t)},$$

where we also applied consistency. With $g(t) = A(0,t)$ and $f(t) = \log(A(0,t))$

$$\delta(t) = \frac{g'(t)}{g(t)} = f'(t) \quad \Rightarrow \quad \log(A(0,t)) = f(t) = \int_0^t \delta(s)ds.$$

Since consistency implies $A(s,t) = A(0,t)/A(0,s)$, we obtain (1). □

We included the apparently unrealistic $A(s,t) < 1$ (accumulated value less than the initial capital) that leads to negative $\delta(\cdot)$. This can be useful for some applications where $\delta(\cdot)$ is not pre-specified, but connected to investment performance where prices can go down as well as up, or to inflation/deflation. Similarly, we allow any $i \in (-1,\infty)$, so that the associated 1-year accumulation factor $1+i$ is positive, but possibly less than 1.

3.4 Time value of money

We have discussed accumulated and discounted values in the constant-i model. In the time-varying $\delta(\cdot)$ model with accumulation factors $A(s,t) = \exp(\int_s^t \delta(r)dr)$, we obtain

$$\mathrm{AVal}_t(c) = \sum_{j=1}^m c_j A(t_j,t) \quad \text{if all } t_j \le t, \qquad \mathrm{DVal}_t(c) = \sum_{j=1}^m c_j V(t,t_j) \quad \text{if all } t_j > t,$$

where $V(s,t) = 1/A(s,t) = \exp(-\int_s^t \delta(r)dr)$ is the discount factor from time t back to time $s \le t$. With $v(t) = V(0,t)$, we get $V(s,t) = v(t)/v(s)$. Notation $v(t)$ is useful, as it is often the *present value*, i.e. the discounted value at time 0, that is of interest, and we then have

$$\mathrm{DVal}_0(c) = \sum_{j=1}^m c_j v(t_j), \quad \text{if all } t_j > 0,$$

where each payment is discounted by $v(t_j)$. Each future payment has a different present value. Note that the formulas for AVal_t and DVal_t are identical, if we express $A(s,t)$ and $V(s,t)$ in terms of $\delta(\cdot)$.

Definition 31 The *time-t value* of a cash-flow c is defined as

$$\mathrm{Val}_t(c) = \mathrm{AVal}_t(c_{\le t}) + \mathrm{DVal}_t(c_{>t}),$$

where $c_{\le t}$ and $c_{>t}$ denote restrictions of c to payments at times $t_j \le t$ resp. $t_j > t$.

Proposition 32 *For all $s \le t$ we have* $\mathrm{Val}_t(c) = \mathrm{Val}_s(c)A(s,t) = \mathrm{Val}_s(c)\dfrac{v(s)}{v(t)}$.

The proof is straightforward and left as an exercise. Note in particular, that if $\mathrm{Val}_t(c) = 0$ for some t, then $\mathrm{Val}_t(c) = 0$ for all t.

Remark 33

1. A sum of money without time specification is meaningless.
2. Do not add or directly compare values at different times.
3. If values of two cash-flows are equal at one time, they are equal at all times.

3.5 Continuous cash-flows

If many small payments are spread evenly over time, it is natural to model them by a continuous stream of payment.

Definition 34 A *continuous cash-flow is a function* $c\colon \mathbb{R} \to \mathbb{R}$. *The total* net *inflow between times* s *and* t *is*

$$\int_s^t c(r)dr,$$

and this may combine periods of inflow and outflow.

As before, we can consider random c. We can also mix continuous and discrete parts. Note that the net inflow "adds" values at different times ignoring the time-value of money. More useful than the net inflow are accumulated and discounted values

$$\mathrm{AVal}_t(c) = \int_0^t c(s)A(s,t)ds \quad \text{and} \quad \mathrm{DVal}_t(c) = \int_t^\infty c(s)V(t,s)ds = \frac{1}{v(t)}\int_t^\infty c(s)v(s)ds.$$

Everything said in the previous section applies in an analogous way.

3.6 Example: withdrawal of interest as a cash-flow

Consider a savings account that does not credit interest to the savings account itself (where it is further compounded), but triggers a cash-flow of interest payments.

1. In a $\delta(\cdot)$-model, $1 \longrightarrow 1 + I = \exp(\int_0^1 \delta(ds))$. Consider the interest cash-flow $(1, -I)$. Then $\mathrm{Val}_1((0,1),(1,-I)) = 1$ is again the capital, at time 1.

2. In the constant-i model, recall the nominal rate $i^{(p)} = p((1+i)^{1/p} - 1)$. Interest on an initial capital 1 up to time $1/p$ is $i^{(p)}/p$. After one or indeed k such $p$thly interest payments of $i^{(p)}/p$, we have $\mathrm{Val}_{k/p}((0,1),(1/p,-i^{(p)}/p),\dots,(k/p,-i^{(p)}/p)) = 1$.

3. In a $\delta(\cdot)$-model, continuous cash-flow $c(s) = \delta(s)$ has $\mathrm{Val}_t((0,1),-c_{\le t}) = 1$ for all t.

We leave as an exercise to check these directly from the definitions.

Note, in particular, that accumulation of interest itself does not correspond to events in the cash-flow. Cash-flows describe external influences on the account. Although interest is not credited continuously or at every withdrawal in practice, our mathematical model does assign a balance=value that changes continuously between instances of external cash-flow. We include the effect of interest in a cash-flow by withdrawal.

Lecture 4

The yield of a cash-flow

Reading: CT1 Core Reading Unit 7, McCutcheon-Scott Section 3.2

Given a cash-flow representing an investment, its yield is the constant interest rate that makes the cash-flow a fair deal. Yields allow to assess and compare the performance of possibly quite different investment opportunities as well as mortgages and loans.

4.1 Definition of the yield of a cash-flow

In that follows, it does not make much difference whether a cash-flow c is discrete, continuous or mixed, whether the time horizon of c is finite or infinite (like e.g. for perpetuities). However, to keep statements and technical arguments simple, we assume:

$$\text{The time horizon of } c \text{ is finite and payment rates of } c \text{ are bounded.} \tag{H}$$

Since we will compare values of cash-flows under different interest rates, we need to adapt our notation to reflect this:

$$\text{NPV}(i) = i\text{-Val}_0(c)$$

denotes the Net Present Value of c discounted in the constant-i interest model, i.e. the value of the cash-flow c at time 0, discounted using discount factors $v(t) = v^t = (1+i)^{-t}$.

Lemma 35 *Given a cash-flow c satifying hypothesis* (H)*, the function $i \mapsto \text{NPV}(i)$ is continuous on $(-1, \infty)$.*

Proof: In the discrete case $c = ((t_1, c_1), \ldots, (t_n, c_n))$, we have $\text{NPV}(i) = \sum_{k=1}^{n} c_k (1+i)^{-t_k}$, and this is clearly continuous in i for all $i > -1$. For a continuous-time cash-flow $c(s)$, $0 \le s \le t$ (and mixed cash-flows) we use the uniform continuity of $i \mapsto (1+i)^{-s}$ on compact intervals $s \in [0, t]$ for continuity to be maintained after integration

$$\text{NPV}(i) = \int_0^t c(s)(1+i)^{-s} ds.$$

□

Corollary 36 *Under hypothesis* (H)*, $i \mapsto i\text{-Val}_t(i)$ is continuous on $(-1, \infty)$ for any t.*

Often the situation is such that an investment deal is profitable ($\text{NPV}(i) > 0$) if the interest rate i is below a certain level, but not above, or vice versa. By the intermediate value theorem, this threshold is a zero of $i \mapsto \text{NPV}(i)$, and we define

Definition 37 Given a cash-flow c, if $i \mapsto \mathrm{NPV}(i)$ has a unique root on $(-1, \infty)$, we define the *yield* $y(c)$ to be this root. If $i \mapsto \mathrm{NPV}(i)$ does not have a root in $(-1, \infty)$ or the root is not unique, we say that the yield is not well-defined.

The yield is also known as the "internal rate of return" or also just "rate of return". We can say that the yield is the fixed interest rate at which c is a "fair deal". The equation $\mathrm{NPV}(i) = 0$ is called *yield equation.*

Example 38 Suppose that for an initial investment of £1,000 you obtain a payment of £400 after one year and 770 after two years. What is the yield of this deal? Clearly $c = ((0, -1000), (1, 400), (2, 770)$. By definition, we are looking for roots $i \in (-1, \infty)$ of

$$\begin{aligned} \mathrm{NPV}(i) = -1,000 + 400(1+i)^{-1} + 800(1+i)^{-2} = 0 \\ \iff \qquad 1,000(i+1)^2 - 400(i+1) - 770 = 0 \end{aligned}$$

The solutions to this quadratic equation are $i_1 = -1.7$ and $i_2 = 0.1$. Since only the second zero lies in $(-1, \infty)$, the yield is $y(c) = 0.1$, i.e. 10%.

Sometimes, it is onvenient to solve for $v = (1+i)^{-1}$, here $1,000 - 400v - 770v^2 = 0$ etc. Note that $i \in (-1, \infty) \iff v \in (0, \infty)$.

Example 39 Consider the security of Example 7 in Lecture 1. The yield equation $\mathrm{NPV}(i) = 0$ can be written as

$$10,000 = 500 \sum_{k=1}^{10} (1+i)^{-k} + 10,000(1+i)^{-10}.$$

We will introduce some short-hand actuarial notation in Lecture 5. Note, however, that we already know a root of this equation, because the cash-flow is the same as for a bank account with capital $£10,000$ and a cash-flow of annual interest payments of $£500$, i.e. at 5%, so $i = 5\%$ solves the yield equation. We will now see in much higher generality that there is usually only one solution to the yield equation for investment opportunities.

4.2 General results ensuring the existence of yields

Since the yield does not always exist, it is useful to have sufficient existence criteria.

Proposition 40 *If c has in- and outflows and all inflows of c precede all outflows of c (or vice versa), then the yield $y(c)$ exists.*

Remark 41 This includes the vast majority of projects that we will meet in this course. Essentially, investment projects have outflows first, and inflows afterwards, while loan schemes (from the borrower's perspective) have inflows first and outflows afterwards.

Proof: By assumption, there is T such that all inflows are strictly before T and all outflows are strictly after T. Then the accumulated value

$$p_i = i\text{-}\mathrm{Val}_T(c_{<T})$$

is positive strictly increasing in i with $p_{-1} = 0$ and the discounted value $p_\infty = \infty$ (by assumption there are inflows) and

$$n_i = i\text{-}\mathrm{Val}_T(c_{>T})$$

is negative strictly increasing with $n_{-1} = -\infty$ (by assumption there are outflows) and $n_\infty = 0$. Therefore

$$b_i = p_i + n_i = i\text{-Val}_T(c)$$

is strictly increasing from $-\infty$ to ∞, continuous by Corollary 36; its unique root i_0 is also the unique root of $i \mapsto \text{NPV}(i) = (1+i)^{-T}\,(i\text{-Val}_T(c))$ by Corollary 32.

For the "vice versa" part, replace c by $-c$ and use $\text{Val}_0(c) = -\text{Val}_0(-c)$ etc. □

Corollary 42 *If all inflows precede all outflows, then*

$$y(c) > i \iff \text{NPV}(i) < 0.$$

If all outflows precede all inflows, then

$$y(c) > i \iff \text{NPV}(i) > 0.$$

Proof: In the first setting assume $y(c) > 0$, we know $b_{y(c)} = 0$ and $i \mapsto b_i$ increases with i, so $i < y(c) \iff b_i < 0$, but $b_i = i\text{-Val}_T(c) = (1+i)^T\text{NPV}(i)$.

The second setting is analogous (substitute $-c$ for c). □

As a useful example, consider $i = 0$, when $\text{NPV}(0)$ is the sum of undiscounted payments.

Example 39 (continued) By Proposition 40, the yield exists and equals $y(c) = 5\%$.

4.3 Example: APR of a loan

A yield that is widely quoted in practice, is the Annual Percentage Rate (APR) of a loan. This is straightforward if the loan agreement is based on a constant interest rate i. Particularly for mortgages (loans to buy a house), it is common, however, to have an initial period of lower interest rates and lower monthly payments followed by a period of higher interest rates and higher payments. The APR then gives a useful summary value:

Definition 43 Given a cash-flow c representing a loan agreement (with inflows preceding outflows), the yield $y(c)$ rounded down to next lower 0.1% is called the *Annual Percentage Rate (APR)* of the loan.

Example 44 Consider a mortgage of £85,000 with interest rates of 2.99% in year 1, 4.19% in year 2 and 5.95% for the remainder of a 20-year term. A Product Fee of £100 is added to the loan amount, and a Funds Transfer Fee is deducted from the Net Amount provided to the borrower. We will discuss in Lecture 6 how this leads to a cash-flow of

$$c = ((0, 84975), (1, -5715), (2, -6339), (3, -7271), (4, -7271), \ldots, (20, -7271)),$$

and how the annual payments are further transformed into equivalent monthly payments. Let us here calculate the APR, which exists by Proposition 40. Consider

$$f(i) = 84,975 - 5,715(1+i)^{-1} - 6,339(1+i)^{-2} - 7,271\sum_{k=3}^{20}(1+i)^{-k}.$$

Solving the geometric progression, or otherwise, we find the root iteratively by evaluation

$$f(5\%) = -3,310, \quad f(5.5\%) = 396, \quad f(5.4\%) = -326, \quad f(5.45\%) = 36.$$

From the last two, we see that $y(c) \approx 5.4\%$, actually $y(c) = 5.44503\ldots\%$. We can see that APR=5.4% already from the middle two evaluations, since we always round down, by definition of the APR.

4.4 Numerical calculation of yields

Suppose we know the yield exists, e.g. by Proposition 40. Remember that $f(i) = \mathrm{NPV}(i)$ is continuous and (usually) takes values of different signs at the boundaries of $(-1, \infty)$.

Interval splitting allows to trace the root of $f\colon (l_0, r_0) = (-1, \infty)$, make successive guesses $i_n \in (l_n, r_n)$, calculate $f(i_n)$ and define

$$(l_{n+1}, r_{n+1}) := (i_n, r_n) \quad \text{or} \quad (l_{n+1}, r_{n+1}) = (l_n, i_n)$$

such that the values at the boundaries $f(l_{n+1})$ and $f(r_{n+1})$ are still of different signs. Stop when the desired accuracy is reached.

The challenge is to make good guesses. Bisection

$$i_n = (l_n + r_n)/2$$

(once $r_n < \infty$) is the ad hoc way, *linear interpolation*

$$i_n = l_n \frac{f(r_n)}{f(r_n) - f(l_n)} + r_n \frac{-f(l_n)}{f(r_n) - f(l_n)}$$

an efficient improvement. There are more efficient variations of this method using some kind of convexity property of f, but that is beyond the scope of this course.

Actually, the iterations are for computers to carry out. For assignment and examination questions, you should make good guesses of l_0 and r_0 and carry out one linear interpolation, then claiming an *approximate* yield.

Example 44 (continued) Good guesses are $r_0 = 6\%$ and $l_0 = 5\%$, since $i = 5.95\%$ is mostly used. [Better, but a priori less obvious guess would be $r_0 = 5.5\%$.] Then

$$\left.\begin{array}{l} f(5\%) = -3,310.48 \\ f(6\%) = 3874.60 \end{array}\right\} \Rightarrow y(c) \approx 5\% \frac{f(6\%)}{f(6\%) - f(5\%)} + 6\% \frac{-f(5\%)}{f(6\%) - f(5\%)} = 5.46\%.$$

Lecture 5

Annuities and fixed-interest securities

Reading: CT1 Core Reading Units 6, 10.1, McCutcheon-Scott 3.3-3.6, 4, 7.2

In this chapter we introduce actuarial notation for discounted and accumulated values of regular payment streams, so-called *annuity symbols.* These are useful not only in the pricing of annuity products, but wherever regular payment streams occur. Our main example here will be fixed-interest securities.

5.1 Annuity symbols

Annuity-certain. An annuity-certain of term n entitles the holder to a cash-flow

$$c = ((1, X), (2, X), \ldots, (n-1, X), (n, X)).$$

Take $X = 1$ for convenience. In the constant-i model, its Net Present Value is

$$a_{\overline{n}|} = a_{\overline{n}|i} = \mathrm{NPV}(i) = \mathrm{Val}_0(c) = \sum_{k=1}^{n} v^k = v\frac{1-v^n}{1-v} = \frac{1-v^n}{i}.$$

The symbols $a_{\overline{n}|}$ and $a_{\overline{n}|i}$ are annuity symbols, pronounced "a angle n (at i)".

The accumulated value at end of term is

$$s_{\overline{n}|} = s_{\overline{n}|i} = \mathrm{Val}_n(c) = v^{-n}\mathrm{Val}_0(c) = \frac{(1+i)^n - 1}{i}.$$

***p*thly payable annuities.** A $p$thly payable annuity spreads (nominal) payment of 1 per unit time equally into p payments of $1/p$, leading to a cash-flow

$$c_p = ((1/p, 1/p), (2/p, 1/p), \ldots, (n - 1/p, 1/p), (n, 1/p))$$

with

$$a^{(p)}_{\overline{n}|} = \mathrm{Val}_0(c_p) = \frac{1}{p}\sum_{k=1}^{np} v^{k/p} = v^{1/p}\frac{1-v^n}{1-v^{1/p}} = \frac{1-v^n}{i^{(p)}},$$

where $i^p = p((1+i)^{1/p} - 1)$ is the nominal rate of interest convertible $p$thly associated with i. This calculation hence the symbol is meaningful for n any integer multiple of $1/p$.

We saw in Section 3.6 that, (now expressed in our new notation)

$$ia_{\overline{n}|} = \mathrm{Val}_0((1,i),(2,i),\ldots,(n,i)) = \mathrm{Val}_0((1/p,i^{(p)}/p),(2/p,i^{(p)}/p),\ldots,(n,i^{(p)}/p)) = i^{(p)}a_{\overline{n}|}^{(p)},$$

since both cash-flows correspond to the income up to time n on 1 unit invested at time 0, at effective rate i.

The accumulated value of c_p at the end of term is

$$s_{\overline{n}|}^{(p)} = \mathrm{Val}_n(c_p) = v^{-n}\mathrm{Val}_0(c_p) = \frac{(1+i)^n - 1}{i^{(p)}}.$$

Perpetuities. As $n \to \infty$, we obtain perpetuities that pay forever

$$a_{\overline{\infty}|} = \mathrm{Val}_0((1,1),(2,1),(3,1),\ldots) = \sum_{k=1}^{\infty} v^k = \frac{v}{1-v} = \frac{1}{i}.$$

Continuously payable annuities. As $p \to \infty$, the cash-flow c_p "tends to" the continuous cash-flow $c(s) = 1$, $0 \le s \le n$, with

$$\overline{a}_{\overline{n}|} = \mathrm{Val}_0(c) = \int_0^n c(s)v^s ds = \int_0^n v^s ds = \int_0^n e^{-\delta s} ds = \frac{1-e^{-\delta n}}{\delta} = \frac{1-v^n}{\delta}.$$

Or $\overline{a}_{\overline{n}|} = \mathrm{Val}_0(c) = \lim_{p\to\infty} a_{\overline{n}|}^{(p)} = \lim_{p\to\infty} \frac{1-v^n}{i^{(p)}} = \frac{1-v^n}{\delta}$. Similarly $\overline{s}_{\overline{n}|} = \mathrm{Val}_n(c) = v^{-n}\overline{a}_{\overline{n}|}$.

Annuity-due. This simply means that the first payment is *now*

$$((0,1),\ldots,(n-1,1))$$

with

$$\ddot{a}_{\overline{n}|} = \mathrm{Val}_0((0,1),(1,1),\ldots,(n-1,1)) = \sum_{k=0}^{n-1} v^k = \frac{1-v^n}{1-v} = \frac{1-v^n}{d}$$

and

$$\ddot{s}_{\overline{n}|} = \mathrm{Val}_n((0,1),(1,1),\ldots,(n-1,1)) = v^{-n}\ddot{a}_{\overline{n}|} = \frac{(1+i)^n - 1}{d}.$$

Similarly

$$\ddot{a}_{\overline{n}|}^{(p)} = \mathrm{Val}_0((0,1/p),(1/p,1/p),\ldots,(n-1/p,1/p))$$

and

$$\ddot{s}_{\overline{n}|} = \mathrm{Val}_n((0,1/p),(1/p,1/p),\ldots,(n-1/p,1/p)).$$

Also $\ddot{a}_{\overline{\infty}|}$, $\ddot{a}_{\overline{\infty}|}^{(p)}$, etc.

Deferred and increasing annuities. Further important annuity symbols dealing with regular cash-flows starting some time in the future, and with cash-flows with regular increasing payment streams, are introduced on Assignment 2. The corresponding symbols are

$${}_{m|}a_{\overline{n}|}, {}_{m|}a_{\overline{n}|}^{(p)}, {}_{m|}\ddot{a}_{\overline{n}|}, {}_{m|}\ddot{a}_{\overline{n}|}^{(p)}, {}_{m|}\overline{a}_{\overline{n}|}, \qquad (Ia)_{\overline{n}|}, (I\ddot{a})_{\overline{n}|}, (I\overline{a})_{\overline{n}|}, (\overline{I}\overline{a})_{\overline{n}|}, \qquad {}_{m|}(Ia)_{\overline{n}|} \text{ etc.}$$

5.2 Fixed-interest securities

Simple fixed-interest securities. A simple fixed-interest security entitles the holder to a cash-flow

$$c = ((1, Nj), (2, Nj), \ldots, (n-1, Nj), (n, Nj + N)),$$

where j is the *coupon rate*, N is the *nominal amount* and n is the *term*. The value in the constant-j model is

$$\mathrm{NPV}(j) = Nja_{\overline{n}|j} + N(1+j)^{-n} = Nj\frac{1-(1+j)^{-n}}{j} + N(1+j)^{-n} = N.$$

This is not a surprise: compare with point 2. of Section 3.6. The value in the constant-i model is

$$\mathrm{NPV}(i) = Nja_{\overline{n}|i} + N(1+i)^{-n} = Nj\frac{1-(1+i)^{-n}}{i} + N(1+i)^{-n} = Nj/i + Nv^n(1-j/i).$$

More general fixed-interest securities. There are fixed-interest securities with $p$thly payable coupons at a *nominal* coupon rate j and with a redemption price of R per unit nominal

$$c = ((1/p, Nj/p), (2/p, Nj/p), \ldots, (n-1/p, Nj/p), (n, Nj/p + NR)),$$

where we say that the security is redeemable *at (resp. above or below) par* if $R = 1$ (resp. $R > 1$ or $R < 1$). We compute

$$\mathrm{NPV}(i) = Nja^{(p)}_{\overline{n}|} + NR(1+i)^{-n}.$$

If $\mathrm{NPV}(i) = N$ (resp. $> N$ or $< N$), we say that the security is valued or traded at (resp. above or below) par. Redemption at par is standard. If redemption is not at par, this is usually expressed as e.g. "redemption at 120%" meaning $R = 1.2$. If redemption is not at par, we can calculate the coupon rate per unit redemption money as $j' = j/R$; with $N' = NR$, the cash-flow of a $p$thly payable security of nominal amount N' with coupon rate j' redeemable at par is identical.

Interest payments are always calculated from the nominal amount. Redemption at par is the standard. In practice, a security is a piece of paper (with coupon strips to cash in the interest) that can change owner (sometimes under some restrictions).

Fixed-interest securities as investments. Fixed-interest securities are issued by Governments and are also called Government bonds as opposed to corporate bonds, which are issued by companies. Corporate bonds are less secure than Government bonds since (in either case, actually) bankruptcy can stop the payment stream. Since Government typically issues large quantities of bonds, they form a very liquid/marketable form of investment that is actively traded on bond markets.

Government bonds are either issued at a fixed price or *by tender*, in which case the highest bidders get the bonds at a set issue date. Government bonds usually have a term of several years. There are also shorter-term Government bills which have no coupons, so they are just offered at a discount on their nominal value.

Example 45 Consider a fixed-interest security of $N = 100$ nominal, coupon rate $j = 3\%$ payable annually and redeemable at par after a term of $n = 2$. If the security is currently trading below par, with a purchase price of $P = £97$, the investment has a cash-flow

$$c = ((0, -97), (1, 3), (2, 103))$$

and we can calculate the yield by solving the equation of value

$$-97 + 3(1+i)^{-1} + 103(1+i)^{-2} = 0 \iff 97(1+i)^2 - 3(1+i) - 103 = 0$$

to obtain $1 + i = 1.04604$, i.e. $i = 4.604\%$ (the second solution of the quadradic is $1 + i = -1.01512$, i.e. $i = -2.01512$, which is not in $(-1, \infty)$; note that we knew already by Proposition 40 that there can only be one admissible solution).

Here, we could solve the quadratic equation explicitly; for fixed-interest securities of longer term, it is useful to note that the yield is composed of two effects, first the coupons payable at rate j/R per unit redemption money (or at rate jN/P per unit purchasing price) and then any capital gain/loss $RN - P$ spread over n years. Here, these rough considerations give

$$j/R + (R - P/N)/n = 4.5\%, \quad \text{or more precisely } (j/R + jN/P)/2 + (R - P/N)/n = 4.546\%,$$

and often even rougher considerations give us an idea of the order of magnitute of a yield that we can then use as good initial guesses for a numerical approximation.

Lecture 6

Mortgages and loans

Reading: CT1 Core Reading Unit 8, McCutcheon-Scott Sections 3.7-3.8

As we indicated in the Introduction, interest-only and repayment loans are the formal inverse cash-flows of securities and annuities. Therefore, most of the last lecture can be reinterpreted for loans. We shall here only translate the most essential formulae and then pass to specific questions and features arising in (repayment) loans and mortgages, e.g. calculations of outstanding capital, proportions of interest/repayment, discount periods and rates used to compare loans/mortgages.

6.1 Loan repayment schemes

Definition 46 A *repayment scheme* for a loan of L in the model $\delta(\cdot)$ is a cash-flow

$$c = ((t_1, X_1), (t_2, X_2), \ldots, (t_n, X_n))$$

such that

$$L = \mathrm{Val}_0(c) = \sum_{k=1}^{n} v(t_k) X_k = \sum_{k=1}^{n} e^{-\int_0^{t_k} \delta(t)dt} X_k. \tag{1}$$

This ensures that, in the model given by $\delta(\cdot)$, the loan is repaid after the nth payment since it ensures that $\mathrm{Val}_0((0, -L), c) = 0$ so also $\mathrm{Val}_t((0, -L), c) = 0$ for all t.

Example 47 A bank lends you £1,000 at an effective interest rate of 8% p.a. initially, but due to rise to 9% after the first year. You repay £400 both after the first and half way through the second year and wish to repay the rest after the second year. How much is the final payment? We want

$$1,000 = 400v(1) + 400v(1.5) + Xv(2) = \frac{400}{1.08} + \frac{400}{(1.09)^{1/2} 1.08} + \frac{X}{(1.09)(1.08)},$$

which gives $X = £323.59$.

Example 48 Often, the payments X_k are constant (*level payments*) and the times t_k are are regularly spaced (so we can assume $t_k = k$). So

$$L = \mathrm{Val}_0((1, X), (2, X), \ldots, (n, X)) = X\ a_{\overline{n|}} \qquad \text{in the constant-}i\text{ model.}$$

6.2 Loan outstanding, interest/capital components

The payments consist both of *interest* and *repayment of the capital.* The distinction can be important e.g. for tax reasons. Earlier in term, there is more capital outstanding, hence more interest payable, hence less capital repaid. In later payments, more capital will be repaid, less interest. Each payment pays *first* for interest due, then repayment of capital.

Example 48 (continued) Let $n = 3$, $L = 1,000$ and assume a constant-i model with $i = 7\%$. Then

$$X = 1,000/a_{\overline{3}|7\%} = 1,000/(2.624316) = 381.05.$$

Furthermore,

	time 1	time 2	time 3
interest due	$1,000 \times 0.07$ $= 70$	688.95×0.07 $= 48.22$	356.12×0.07 $= 24.93$
capital repaid	$381.05 - 70$ $= 311.05$	$381.05 - 48.22$ $= 332.83$	$381.05 - 24.93$ $= 356.12$
amount outstanding	$1,000 - 311.05$ $= 688.95$	$688.95 - 332.83$ $= 356.12$	0

Let us return to the general case. In our example, we kept track of the amount outstanding as an important quantity. In general, for a loan L in a $\delta(\cdot)$-model with payments $c_{\leq t} = ((t_1, X_1), \ldots, (t_m, X_m))$, the outstanding debt at time t is L_t such that

$$\mathrm{Val}_t((0, -L), c_{\leq t}, (t, L_t)) = 0,$$

i.e. a single payment of L_t would repay the debt.

Proposition 49 (Retrospective formula) *Given L, $\delta(\cdot)$, $c_{\leq t}$,*

$$L_t = \mathrm{Val}_t((0, L)) - \mathrm{Val}_t(c_{\leq t}) = A(0, t)L - \sum_{k=1}^{m} A(t_k, t)X_k.$$

Recall here that $A(s, t) = e^{\int_s^t \delta(r)dr}$.

Alternatively, for a given repayment scheme, we can also use the following *prospective formula.*

Proposition 50 (Prospective formula) *Given L, $\delta(\cdot)$) and a repayment scheme c,*

$$L_t = \mathrm{Val}_t(c_{>t}) = \frac{1}{v(t)} \sum_{k:t_k>t} v(t_k)X_k. \qquad (2)$$

Proof: $\mathrm{Val}_t((0, -L), c_{\leq t}, (t, L_t)) = 0$ and $\mathrm{Val}_t((0, -L), c_{\leq t}, c_{>t}) = 0$ (since c is a repayment scheme), so

$$L_t = \mathrm{Val}_t((t, L_t)) = \mathrm{Val}_t(c_{>t}).$$

□

Corollary 51 *In a repayment scheme* $c = ((t_1, X_1), (t_2, X_2), \ldots, (t_n, X_n))$*, the* j*th payment consists of*

$$R_j = L_{t_{j-1}} - L_{t_j}$$

capital repayment and

$$I_j = X_j - R_j = L_{t_{j-1}}(A(t_{j-1}, t_j) - 1)$$

interest payment.

Note I_j represents the interest payable on a sum of $L_{t_{j-1}}$ over the period (t_{j-1}, t).

6.3 Fixed, capped and discount mortgages

In practice, the interest rate of a mortgage is rarely fixed for the whole term and the lender has some freedom to change their Standard Variable Rate (SVR). Usually changes are made in accordance with changes of the UK base rate fixed by the Bank of England. However, there is often a special "initial period":

Example 52 (Fixed period) For an initial 2-10 years, the interest rate is fixed, usually below the current SVR, the shorter the period, the lower the rate.

Example 53 (Capped period) For an initial 2-5 years, the interest rate can fall parallel to the base rate or the SVR, but cannot rise above the initial level.

Example 54 (Discount period) For an initial 2-5 years, a certain discount on the SVR is given. This discount may change according to a prescribed schedule.

Regular (e.g. monthly) payments are always calculated as if the current rate was valid for the whole term (even if changes are known in advance). So e.g. a discount period leads to lower initial payments. Any change in the interest rate leads to changes in the monthly payments.

Initial advantages in interest rates are usually combined with early redemption penalties that may or may not extend beyond the initial period (e.g. 6 months of interest on the amount redeemed early).

Example 55 We continue Example 44 and consider the discount mortgage of £85,000 with interest rates of $i_1 = \text{SVR} - 2.96\% = 2.99\%$ in year 1, $i_2 = \text{SVR} - 1.76\% = 4.19\%$ in year 2 and SVR of $i_3 = 5.95\%$ for the remainder of a 20-year term; a £100 Product Fee is added to the initial loan amount, a £25 Funds Transfer Fee is deducted from the Net Amount provided to the borrower. Then the borrower receives £84,975, but the initial loan outstanding is $L_0 = 85,100$. With annual payments, the repayment scheme is $c = ((1, X), (2, Y), (3, Z), \ldots, (20, Z))$, where

$$X = \frac{L_0}{a_{\overline{20}|2.99\%}}, \quad Y = \frac{L_1}{a_{\overline{19}|4.19\%}}, \quad Z = \frac{L_0}{a_{\overline{18}|5.95\%}}.$$

With

$$a_{\overline{20}|2.99\%} = \frac{1 - (1.0299)^{-20}}{0.0299} = 14.89124 \qquad \Rightarrow \qquad X = \frac{L_0}{a_{\overline{20}|2.99\%}} = 5,714.77,$$

we will have a loan outstanding of $L_1 = L_0(1.0299) - X = 81,929.72$ and then

$$Y = \frac{L_1}{a_{\overline{19}|4.19\%}} = 6,339.11, \quad L_2 = L_1(1.0419) - Y = 79,023.47, \quad Z = \frac{L_0}{a_{\overline{18}|5.95\%}} = 7270.96.$$

With monthly payments, we can either repeat the above with $12\widetilde{X} = L_0/a^{(12)}_{\overline{20}|2.99\%}$ etc. to get monthly payments $((1/12,\widetilde{X}),(2/12,\widetilde{X}),\ldots,(11/12,\widetilde{X}),(1,\widetilde{X}))$ for the first year and then proceed as above. But, of course, L_1 and L_2 will be exactly as above, and we can in fact replace parts of the repayment scheme $c = ((1,X),(2,Y),(3,Z),\ldots,(20,Z))$ by *equivalent cash-flows*, where equivalence means same discounted value. Using in each case the appropriate interest rate in force at the time

- $((1,X))$ is equivalent to $((1/12,\widetilde{X}),\ldots,(11/12,\widetilde{X}),(1,\widetilde{X}))$ in the constant i_1-model, where $12\widetilde{X}s^{(12)}_{\overline{1}|2.99\%} = X$, so $\widetilde{X} = X/12s^{(12)}_{\overline{1}|2.99\%} = \frac{1}{12}Xi_1^{(12)}/i_1 = 469.83$.
- $((2,Y))$ is equivalent to $((1+1/12,\widetilde{Y}),\ldots,(1+11/12,\widetilde{Y}),(2,\widetilde{Y}))$ in the constant i_2-model, where $\widetilde{Y} = \frac{1}{12}Yi_2^{(12)}/i_2 = 518.38$.
- $((k,Z))$ is equivalent to $((k-11/12,\widetilde{Z}),(k-10/12,\widetilde{Z}),\ldots,(k-1/12,\widetilde{Z}),(k,\widetilde{Z}))$ in the constant i_3-model, where $\widetilde{Z} = \frac{1}{12}Zi_3^{(12)}/i_3 = 589.99$.

6.4 Comparison of mortgages

How can we compare deals, e.g. describe the "overall rate" of variable-rate mortgages?

A method that is still used sometimes, is the "flat rate"

$$F = \frac{\text{total interest}}{\text{total term} \times \text{initial loan}} = \frac{\sum_{j=1}^{n} I_j}{t_n L} = \frac{\sum_{j=1}^{n} X_j - L}{t_n L}.$$

This is *not* a good method: we should think of interest paid on *outstanding debt* L_t, not on all of L. E.g. loans of different terms but same constant rate have different flat rates.

A better method to use is the Annual Percentage Rate (APR) of Section 4.3.

Example 55 (continued) The Net Amount provided to the borrower is $L = 84,975$, so the flat rate with annual payments is

$$F_{\text{annual}} = \frac{X + Y + 18Z - L}{20 \times L} = 3.41\%,$$

while the yield is 5.445%, i.e. the APR is 5.4% – we calculated this in Example 44.

With monthly payments we obtain

$$F_{\text{monthly}} = \frac{12\widetilde{X} + 12\widetilde{Y} + 18 \times 12\widetilde{Z} - L}{20 \times L} = 3.196\%,$$

while the yield is 5.434%, i.e. the APR is still 5.4%.

The yield is also more stable under changes of payment frequency than the flat rate.

Lecture 7

Modelling future lifetimes

Reading: Gerber Sections 2.1, 2.2, 2.4, 3.1, 3.2, 4.1

In this lecture we introduce and apply actuarial notation for lifetime distributions.

7.1 Introduction to life insurance

The lectures that follow are motivated by the following problems.

1. An individual aged x would like to buy a *life annuity* (e.g. a pension) that pays him a fixed amount N p.a. for the rest of his life. How can a life insurer determine a fair price for this product?

2. An individual aged x would like to buy a *whole life insurance* that pays a fixed amount S to his dependants upon his death. How can a life insurer determine a fair single or annual premium for this product?

3. Other related products include *pure endowments* that pay an amount S at time n provided an individual is still alive, an *endowment assurance* that pays an amount S either upon an individual's death or at time n whichever is earlier, and a *term assurance* that pays an amount S upon an individual's death only if death occurs before time n.

The answer to these questions will depend on the chosen model of the future lifetime T_x of the individual.

7.2 Lives aged x

For a continuously distributed random lifetime T, we write $F_T(t) = \mathbb{P}(T \leq t)$ for the cumulative distribution function, $f_T(t) = F_T'(t)$ for the probability density function, $\overline{F}_T(t) = \mathbb{P}(T > t)$ for the survival function and $\omega_T = \inf\{t \geq 0 : \overline{F}_T(t) = 0\}$ for the maximal possible lifetime.

Definition 56 The function

$$\mu_T(t) = \frac{f_T(t)}{\overline{F}_T(t)}, \qquad t \geq 0,$$

specifies the *force of mortality* or *hazard rate* at (time) t.

Proposition 57 *We have* $\frac{1}{\varepsilon}\mathbb{P}(T \leq t + \varepsilon | T > t) \to \mu_T(t)$ *as* $\varepsilon \to 0$.

Proof: We use the definitions of conditional probabilities and differentiation:

$$\begin{aligned}\frac{1}{\varepsilon}\mathbb{P}(T \le t+\varepsilon | T > t) &= \frac{1}{\varepsilon}\frac{\mathbb{P}(T > t, T \le t+\varepsilon)}{\mathbb{P}(T > t)} = \frac{1}{\varepsilon}\frac{\mathbb{P}(T \le t+\varepsilon) - \mathbb{P}(T \le t)}{\mathbb{P}(T > t)} \\ &= \frac{1}{\overline{F}_T(t)}\frac{F_T(t+\varepsilon) - F_T(t)}{\varepsilon} \longrightarrow \frac{F_T'(t)}{\overline{F}_T(t)} = \frac{f_T(t)}{\overline{F}_T(t)} = \mu_T(t).\end{aligned}$$

□

So, we can say informally that $\mathbb{P}(T \in (t, t+dt)|T > t) \approx \mu_T(t)dt$, i.e. $\mu_T(t)$ represents for each $t \ge 0$ the current "rate of death" given survival up to t.

Example 58 The exponential distribution of rate μ is given by

$$F_T(t) = 1 - e^{-\mu t}, \quad \overline{F}_T(t) = e^{-\mu t}, \quad f_T(t) = \mu e^{-\mu t}, \quad \mu_T(t) = \frac{\mu e^{-\mu t}}{e^{-\mu t}} = \mu \quad \text{constant.}$$

Lemma 59 *We have* $\overline{F}_T(t) = \exp\left(-\int_0^t \mu_T(s)ds\right)$.

Proof: First note that $\overline{F}_T(0) = \mathbb{P}(T > 0) = 1$. Also

$$\frac{d}{dt}\log \overline{F}_T(t) = \frac{\overline{F}_T'(t)}{\overline{F}_T(t)} = \frac{-f_T(t)}{\overline{F}_T(t)} = -\mu_T(t).$$

So

$$\log \overline{F}_T(t) = \log \overline{F}_T(0) + \int_0^t \frac{d}{ds}\log \overline{F}_T(s)ds = 0 + \int_0^t -\mu_T(s)ds.$$

□

Suppose now that T models the future lifetime of a new-born person. In life insurance applications, we are often interested in the future lifetime of a person aged x, or more precisely the *residual lifetime* $T - x$ given $\{T > x\}$, i.e. given survival to age x. For life annuities this determines the random number of annuity payments that are payable. For a life assurance contract, this models the time of payment of the sum assured. In practice, insurance companies perform medical tests and/or collect employment/geographical/medical data that allow more accurate modelling. However, let us here assume that no such other information is available. Then we have, for each $x \in [0, \omega)$,

$$\mathbb{P}(T - x > y | T > x) = \frac{\mathbb{P}(T > x+y)}{\mathbb{P}(T > x)} = \frac{\overline{F}(x+y)}{\overline{F}(x)}, \qquad y \ge 0,$$

by the definition of conditional probabilities $\mathbb{P}(A|B) = \mathbb{P}(A \cap B)/\mathbb{P}(B)$.

It is natural to directly model the residual lifetime T_x of an individual (a *life*) aged x as

$$\overline{F}_x(y) = \mathbb{P}(T_x > y) = \frac{\overline{F}(x+y)}{\overline{F}(x)} = \exp\left(-\int_x^{x+y}\mu(s)ds\right) = \exp\left(-\int_0^y \mu(x+t)dt\right).$$

We can read off $\mu_x(t) = \mu_{T_x}(t) = \mu(x+t)$, i.e. the force of mortality is still the same, just shifted by x to reflect the fact that the individual aged x and dying at time T_x is aged $x + T_x$ at death. We can also express cumulative distribution functions

$$F_x(y) = 1 - \overline{F}_x(y) = \frac{\overline{F}(x) - \overline{F}(x+y)}{\overline{F}(x)} = \frac{F(x+y) - F(x)}{1 - F(x)},$$

and probability density functions

$$f_x(y) = \begin{cases} F_x'(y) = \frac{f(x+y)}{1-F(x)} = \frac{f(x+y)}{\overline{F}(x)}, & 0 \le y \le \omega - x, \\ 0, & \text{otherwise.} \end{cases}$$

There is also actuarial lifetime notation, as follows

$${}_tq_x = F_x(t), \quad {}_tp_x = 1 - {}_tq_x = \overline{F}_x(t), \qquad q_x = {}_1q_x, \quad p_x = {}_1p_x, \qquad \mu_x = \mu(x) = \mu_x(0).$$

In this notation, we have the following *consistency condition* on lifetime distributions for different ages:

Proposition 60 *For all $x \ge 0$, $s \ge 0$ and $t \ge 0$, we have*

$${}_{s+t}p_x = {}_sp_x \times {}_tp_{x+s}.$$

By general reasoning, the probability that a life aged x survives for $s+t$ years is the same as the probability that it first survives for s years and then, aged $x+s$, survives for another t years.

Proof: Formally, we calculate the right-hand side

$$\begin{aligned} {}_sp_x \times {}_tp_{x+s} &= \mathbb{P}(T_x > s)\mathbb{P}(T_{x+s} > t) = \frac{\mathbb{P}(T > x+s)}{\mathbb{P}(T > x)} \frac{\mathbb{P}(T > x+s+t)}{\mathbb{P}(T > x+s)} = \mathbb{P}(T_x > s+t) \\ &= {}_{s+t}p_x. \end{aligned}$$

□

We can also express other formulas, which we have already established, in actuarial notation:

$$f_x(t) = {}_tp_x\mu_{x+t}, \qquad {}_tp_x = \exp\left(-\int_0^t \mu_{x+s}ds\right), \qquad {}_tq_x = \int_0^t {}_sp_x\mu_{x+s}ds.$$

The first one says that to die at t, life x must survive for time t and then die instantaneously.

7.3 Curtate lifetimes

In practice, many cash-flows pay at discrete times, often at the end of each month. Let us begin here by discretising continuous lifetimes to integer-valued lifetimes. This is often done in practice, with interpolation being used for finer models.

Definition 61 Given a continuous lifetime random variable T_x, the random variable $K_x = [T_x]$, where $[\cdot]$ denotes the integer part, is called the associated *curtate lifetime.*

Of course, one can also model curtate lifetimes directly. Note that, for continuously distributed T_x

$$\mathbb{P}(K_x = k) = \mathbb{P}(k \le T_x < k+1) = \mathbb{P}(k < T_x \le k+1) = {}_kp_x \times q_{x+k}.$$

Also, by Proposition 60,

$$\mathbb{P}(K_x \ge k) = {}_kp_x = \prod_{j=0}^{k-1} p_{x+j},$$

i.e. K_x can be thought of as the number of successes before the first failure in a sequence of independent Bernoulli trials with varying success probabilities p_{x+j}, $j \ge 0$. Here, success is the survival of a year, while failure is death during the year.

Proposition 62 *We have* $\mathbb{E}(K_x) = \sum_{k=1}^{[\omega-x]} {}_kp_x$.

Proof: By definition of the expectation of a discrete random variable,

$$\begin{aligned}\sum_{k=1}^{[\omega-x]} {}_kp_x = \sum_{k=1}^{[\omega-x]} \mathbb{P}(K_x \geq k) &= \sum_{k=1}^{[\omega-x]} \sum_{m=k}^{[\omega-x]} \mathbb{P}(K_x = m) \\ &= \sum_{m=1}^{[\omega-x]} \sum_{k=1}^{m} \mathbb{P}(K_x = m) = \sum_{m=0}^{[\omega-x]} m\mathbb{P}(K_x = m) = \mathbb{E}(K_x).\end{aligned}$$

□

Example 63 If T is exponentially distributed with parameter $\mu \in (0, \infty)$, then $K = [T]$ is geometrically distributed:

$$\mathbb{P}(K = k) = \mathbb{P}(k \leq T < k+1) = e^{-k\mu} - e^{-(k+1)\mu} = (e^{-\mu})^k(1 - e^{-\mu}), \qquad k \geq 0.$$

We identify the parameter of the geometric distribution as $e^{-\mu}$. Note also that here $p_x = e^{-\mu}$ and $q_x = 1 - e^{-\mu}$ for all x.

In general, we get

$$\mathbb{P}(K_x \geq k) = \prod_{j=0}^{k-1} p_{x+j} = {}_kp_x = \exp\left(-\int_0^k \mu(x+t)dt\right) = \prod_{j=0}^{k-1} \exp\left(-\int_{x+j}^{x+j+1} \mu(s)ds\right),$$

so that we read off

$$p_x = \exp\left(-\int_x^{x+1} \mu(s)ds\right), \qquad x \geq 0.$$

In practice, μ is often assumed constant between integer points (denoted $\mu_{x+0.5}$) or continuous piecewise linear between integer points.

7.4 Examples

Let us now return to our motivating problem.

Example 64 (Whole life insurance) Let K_x be a curtate future lifetime. A whole life insurance pays one unit at the end of the year of death, i.e. at time $K_x + 1$. In the model of a constant force of interest δ, the random discounted value at time 0 is $Z = e^{-\delta(K_x+1)}$, so the fair premium for this random cash-flow is

$$A_x = \mathbb{E}(Z) = \mathbb{E}(e^{-\delta(K_x+1)}) = \sum_{k=1}^{\infty} e^{-\delta k}\mathbb{P}(K_x = k-1) = \sum_{m=0}^{\infty} (1+i)^{-m-1} {}_mp_x q_{x+m},$$

where $i = e^{\delta} - 1$ and A_x is just the actuarial notation for this expected discounted value.

Lecture 8

Lifetime distributions and life-tables

Reading: Gerber Sections 2.3, 2.5, 3.2

In this lecture we give an introduction to life-tables. We will not go into the details of constructing life-tables, which is one of the subjects of BS3b Statistical Lifetime-Models. What we are mostly interested in is the use of life-tables to price life insurance products.

8.1 Actuarial notation for life products.

Recall the motivating problems. Let us introduce the associated notation.

Example 65 (Term assurance, pure endowment and endowment) The fair premium of a term insurance is denoted by

$$A^1_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^{k+1}{}_kp_x q_{x+k}.$$

The superscript 1 above the x indicates that 1 is only paid in case of death within the period of n years.

The fair premium of a pure endowment is denoted by $A_{x:\overline{n}|}^{\ \ 1} = v^n{}_np_x$. Here the superscript 1 indicates that 1 is only paid in case of survival of the period of n years.

The fair premium of an endowment is denoted by $A_{x:\overline{n}|} = A^1_{x:\overline{n}|} + A_{x:\overline{n}|}^{\ \ 1}$, where we could have put a 1 above both x and n, but this is omitted being the default, like in previous symbols.

Example 66 (Life annuities) Given a constant i interest model, the fair premium of an ordinary (respectively temporary) life annuity for a life aged x is given by

$$a_x = \sum_{k=1}^{\infty} v^k{}_kp_x \quad \text{respectively} \quad a_{x:\overline{n}|} = \sum_{k=1}^{n} v^k{}_kp_x.$$

For an ordinary (respectively temporary) life annuity-due, an additional certain payment at time 0 is made (and any payment at time n omitted):

$$\ddot{a}_x = 1 + a_x \quad \text{respectively} \quad \ddot{a}_{x:\overline{n}|} = 1 + a_{x:\overline{n-1}|}.$$

8.2 Simple laws of mortality

As we have seen, the theory is nicest for exponentially distributed lifetimes. However, the exponential distribution is not actually a good distribution to model human lifetimes:

- One reason that we have seen is that the curtate lifetime is geometric, i.e. each year given survival up to then, there is the same probability of dying in the next year. In practice, you would expect that this probability increases for higher ages.
- There is clearly significantly positive probability to survive up to age 70 and zero probability to survive to age 140, and yet the exponential distribution suggests that

$$\mathbb{P}(T > 140) = e^{-140\mu} = (e^{-70\mu})^2 = (\mathbb{P}(T > 70))^2.$$

  Specifically, if we think there is at least 50% chance of a newborn to survive to age 70, there would be at least 25% chance to survive to age 140; if we think that the average lifetime is more than 70, then $\mu < 1/70$, so $\mathbb{P}(T > 140) > e^{-2} > 10\%$.
- More formally, the exponential distribution has the lack of memory property, which here says that the distribution of T_x is still exponential with the same parameter, independent of x. This would mean that there is no ageing.

These observations give some ideas for more realistic models. Generally, we would favour models with an eventually increasing force of mortality (in reliability theory such distributions are called IFR distributions – increasing failure rate).

1. Gompertz' Law: $\mu(t) = Bc^t$ for some $B > 0$ and $c > 1$.
2. Makeham's Law: $\mu(t) = A + Bc^t$ for some $A \geq 0$, $B > 0$ and $c > 1$ (or $c > 0$ to include DFR – decreasing failure rate cases).
3. Weibull: $\mu(t) = kt^\beta$ for some $k > 0$ and $\beta > 0$.

Makeham's Law actually gives a reasonable fit for ages 30-70.

8.3 The life-table

Suppose we have a population of newborn individuals (or individuals aged $\alpha > 0$, some lowest age in the table). Denote the size of the population by ℓ_α. Then let us observe each year the number ℓ_x of individuals still alive, until the age when the last individual dies reaching $\ell_\omega = 0$. Then out of ℓ_x individuals, ℓ_{x+1} survived age x, and the proportion ℓ_{x+1}/ℓ_x can be seen as the probability for each individual to survive. So, if we set

$$p_x = \ell_{x+1}/\ell_x \qquad \text{for all } \alpha \leq x \leq \omega - 1.$$

we specify a curtate lifetime distribution. The function $x \mapsto \ell_x$ is usually called the *life-table* in the strict sense. Note that also vice versa, we can specify a life-table with any given curtate lifetime distribution by choosing $\ell_0 = 100,000$, say, and setting $\ell_x = \ell_0 \times {}_xp_0$. In this case, we should think of ℓ_x as the *expected* number of individuals alive at age x. Strictly speaking, we should distinguish p_x and its estimate $\widehat{p}_x = \ell_{x+1}/\ell_x$, but this notion had not been developed when actuaries first did this, and rather leave the link to statistics and the interpretation of ℓ_x

as an observed quantity, which it usually isn't anyway for real life-tables that were set up by rather more sophisticated methods.

There is a lot of notation associated with life-tables, and the 1967/70 life-table that we will work with actually consists of several pages of tables of different life functions. We work with the table for q_x, and there is also a table containing all ℓ_x, or indeed $d_x = \ell_x - \ell_{x+1}$, $\alpha \leq x \leq \omega - 1$. Observe that, with this notation, $q_x = d_x/\ell_x$.

See Figure 8.1 for plots of hazard function ($x \mapsto q_x$) and probability mass functions ($x \mapsto \mathbb{P}(K = x)$). Note that initially and up to age 35, the one-year death probabilities are below 0.001, then increase to cross 0.1 at age 80 and 0.5 at age 100.

8.4 Example

We will look at the 1967/70 life-table in more detail in the next lecture. Let us finish this lecture by giving an (artificial) example of a life-table constructed from a parametric family. A more realistic example is on the next assignment sheet.

Example 67 We find records about a group of 10,000 from when they were all aged 20, which reveal that 8,948 were still alive 5 years later and 7,813 another 5 years later. This means, that we are given

$$\ell_{20} = 10,000, \qquad \ell_{25} = 8,948 \qquad \text{and} \qquad \ell_{30} = 7,813.$$

Of course, this is not enough to set up a life-table, since all we would be able to calculate is

$${}_5p_{20} = \frac{\ell_{25}}{\ell_{20}} = 0.895 \qquad \text{and} \qquad {}_5p_{25} = \frac{\ell_{30}}{\ell_{25}} = 0.873.$$

However, if we assume that the population has survival function

$$\overline{F}_T(t) = \exp(-Ax - Bx^2), \qquad \text{for some } A \geq 0 \text{ and } B \geq 0,$$

we can solve two equations to identify the parameters that exactly replicate these two probabilities:

$$\begin{aligned} {}_5p_{20} &= \mathbb{P}(T > 25 | T > 20) = \frac{\exp(-25A - 25^2 B)}{\exp(-20A - 20^2 B)} = \exp(-5A - 225B) \\ {}_pp_{25} &= \exp(-5A - 275B). \end{aligned}$$

(More sophisticated statistical techniques are beyond the scope of this course.) Here, we obtain that

$$B = \frac{\ln(0.895) - \ln(0.873)}{50} = 0.00049 \qquad \text{and} \qquad A = \frac{-\ln(0.895) - 225B}{5} = 0.00018,$$

and we can then construct the whole associated lifetable as

$$\ell_x = \ell_{20} \times {}_{x-20}p_{20} = 10,000 \exp(-(x-20)A - (x^2 - 20^2)B).$$

The reliability away from $x \in [20, 30]$ must be questioned, but we can compute the implied $\ell_{100} = 89.29$.

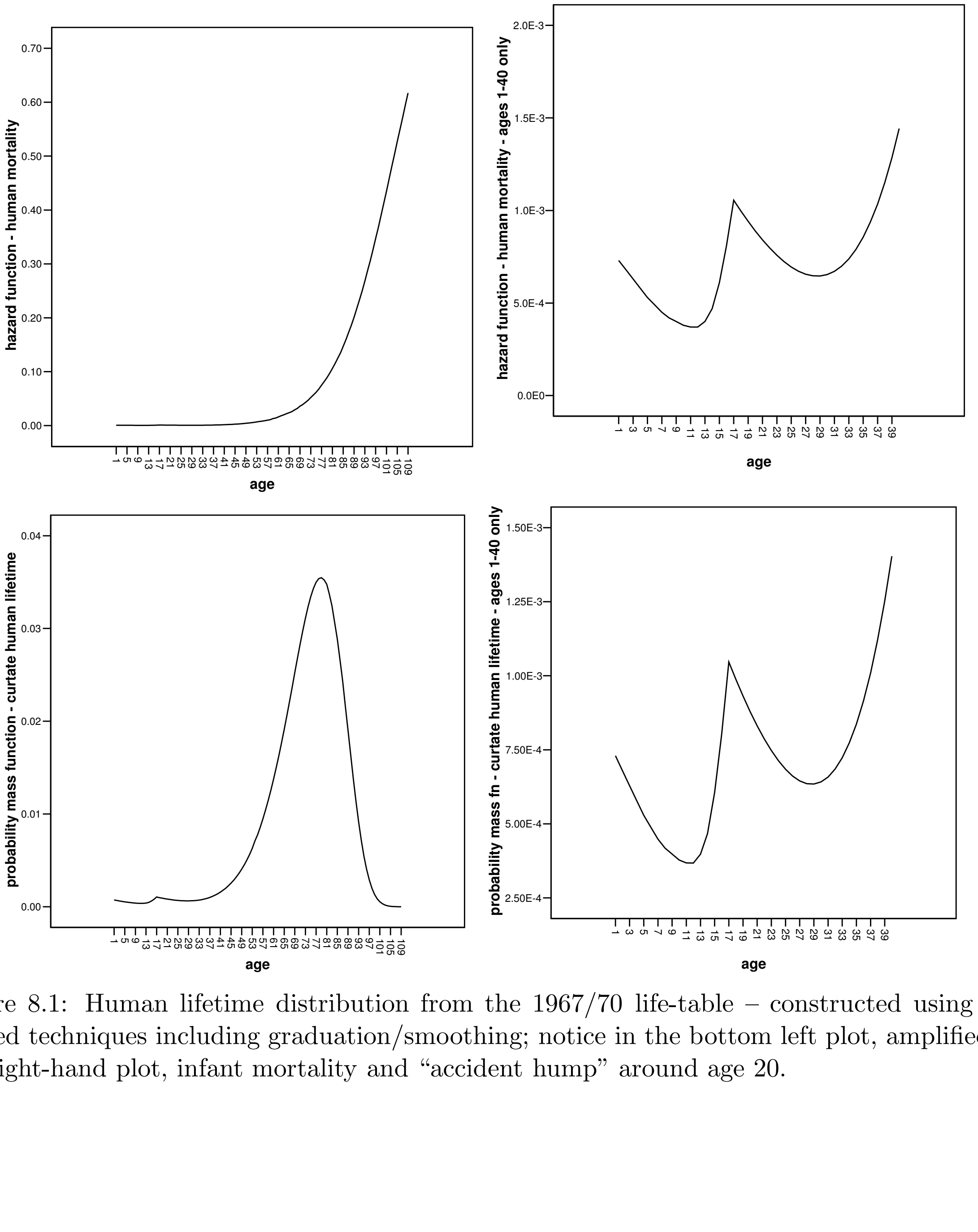


Figure 8.1: Human lifetime distribution from the 1967/70 life-table – constructed using advanced techniques including graduation/smoothing; notice in the bottom left plot, amplified in the right-hand plot, infant mortality and "accident hump" around age 20.

Lecture 9

Select life-tables and applications

Reading: Gerber 2.5, 2.6

9.1 Select life-tables

Some life-tables are select tables, meaning that mortality rates depend on the duration from joining the population. This is relevant if there is e.g. a medical check at the policy start date, which can reject applicants with poor health and only offer policies to applicants with better than average health and lower than average mortality, at least for the next few years. We denote by

- $[x]$ the age at the date of joining the population;
- $q_{[x]}$ the mortality rate for a life aged x having joined the population at age x;
- $q_{[x]+j}$ the mortality rate for a life aged $x+j$ having joined the population at age x;
- q_{x+r} the ultimate mortality for a life aged x having joined the population at age x;

It is supposed that after some select period of r years, mortality no longer varies significantly with duration since joining, but only with age: this is the *ultimate* portion of the table.

Example 68 The 1967/70 life-table has three columns, two select columns for $q_{[x]}$ and $q_{[x]+1}$ and one ultimate column for q_{x+2}. To calculate probabilities for $K_{[x]}$, the relevant entries are the three entries in row $[x]$ and all ultimate entries below this row. E.g.

$$\mathbb{P}(K_{[x]} = 4) = (1 - q_{[x]})(1 - q_{[x]+1}(1 - q_{x+2})(1 - q_{x+3})q_{x+4}.$$

Using the excerpt of Figure 9.1 of the table for an age [55], we obtain specifically

$$\mathbb{P}(K_{[55]} = 4) = (1 - 0.00447)(1 - 0.00625)(1 - 0.01050)(1 - 0.01169)0.01299 \approx 0.01257.$$

Example 69 Consider a 4-year temporary life assurance issued to a checked life [55]. Assuming a constant interest rate of $i = 4\%$, we calculate

$$A^{\,1}_{[55]:\overline{4}|} = vq_{[55]} + v^2 p_{[55]} q_{[55]+1} + v^3 p_{[55]} p_{[55]+1} q_{57} + v^4 p_{[55]} p_{[55]+1} p_{57} q_{58} = 0.029067.$$

Therefore, the fair price of an assurance with sum assured $N = £100,000$ is $£2,906.66$, payable as a single up-front premium.

Age $[x]$	$q_{[x]}$	$q_{[x]+1}$	q_{x+2}	Age $x+2$
53	.00376288	.00519413	.00844128	55
54	.00410654	.00570271	.00941902	56
55	.00447362	.00625190	.01049742	57
56	.00486517	.00684424	.01168566	58
57	.00528231	.00748245	.01299373	59
58	.00572620	.00816938	.01443246	60
59	.00619802	.00890805	.01601356	61
60	.00669904	.00970168	.01774972	62
61	.00723057	.01055365	.01965464	63
62	.00779397	.01146756	.02174310	64

Figure 9.1: Extract from mortality tables of assured lives based on 1967-70 data.

9.2 Multiple premiums

We will return to a more complete discussion of multiple premiums later, but for the purpose of this section, let us define:

Definition 70 Given a constant i interest model, let C be the cash-flow of insurance benefits, the annual fair level premium of C is defined to be

$$P_x = \frac{E(DVal_0(C))}{\ddot{a}_x}.$$

This definition is based on the following principles:

Proposition 71 *In the setting of Definition 70, the expected dicounted benefits equal the expected discounted premium payments. Premium payments stop upon death.*

Proof: The expected discounted value of premium payments $((0, P_x), \ldots, (K_x, P_x))$ is $P_x \ddot{a}_x = E(DVal_0(C))$. □

Example 72 We calculate the fair annual premium in Example 69. Since there should not be any premium payments beyond when the policy ends either by death or by reaching the end of term, the premium payments form a effectively a temporary life annuity-due, i.e. cash-flow $((0, 1), (1, 1), (2, 1), (3, 1))$ restricted to the lifetime $K_{[55]}$. First

$$\ddot{a}_{[55]:\overline{4}|} = 1 + vp_{[55]} + v^2 p_{[55]} p_{[55]+1} + v^3 p_{[55]} p_{[55]+1} p_{57} \approx 3.742157$$

and the annual premium is therefore calculated from the life table in Figure 9.1 as

$$P = \frac{NA^{1}_{[55]:\overline{4}|}}{\ddot{a}_{[55]:\overline{4}|}} = 776.73.$$

9.3 Interpolation for non-integer ages $x+u$, $x \in \mathbb{N}$, $u \in (0,1)$

There are two popular models. Model 1 is to assume that the force of mortality μ_t is constant between each $(x, x+1)$, $x \in \mathbb{N}$. This implies

$$p_x = \exp\left(-\int_x^{x+1} \mu_t dt\right) = \exp\left(-\mu_{x+0.5}\right) \qquad \Rightarrow \qquad \mu_{x+\frac{1}{2}} = -\ln(p_x).$$

Also, for $0 \le u \le 1$,

$$\mathbb{P}(T > x+u | T > x) = \exp\left\{-\int_x^{x+u} \mu_t dt\right\} = \exp\left\{-u\mu_{x+0.5}\right\} = (1-q_x)^u,$$

and with notation $T = K+S$, where $K = [T]$ is the integer part and $S = \{T\} = T-[T] = T-K$ the fractional part of T, this means that

$$\mathbb{P}(S \le u | K = x) = \frac{\mathbb{P}(x \le T \le x+u)/\mathbb{P}(T > x)}{\mathbb{P}(x \le T < x+1)/\mathbb{P}(T > x)} = \frac{1-\exp\left\{-u\mu_{x+0.5}\right\}}{1-\exp\left\{-\mu_{x+0.5}\right\}}, \qquad 0 \le u \le 1,$$

a distribution that is in fact the exponential distribution with parameter $\mu_{x+0.5}$, truncated at $\omega = 1$: for exponentially distributed E

$$\mathbb{P}(E \le u | E \le 1) = \frac{\mathbb{P}(E \le u)}{\mathbb{P}(E \le 1)} = \frac{1-\exp\left\{-u\mu_{x+0.5}\right\}}{1-\exp\left\{-\mu_{x+0.5}\right\}}, \qquad 0 \le u \le 1.$$

Since the parameter depends on x, S is not independent of K here.

Model 2 is convenient e.g. when calculating variances of continuous lifetimes: we assume that S and K are independent, and that S has a uniform distribution on $[0,1]$. Mathematically speaking, these models are not compatible: in Model 2, we have, instead, for $0 \le u \le 1$,

$$\begin{aligned}\bar{F}_{T_x}(u) &= \mathbb{P}(T > x+u | T > x) \\ &= \mathbb{P}(K \ge x+1 | K \ge x) + \mathbb{P}(S > u | K = x)\mathbb{P}(K = x | T > x) \\ &= (1-q_x) + (1-u)q_x = 1 - uq_x.\end{aligned}$$

We then calculate the force of mortality at $x+u$ as

$$\mu_{x+u} = -\frac{\bar{F}'_{T_x}(u)}{\bar{F}_{T_x}(u)} = \frac{q_x}{1-uq_x}$$

and this is increasing in u. Note that μ is discontinuous at (some if not all) integer times. The only exception is very artificial, as we require $q_{x+1} = q_x/(1-q_x)$, and in order for this to not exceed 1 at some point, we need $q_0 = \alpha = 1/n$, and then $q_k = \frac{\alpha}{1-k\alpha}$, $k = 1, \ldots, n-1$, with $\omega = n$ maximal age. Usually, one accepts discontinuities.

If one of the two assumptions is satisfied, the above formulas allow to reconstruct the full distribution of a lifetime T from the entries $(q_x)_{x\in\mathbb{N}}$ of a life-table: from the definition of conditional probabilities

$$\mathbb{P}(S \le t-[t] | K = [t]) = \frac{\mathbb{P}(K = [t], S \le t-[t])}{\mathbb{P}(K = [t])} = \begin{cases} \dfrac{1-e^{-(t-[t])\mu_{x+0.5}}}{1-e^{-\mu_{x+0.5}}} & \text{in Model 1,} \\ t-[t] & \text{in Model 2,}\end{cases}$$

we deduce that

$$\mathbb{P}(T \le t) = \mathbb{P}(K \le [t]-1) + \mathbb{P}(K = [t])\mathbb{P}(S \le t-[t] | K = [t]),$$

and we have already expressed the distribution of K in terms of $(q_x)_{x\in\mathbb{N}}$.

9.4 Practical concerns

- Mortality depends on individual characteristics (rich, athletic, adventurous). Effects on mortality can be studied. Models include scaling or shifting already existing tables as a function of the characteristics
- We need to estimate future mortality, which may not be the same as current or past mortality. Note that this is inherently two-dimensional: a 60-year old in 2060 is likely to have a different mortality from a 60-year old in 2010. A two-dimensional life-table should be indexed separately by calender year of birth and age, or current calender year and age. Prediction of future mortality is a major actuarial problem. Extrapolating substantially into the future is subject to considerable uncertainty.
- There are other risks, such as possible effects of pills to halt ageing, or major influenza outbreak. How can such risks be hedged?

Lecture 10

Evaluation of life insurance products

Reading: Gerber Sections 3.2, 3.3, 3.4, 3.5, 3.6, 4.2

10.1 Life assurances

Recall that the fair single premium for a whole life assurance is

$$A_x = \mathbb{E}(v^{K_x+1}) = \sum_{k=0}^{\infty} v^{k+1} {}_kp_x q_{x+k},$$

where $v = e^{-\delta} = (1+i)^{-1}$ is the discount factor in the constant-i model. Note also that higher moments of the present value are easily calculated. Specifically, the second moment

$${}^2A_x = \mathbb{E}(v^{2(K_x+1)}),$$

associated with discount factor v^2, is the same as A_x calculated at a rate of interest $i' = (1+i)^2 - 1$, and hence

$$\mathrm{Var}(v^{K_x+1}) = \mathbb{E}(v^{2(K_x+1)}) - \left(\mathbb{E}(v^{K_x+1})\right)^2 = {}^2A_x - (A_x)^2.$$

Remember that the variance as expected *squared* deviation from the mean is a quadratic quantity and mean and variance of a whole life assurance of sum assured S are

$$\mathbb{E}(Sv^{K_x+1}) = SA_x \qquad \text{and} \qquad \mathrm{Var}(Sv^{K_x+1}) = S^2\mathrm{Var}(v^{K_x+1}) = S^2({}^2A_x - (A_x)^2).$$

Similarly for a term assurance,

$$\mathbb{E}\left(Sv^{K_x+1}1_{\{K_x<n\}}\right) = SA^1_{x:\overline{n}|} \quad \text{and} \quad \mathrm{Var}\left(Sv^{K_x+1}1_{\{K_x<n\}}\right) = S^2\left({}^2A^1_{x:\overline{n}|} - (A^1_{x:\overline{n}|})^2\right)$$

for a pure endowment

$$\mathbb{E}\left(Sv^n 1_{\{K_x\geq n\}}\right) = SA_{x:\overline{n}|}^{\;\;1} = Sv^n {}_np_x \quad \text{and} \quad \mathrm{Var}(Sv^n 1_{\{K_x\geq n\}}) = S^2\left({}^2A_{x:\overline{n}|}^{\;\;1} - (A_{x:\overline{n}|}^{\;\;1})^2\right)$$

and for an endowment assurance

$$\mathbb{E}(Sv^{\min(K_x+1,n)}) = SA_{x:\overline{n}|} \quad \text{and} \quad \mathrm{Var}(Sv^{\min(K_x+1,n)}) = S^2\left({}^2A_{x:\overline{n}|} - (A_{x:\overline{n}|})^2\right)$$

Note that

$$S^2\left({}^2A_{x:\overline{n}|}-(A_{x:\overline{n}|})^2\right)\neq S^2\left({}^2A^1_{x:\overline{n}|}-(A^1_{x:\overline{n}|})^2\right)+S^2\left({}^2A_{x:\overline{n}|}^{\ \ 1}-(A_{x:\overline{n}|}^{\ \ 1})^2\right),$$

because term assurance and pure endowment are *not* independent, quite the contrary, the product of their discounted values always vanishes, so that their covariance $-S^2A^1_{x:\overline{n}|}A_{x:\overline{n}|}^{\ \ 1}$ is maximally negative. In other notation, from the variance formula for sums of dependent random variables,

$$\begin{aligned}
\mathrm{Var}(Sv^{\min(K_x+1,n)}) &= \mathrm{Var}(Sv^{K_x+1}1_{\{K_x<n\}}+Sv^n1_{\{K_x\geq n\}})\\
&= \mathrm{Var}(Sv^{K_x+1}1_{\{K_x<n\}})+\mathrm{Var}(Sv^n1_{\{K_x\geq n\}})+2\mathrm{Cov}(Sv^{K_x+1}1_{\{K_x<n\}},Sv^n1_{\{K_x\geq n\}})\\
&= \mathrm{Var}(Sv^{K_x+1}1_{\{K_x<n\}})+\mathrm{Var}(Sv^n1_{\{K_x\geq n\}})-2\mathbb{E}(Sv^{K_x+1}1_{\{K_x<n\}})\mathbb{E}(Sv^n1_{\{K_x\geq n\}})\\
&= S^2\left({}^2A^1_{x:\overline{n}|}-(A^1_{x:\overline{n}|})^2\right)+S^2\left({}^2A_{x:\overline{n}|}^{\ \ 1}-(A_{x:\overline{n}|}^{\ \ 1})^2\right)-S^2A^1_{x:\overline{n}|}A_{x:\overline{n}|}^{\ \ 1}.
\end{aligned}$$

10.2 Life annuities and premium conversion relations

Recall present values of whole-life annuities, temporary annuities and their due versions

$$a_x=\sum_{k=1}^{\infty}v^k{}_kp_x,\quad a_{x:\overline{n}|}=\sum_{k=1}^{n}v^k{}_kp_x,\qquad \ddot{a}_x=1+a_x\quad\text{and}\quad \ddot{a}_{x:\overline{n}|}=1+a_{x:\overline{n-1}|}.$$

Also note the simple relationships (that are easily proved algebraically)

$$a_x=vp_x\ddot{a}_{x+1}\qquad\text{and}\qquad a_{x:\overline{n}|}=vp_x\ddot{a}_{x+1:\overline{n}|}.$$

By general reasoning, they can be justified by saying that the expected discounted value of regular payments in arrears for up to n years contingent on a life x is the same as the expected discounted value of up to n payments in advance contingent on a life $x+1$, discounted by a further year, and given survival of x for one year (which happens with probability p_x).

To calculate variances of discounted life annuity values, we use premium conversion relations:

Proposition 73 *$A_x=1-d\ddot{a}_x$ and $A_{x:\overline{n}|}=1-d\ddot{a}_{x:\overline{n}|}$, where $d=1-v$.*

Proof: The quickest proof is based on the formula $\ddot{a}_{\overline{n}|}=(1-v^n)/d$ from last term

$$\ddot{a}_x=\mathbb{E}(\ddot{a}_{\overline{K_x+1}|}=\mathbb{E}\left(\frac{1-v^{K_x+1}}{d}\right)=\frac{1-\mathbb{E}(v^{K_x+1})}{d}=\frac{1-A_x}{d}.$$

The other formula is similar, with K_x+1 replaced by $\min(K_x+1,n)$. □

Now for a whole life annuity,

$$\mathrm{Var}(a_{\overline{K_x}|})=\mathrm{Var}\left(\frac{1-v^{K_x}}{i}\right)=\mathrm{Var}\left(\frac{v^{K_x+1}}{d}\right)=\frac{1}{d^2}\mathrm{Var}(v^{K_x+1})=\frac{1}{d^2}\left({}^2A_x-(A_x)^2\right).$$

For a whole life annuity-due,

$$\mathrm{Var}(\ddot{a}_{\overline{K_x+1}|})=\mathrm{Var}(1+a_{\overline{K_x}|})=\frac{1}{d^2}\left({}^2A_x-(A_x)^2\right).$$

Similarly,

$$\begin{aligned}\mathrm{Var}(a_{\overline{\min(K_x,n)}|}) &= \mathrm{Var}\left(\frac{1-v^{\min(K_x,n)}}{i}\right) = \mathrm{Var}\left(\frac{v^{\min(K_x+1,n+1)}}{d}\right) \\ &= \frac{1}{d^2}\left({}^2A_{x:\overline{n+1}|} - (A_{x:\overline{n+1}|})^2\right)\end{aligned}$$

and

$$\mathrm{Var}(\ddot{a}_{\overline{\min(K_x+1,n)}|}) = \mathrm{Var}\left(1 + a_{\overline{\min(K_x,n-1)}|}\right) = \frac{1}{d^2}\left({}^2A_{x:\overline{n}|} - (A_{x:\overline{n}|})^2\right).$$

10.3 Continuous life assurance and annuity functions

A whole of life assurance with payment exactly at date of death has expected present value

$$\overline{A}_x = \mathbb{E}(v^{T_x}) = \int_0^\infty v^t f_x(t)dt = \int_0^\infty v^t {}_tp_x \mu_{x+t} dt.$$

An annuity payable continuously until the time of death has expected present value

$$\overline{a}_x = \mathbb{E}\left(\frac{1-v^{T_x}}{\delta}\right) = \int_0^\infty v^t {}_tp_x dt.$$

Note also the premium conversion relation $\overline{A}_x = 1 - \delta\overline{a}_x$.

For a term assurance with payment exactly at the time of death, we obtain

$$\overline{A}^1_{x:\overline{n}|} = \mathbb{E}(v^{T_x} 1_{\{T_x \le n\}}) = \int_0^n v^t {}_tp_x \mu_{x+t} dt.$$

Similarly, variances can be expressed, as before, e.g.

$$\mathrm{Var}(v^{T_x} 1_{\{T_x \le n\}}) = {}^2\overline{A}^1_{x:\overline{n}|} - (\overline{A}^1_{x:\overline{n}|})^2.$$

10.4 More general types of life insurance

In principle, we can find appropriate premiums for any cash-flow of benefits that depend on T_x, by just taking expected discounted values. An example of this was on Assignment 4, where an increasing whole life assurance was considered that pays $K_x + 1$ at time $K_x + 1$. The purpose of the exercise was to establish the premium conversion relation

$$(IA)_x = \ddot{a}_x - d(I\ddot{a})_x$$

that relates the premium $(IA)_x$ to the increasing life annuity-due that pays $k+1$ at time k for $0 \le k \le K_x$. It is natural to combine such an assurance with a regular savings plan and pay annual premiums. The principle that the total expected discounted premium payments coincide with the total expected discounted benefits yield a level annual premium $(IP)_x$ that satisfies

$$(IP)_x \ddot{a}_x = (IA)_x = \ddot{a}_x - d(I\ddot{a})_x \quad \Rightarrow \quad (IP)_x = 1 - d\frac{(I\ddot{a})_x}{\ddot{a}_x}.$$

Similarly, there are decreasing life assurances. A regular decreasing life assurance is useful to secure mortgage payments. The (simplest) standard case is where a payment of $n - K_x$ is due at time $K_x + 1$ provided $K_x < n$. This is a term assurance. We denote its single premium by

$$(DA)^1_{x:\overline{n}|} = \sum_{k=0}^{n-1} (n-k) v^{k+1} {}_k p_x q_{x+k}$$

and note that

$$(DA)^1_{x:\overline{n}|} = A^1_{x:\overline{n}|} + A^1_{x:\overline{n-1}|} + \cdots + A^1_{x:\overline{1}|} = nA^1_{x:\overline{n}|} - (IA)^1_{x:\overline{n}|},$$

where $(IA)^1_{x:\overline{n}|}$ denotes the present value of an increasing term-assurance.

Lecture 11

Premiums

Reading: Gerber Sections 5.1, 5.3, 6.2, 6.5, 10.1, 10.2

In this lecture we incorporate expenses into premium calculations. On average, such expenses are to cover the insurer's administration cost, contain some risk loading and a profit margin for the insurer. We assume here that expenses are incurred for each policy separately. In practice, actual expenses per policy also vary with the total number of policies underwritten. There are also strategic variations due to market forces.

11.1 Different types of premiums

Consider the future benefits payable under an insurance contract, modelled by a random cash-flow C. Recall that typically payment for the benefits are either made by a single lump sum premium payment at the time the contract is effected (a *single-premium contract*) or by a regular annual (or monthly) premium payments of a level amount for a specified term (a *regular premium contract*). Note that we will be assuming that all premiums are paid *in advance*, so the first payment is always due at the time the policy is effected.

Definition 74

- The *net premium* (or pure premium) is the premium amount required to meet the expected benefits under a contract, given mortality and interest assumptions.
- The *office premium* (or gross premium) is the premium required to meet all the costs under an insurance contract, usually including expected benefit cost, expenses and profit margin. This is the premium which the policy holder pays.

In this terminology, the *net premium for a single premium contract* is the expected cost of benefits $\mathbb{E}(\mathrm{Val}_0(C))$. E.g., the net premium for a single premium whole life assurance policy of sum assured 1 issued to a life aged x is A_x.

In general, recall the principle that the expected present value of *net* premium payment equals the expected present value of benefit payments. For office premiums, and later premium reserves, it is more natural to write this from the insurer's perspective as

$$\text{expected present value of net premium income} = \text{expected present value of benefit outgo.}$$

Then, we can say similarly

expected present value of office premium income
= expected present value of benefit outgo
+ expected present value of outgo on expenses
+ expected present value of required profit loading.

Definition 75 A *(policy) basis* is a set of assumptions regarding future mortality, investment returns, expenses etc.

The basis used for calculating premiums will usually be more cautious than the best estimate for a number of reasons, including to allow for a contingency margin (the insurer does not want to go bust) and to allow for uncertainty in the estimates themselves.

11.2 Net premiums

We will use the following notation for the regular net premium payable annually throughout the duration of the contract:

$P_{x:\overline{n}|}$ for an endowment assurance
$P^1_{x:\overline{n}|}$ for a term assurance
P_x for a whole life assurance.

In each case, the understanding is that we apply a second principle which stipulates that the premium payments end upon death, making the premium payment cash-flow a random cash-flow. We also introduce

${}_nP_x$ as regular net premium payable for a maximum of n years.

To calculate net premiums, recall that premium payments form a life annuity (temporary or whole-life), so we obtain net annual premiums from the first principle of equal expected discounted values for premiums and benefits, e.g.

$$\ddot{a}_{x:\overline{n}|}P_{x:\overline{n}|} = A_{x:\overline{n}|} \qquad \Rightarrow \qquad P_{x:\overline{n}|} = \frac{A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}.$$

Similarly, $P^1_{x:\overline{n}|} = \dfrac{A^1_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}$, $P_x = \dfrac{A_x}{\ddot{a}_x}$, ${}_nP_x = \dfrac{A_x}{\ddot{a}_{x:\overline{n}|}}$.

11.3 Office premiums

For office premiums, the *basis* for their calculation is crucial. We are already used to making assumptions about an interest rate model and about mortality. Expenses can be set in a variety of ways, and often it is a combination of several expenses that are charged differently. It is of little value to categorize expenses by producing a list of possibilities, because whatever their form, they describe nothing else than a cash-flow, sometimes involving the premium to be determined. Finding the premium is solving an equation of value, which is usually a linear equation in the unknown. We give an example.

Example 76 Calculate the premium for a whole-life assurance for a sum assured of £10,000 to a life aged 40, where we have

Expenses:	£100 to set up the policy,
	30% of the first premium as a commission,
	1.5% of subsequent premiums as renewal commission,
	£10 per annum maintenance expenses (after first year).

If we denote the gross premium by P, then the equation of value that sets expected discounted premium payments equal to expected discounted benefits plus expected discounted expenses is

$$P\ddot{a}_{40} = 10,000A_{40} + 100 + 0.3P + 0.015Pa_{40} + 10a_{40}.$$

Therefore, we obtain

$$\text{gross premium } P = \frac{10,000A_{40} + 100 + 10a_{40}}{\ddot{a}_{40} - 0.3 - 0.015a_{40}}.$$

In particular, we see that this exceeds the net premium

$$\frac{10,000A_{40}}{\ddot{a}_{40}}.$$

11.4 Prospective policy values

Consider the benefit and premium payments under a life insurance contract. Given a policy basis and given survival to time t, we can specify the expected present value of the contract (for the insured) at a time t during the term of the contract as

$$\begin{aligned}\text{Prospective policy value} \;&=\; \text{expected time-}t\text{ value of future benefits}\\ &\quad - \text{expected time-}t\text{ value of future premiums.}\end{aligned}$$

We call *net premium policy value* the prospective policy value when no allowance is made for future expenses and where the premium used in the calculation is a notional premium, using the policy value basis. For the net premium policy values of the standard products at time t we write

$${}_tV_{x:\overline{n}|}, \quad {}_tV^1_{x:\overline{n}|}, \quad {}_tV_x, \quad {}_t\overline{V}_{x:\overline{n}|}, \quad {}_t\overline{V}_x, \quad \text{etc.}$$

Example 77 (n-year endowment assurance) The contract has term $\max\{K_x + 1, n\}$. We assume annual level premiums. When we calculate the net premium policy value at time $k = 1, \dots, n-1$, this is for a life aged $x + k$, i.e. a life aged x at time 0 that survived to time k. The residual term of the policy is up to $n - k$ years, and premium payments are still at rate $P_{x:\overline{n}|}$. Therefore,

$${}_kV_{x:\overline{n}|} = A_{x-k:\overline{n-k}|} - P_{x:\overline{n}|}\ddot{a}_{x+k:\overline{n-k}|}$$

But from earlier calculations of premiums and associated premium conversion relations, we have

$$P_{x:\overline{n}|} = \frac{A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}} \quad \text{and} \quad A_{y:\overline{m}|} = 1 - d\ddot{a}_{y:\overline{m}|},$$

so that the value of the endowment assurance contract at time k is

$$
\begin{aligned}
{}_kV_{x:\overline{n}|} &= A_{x+k:\overline{n-k}|} - A_{x:\overline{n}|}\frac{\ddot{a}_{x+k:\overline{n-k}|}}{\ddot{a}_{x:\overline{n}|}} \\
&= 1 - d\ddot{a}_{x+k:\overline{n-k}|} - (1 - d\ddot{a}_{x:\overline{n}|})\frac{\ddot{a}_{x+k:\overline{n-k}|}}{\ddot{a}_{x:\overline{n}|}} = 1 - \frac{\ddot{a}_{x+k:\overline{n-k}|}}{\ddot{a}_{x:\overline{n}|}},
\end{aligned}
$$

where we recall that this quantity refers to a surviving life, while the prospective value for a non-surviving life is zero, since the contract will have ended.

As an aside, for a life aged x at time 0 that did not survive to time k, there are no future premium or benefit payments, so the prospective value of such an (expired) policy at such time k is zero. The insurer may have made a loss on this individual policy, such loss is paid for by parts of premiums under other policy contracts (in the same portfolio).

Example 78 (Whole-life policies) Similarly, for whole-life policies with payment at the end of the year of death, for $k = 1, 2, \ldots,$

$$
{}_kV_x = A_{x+k} - P_x\ddot{a}_{x+k} = A_{x+k} - \frac{A_x}{\ddot{a}_x}\ddot{a}_{x+k} = (1 - d\ddot{a}_{x+k}) - (1 - d\ddot{a}_x)\frac{\ddot{a}_{x+k}}{\ddot{a}_x} = 1 - \frac{\ddot{a}_{x+k}}{\ddot{a}_x},
$$

or with payment at death for any real $t \geq 0$.

$$
{}_t\overline{V}_x = \overline{A}_{x+t} - \overline{P}_x\overline{a}_{x+t} = \overline{A}_{x+t} - \frac{\overline{A}_x}{\overline{a}_x}\overline{a}_{x+t} = (1 - \delta\overline{a}_{x+t}) - (1 - \delta\overline{a}_x)\frac{\overline{a}_{x+t}}{\overline{a}_x} = 1 - \frac{\overline{a}_{x+t}}{\overline{a}_x}.
$$

While for whole-life and endowment policies the prospective policy values are increasing (because there will be a benefit payment at death, and death is more likely to happen soon, as the policyholder ages), the behaviour is quite different for temporary assurances (because it is also getting more and more likely that no benefit payment is made):

Example 79 (Term assurance policy) Consider a 40-year policy issued to a life aged 25 subject to A1967/70 mortality. For a sum assured of £100,000 and $i = 4\%$, the net premium of this policy can best be worked out by a computer: $£100,000P_{25:\overline{40}|} = £310.53$. The prospective policy values ${}_kV^1_{25:\overline{40}|} = A^1_{25+k,\overline{40-k}|} - P^1_{25:\overline{40}|}\ddot{a}_{25+k:\overline{40-k}|}$ per unit sum assured give policy values as in Figure 11.1, plotted against age, rising up to age 53, then falling.

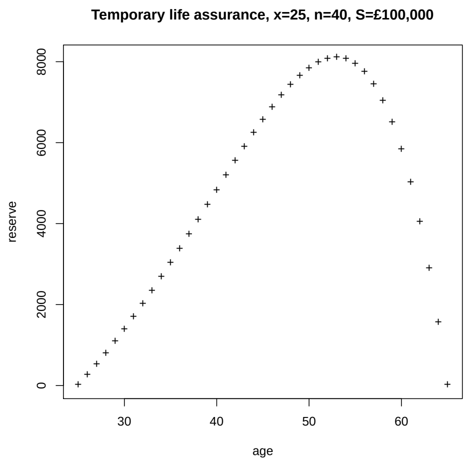


Figure 11.1: Prospective policy values for a temporary assurance.

Lecture 12

Reserves and risk

Reading: Gerber Sections 6.1, 6.3, 6.11

In many long-term life insurance contracts the cost of benefits is increasing over the term but premiums are level (or single). Therefore, the insurer needs to set aside part of early premium payments to fund a shortfall in later years of contracts. In this lecture we calculate such reserves.

12.1 Reserves and random policy values

Recall

$$\text{Prospective policy value} = \text{expected time-}t\text{ value of future benefits} - \text{expected time-}t\text{ value of future premiums.}$$

If this prospective policy value is positive, the life office needs a *reserve* for that policy, i.e. an amount of funds held by the life office at time t in respect of that policy. Apart possibly from an initial reserve that the insurer provides for solvency reasons, such a reserve typically consists of parts of earlier premium payments.

If the reserve exactly matches the prospective policy value and if experience is exactly as expected in the policy basis then reserve plus future premiums will exactly meet future liabilities. Note, however, that the mortality assumptions in the policy basis usually build a stochastic model that for a single policy will produce some spread around expected values. If life offices hold reserves for portfolios of policies, where premiums for each policy are set to match expected values, randomness will mean that some policies will generate surplus that is needed to pay for the shortfall of other policies.

When calculating present values of insurance policies and annuity contracts in the first place, it was convenient to work with expectations of present values of random cashflows depending on a lifetime random variable T_x or $K_x = [T_x]$. We can formalise prospective policy values in terms of an underlying stochastic lifetime model, and a constant-i (or constant-δ) interest model. A life insurance contract issued to a life aged x gives rise to a random cash-flow $C = C^B - C^P$ of benefit inflows C^B and premium outflows $-C^P$. The associated prospective policy value is

$$\mathbb{E}(L_t | T_x > t), \qquad \text{where } L_t = \mathrm{Val}_t(C^B_{|(t,\infty)} - C^P_{|[t,\infty)}),$$

where the subtlety of restricting to times (t, ∞) and $[t, \infty)$, respectively, arises naturally (and was implicit in calculations for Examples 77 and 78), because premiums are paid in advance and benefits in arrears in the discrete model, so for $t = k \in \mathbb{N}$, a premium payment at time k is in advance (e.g. for the year $(k, k+1]$), while a benefit payment at time k is in arrears (e.g. for death in $[k-1, k)$). Note, in particular, that if death occurs during $[k-1, k)$, then $L_k = 0$ and $L_{k-1} = v - P_x$, where $v = (1+i)^{-1}$.

Proposition 80 (Recursive calculation of policy values) *For a whole-life assurance, we have*

$$({}_kV_x + P_x)(1+i) = q_{x+k} + p_{x+k\ k+1}V_x.$$

By general reasoning, the value of the policy at time k plus the annual premium for year k payable in advance, all accumulated to time $k+1$ will give the death benefit of 1 for death in year $k+1$, i.e., given survival to time k, a payment with expected value q_{x+k} and, for survival, the policy value ${}_{k+1}V_x$ at time $k+1$, a value with expectation $p_{x+k\ k+1}V_x$.

Proof: The most explicit actuarial proof exploits the relationships between both assurance and annuity values for consecutive ages (which are obtained by partitioning according to one-year death and survival)

$$A_{x+k} = vq_{x+k} + vp_{x+k}A_{x+k+1} \quad \text{and} \quad \ddot{a}_{x+k} = 1 + vp_{x+k}\ddot{a}_{x+k+1}.$$

Now we obtain

$$\begin{aligned} {}_kV_x + P_x &= A_{x+k} - P_x\ddot{a}_{x+k} + P_x \\ &= vq_{x+k} + vp_{x+k}A_{x+k+1} - (1 + vp_{x+k}\ddot{a}_{x+k+1})P_x + P_x \\ &= v(q_{x+k} + p_{x+k}(A_{x+k+1} - P_x\ddot{a}_{x+k+1}) \\ &= v(q_{x+k} + p_{x+k\ k+1}V_x). \end{aligned}$$

□

An alternative proof can be obtained by exploiting the premium conversion relationship, which reduces the recursive formula to $\ddot{a}_{x+k} = 1 + vp_{x+k}\ddot{a}_{x+k+1}$. However, such a proof does not use insight into the cash-flows underlying the insurance policy.

A probabilistic proof can be obtained using the underlying stochastic model: by definition, ${}_kV_x = \mathbb{E}(L_k|T_x > k)$, but we can split the cash-flow underlying L_k as

$$C^B_{|(k,\infty)} - C^P_{|[k,\infty)} = ((0, -P_x), (1, 1_{\{k \le T_k < k+1\}}), C^B_{|(k+1,\infty)} - C^P_{|[k+1,\infty)}).$$

Taking Val_k and $\mathbb{E}(\cdot|T_x > k)$ then using $\mathbb{E}(1_{\{k \le T_x < k+1\}}|T_x > k) = \mathbb{P}(T_x < k+1|T_k > k) = q_{x+k}$, we get

$${}_kV_x = -P_x + vq_{x+k} + v\mathbb{E}(L_{k+1}|T_x > k) = -P_x + vq_{x+k} + vp_{x+k\ k+1}V_x,$$

where the last equality uses $\mathbb{E}(X|A) = \mathbb{P}(B|A)\mathbb{E}(X|A \cap B) + \mathbb{P}(B^c|A)\mathbb{E}(X|A \cap B^c)$, the partition theorem, where here $X = L_{k+1} = 0$ on $B^c = \{T_x < k+1\}$.

There are similar recursive formulas for the other types of life insurance contracts.

12.2 Risk and the pooling of risk

What is risk? We attempt answers at two levels, real life and stochastic models. In real life, the following provide typical examples of risk and how insurance products respond to the endeavour of passing on risk to somebody else, typically to an insurance company.

- Risk of death. Particularly untimely death (beyond the emotional dimension) often cuts income streams that dependants of the deceased rely on. A life assurance provides a lump sum to replace any/some lost income in the case of premature death, against regular premium payments while alive. Such products are sold because people prefer regular premium payments (for no benefits in the likely case of survival) to serious financial problems of dependants (in the unlikely case of death).

before retirement	no death before age $x+n$	death before age $x+n$
no life assurance	no premium	serious financial problems
life assurance	small premium	no financial problems

- Risk of long life. Even the most basic retirement life costs money, any higher standard of living costs more. Given such cost, typical savings only last for a certain number of years. A pension pays a regular income for life.

after retirement	no death before age $x+n$	death before age $x+n$
savings	run out of money	leave behind some money
pension	regular pension income	regular pension income

- Risk of poor health. In some countries, health insurance is voluntary. Without insurance, every visit to a doctor/hospital is expensive, the cost over a lifetime highly unpredictable. With insurance, premium payments are regular and predictable.

	good health	poor health
without insurance	cheap	expensive/unaffordable
with insurance	regular premium	regular premium

We could similarly list home insurance, car insurance, personal liability insurance, disability insurance, valuable items insurance etc. The principle is usually the same. Most people prefer regular premium payments (for no benefits in the likely case of no claim) to "large" personal expenditure (in the unlikely case of a claim). But scale matters; should you insure your washing machine or even smaller items if you can easily afford to buy a new one when the old one fails? Since actual premiums exceed net premiums, you end up paying more premiums than you get benefits, not just on average, but also in the long term, by the Law of Large Numbers, which applies either for a large number of small items or for a large number of years. We can/must keep some (financial and other) risks. Note however that joint failure of several items due to burglary, fire, etc. is covered by a (useful) home/contents insurance .

It is instructive to consider stochastic models. We have seen stochastic models for lifetimes, based on lifetables estimated from actual mortality observations. We can similarly set up stochastic models for health expenditure, based on suitable observations of actual health expenditure, etc. In the end, we can specify, for each insurance contract, a random variable X of the loss without insurance, and a random variable Y of expenditure with insurance. Often, Y is a deterministic premium payment or a premium payment plus a small excess payment when a claim is made. If it is deterministic, the risk of X has been completely eliminated for payment of Y. In general, $\mathbb{E}(Y) > \mathbb{E}(X)$, since the net premium principle ignores expenses etc., but Y has other preferable features, often the following

- $\mathbb{P}(Y > c) = 0$, while $\mathbb{P}(X > c) > 0$, for some threshold c where hardship starts;
- $\mathrm{Var}(Y) < \mathrm{Var}(X)$, or maybe more appropriately $\mathbb{E}((Y - \mathbb{E}(X))^2) \leq \mathbb{E}((X - \mathbb{E}(X))^2)$.

The term "hardship" needs defining. It certainly covers "bankruptcy", and this is why some insurances are compulsory (health, car, professional liability insurance for certain professions, personal liability, etc. – there are, however, differences between countries). More appropriate would be "loss of lifestyle".

Ultimately, the part of the risk we can capture here is financial uncertainty. In a scenario-based analysis of the real world, we should try to pass on risk for which there are scenarios that cause financial hardship. In a stochastic model of the real world, we may decide to pass on risk for which there is positive probability of financial hardship.

Why would an insurer want to take on the risk? One incentive clearly is $\mathbb{E}(Y) > \mathbb{E}(X)$. But also, suppose an insurer takes on many risks $X_1, \ldots, X_n$, say. In the simplest case, these can be assumed to be independent and identically distributed. This is a realistic assumption for a portfolio of identical life assurance policies issued to a homogeneous population. This is not a reasonable assumption for a flood insurance, because if one property is flooded due to extreme weather that must be expected to also affect other properties.

The random total claim amount $S = X_1 + \cdots + X_n$ must be ensured by premium payments $A = Y_1 + \cdots + Y_n$, say, to be determined. Usually, the fair premium $\mathbb{E}(S)$ leaves too much risk to the insurer. E.g. the loss probabilities $\mathbb{P}(S > \mathbb{E}(S))$ is usually too high. The following result suggests to set (deterministic) premiums Y such that the probability of a loss for the insurer does not exceed $\varepsilon > 0$.

Proposition 81 *Given a random variable X_1 with mean μ and variance σ^2, representing the benefits from an insurance policy, we have*

$$\mathbb{P}\left(X_1 \geq \mu + \frac{\sigma}{\sqrt{\varepsilon}}\right) \leq \varepsilon,$$

and $A_1(\varepsilon) = \mu + \sigma/\sqrt{\varepsilon}$ is the premium to be charged to achieve a loss probability below ε.

Given independent and identically distributed $X_1, \ldots, X_n$ from n independent policies, we obtain

$$\mathbb{P}\left(\sum_{j=1}^{n} X_j \geq n\left(\mu + \frac{\sigma}{\sqrt{n\varepsilon}}\right)\right) \leq \varepsilon,$$

i.e. $A_n(\varepsilon) = \mu + \sigma/\sqrt{n\varepsilon}$ suffices if the risk of n policies is pooled.

Proof: The statements follow as consequences of Tchebychev's inequality:

$$\mathbb{P}\left(\sum_{j=1}^{n} X_j \geq n\left(\mu + \frac{\sigma}{\sqrt{n\varepsilon}}\right)\right) \leq \mathbb{P}\left(\left|\frac{1}{n}\sum_{j=1}^{n} X_j - \mu\right| \geq \frac{\sigma}{\sqrt{n\varepsilon}}\right) \leq \frac{\mathrm{Var}\left(\frac{1}{n}\sum_{j=1}^{n} X_j\right)}{\left(\frac{\sigma}{\sqrt{n\varepsilon}}\right)^2} = \frac{\sigma^2}{n\frac{\sigma^2}{n\varepsilon}} = \varepsilon.$$

□

The estimates used in this proposition are rather weak, and the premiums suggested require some modifications in practice, but adding a multiple of the standard deviation is one important method, also since often the variance, and hence the standard deviation, can be easily calculated. However, for large n, so-called safety loadings $A_n(\varepsilon) - \mu$ proportional to $n^{-1/2}$ are of the right order, e.g. for normally distributed risks, or in general by the Central Limit Theorem for large n, when

$$\mathbb{P}\left(\frac{X_1 + \cdots + X_n - n\mu}{\sqrt{n\sigma^2}} \geq c\right) \approx \mathbb{P}(Z > c), \qquad \text{with } Z \text{ standard normally distributed.}$$

The important observation in these result is that the premiums $A_n(\varepsilon)$ decrease with n. This means, that the more policies an insurer can sell the smaller gets the (relative) risk, allowing him to reduce the premium. The proposition indicates this for identical policies, but in fact, this is a general rule about risks with sufficient independence and no unduly large risks.