4 Reserving and further life insurance products

A.1. Let T_x and T_y be two independent residual lifetimes and i a constant interest rate. Describe the benefit which has the present value random variable given by Z below:

$$Z = \begin{cases} 0 & T_x < T_y \\ \overline{a}_x - \overline{a}_y & T_x \geq T_y. \end{cases}$$

A.2. Consider a constant interest rate i and a multiple decrement model. Express in terms of the hazard rates $\mu_x^{(j)}$, $x \geq 0$, of causes $1 \leq j \leq m$ the expected present value of a life assurance that pays 1 only on death due to cause 1.

A.3. Consider the interest model of an effective annual rate of $i > 0$.

(i) Consider a lifetime random variable K with geometric distribution

$$\mathbb{P}(K = k) = (1-q)^{k-1}q, \qquad k \geq 1.$$

Show that the price of a whole-life annuity is $(1-q)/(i+q)$.

(ii) Consider a discounted dividend model, in which dividend payments D_k, $k \geq 1$, are such that $\mathbb{E}(D_k) = (1+g)^k$, $k \geq 1$. Show that the total expected present value of all dividend payments is $(1+g)/(i-g)$.

(iii) Consider a discounted dividend model for company shares, in which the company has a geometrically distributed insolvency time K. Dividends satisfy $\mathbb{E}(D_k|K \geq k) = (1+h)^k$. Show that the total expected present value of all dividend payments is $(1+h)(1-q)/(i+q-h(1-q))$.

B.1. Some time ago, a life office issued an assurance policy to a life now aged exactly 55. Premiums are payable annually in advance, and death benefits are paid at the end of the year of death. The office calculates reserves using gross premium policy values. The following information gives the reserve assumptions for the policy year just completed. Expenses are assumed to be incurred at the start of the policy year.

Reserve brought forward at the start of the policy year: £12,500

Annual premium: £1,150

Annual expenses: £75

Death benefit: £50,000

Mortality: A1967/70

Interest 5.5% per annum

Calculate the reserve at the end of the policy year.

B.2. (i) Show that $\overline{A}_x = 1 - \delta\overline{a}_x$.

(ii) Consider a whole life assurance with sum assured 1 payable at the point of death, with a constant premium paid continuously. Show that the reserve at time t satisfies

$${}_t\overline{V}_x = 1 - \frac{\overline{a}_{x+t}}{\overline{a}_x}$$

B.3. An insurer issues n identical policies. Let Y_j be the claim amount from the jth policy, and suppose that the random variables Y_j, $j = 1, \ldots, n$ are i.i.d. with mean $\mu > 0$ and variance σ^2. The insurer charges a premium of A for each policy.

(a) Show that if $A = \mu + 10\sigma n^{-1/2}$, then the probability that total claims exceed total premiums is no more than 1%, for any value of n.

(b) Use the Central Limit Theorem to show that if instead $A = \mu + 3\sigma n^{-1/2}$, then this probability is still less than 1%, provided n is large enough.

B.4 Consider the constant interest rate model at rate i, and consider m independent lifetime random variables $T^{(j)}$, with respective force of mortality $\mu_t^{(j)}$, $t \geq 0$, $1 \leq j \leq m$. Suppose that the forces of mortality are constant equal to $\mu_{x+1/2}^{(j)}$ for $x \leq t < x+1$. Let $T_{\min} = \min\{T^{(1)}, \ldots, T^{(m)}\}$ be the time of the first death and $J = j$ if $T_{\min} = T^{(j)}$, $1 \leq j \leq m$. Let $K^{(j)} = [T^{(j)}]$, $1 \leq j \leq m$, the associated curtate lifetimes.

(a) Suppose that the $T^{(j)}$ are lives aged x_j, $1 \leq j \leq m$, with common force of mortality μ_t, $t \geq 0$. Express the force of mortality $\mu^{(j)}$ of $T^{(j)}$ in terms of μ.

(b) Consider a life assurance that pays S_j at time $T_{\min}$ if $J = j$. Determine the expected discounted value of this assurance in terms of $\mu^{(k)}$, $1 \leq k \leq m$.

(c) Consider a life assurance that pays S_j at $[T_{\min}]+1$ if $J = j$. Determine the expected discounted value of this assurance in terms of $\mu^{(k)}$, $1 \leq j \leq m$, and show that this answer can be written in the form $\sum_{n\geq 0} v^{n+1} {}_np\, q_n^{(j)}$, where you should explain notation.

(d) Let $m = 2$ in the setting of (a). Consider two life assurances that pay, respectively,

- 1 at the end of the year of the first death,
- 1 at the end of the year of the second death.

Calculate their expected present values A_{xy} and $A_{\overline{xy}}$ in terms of one-year death probabilities q_x, $x \geq 0$. Show that $A_{\overline{xy}} = A_x + A_y - A_{xy}$.

B.5 In a particular accumulation fund income is retained and used to increase the value of the fund unit. The 'middle price' of the unit on 1 April in each of the years 1999 to 2005 is given in the following table:

Year 1 April	1999	2000	2001	2002	2003	2004	2005
Middle price of unit in £	1.86	2.11	2.55	2.49	2.88	3.18	3.52

(a) On the basis of the above prices and ignoring charges:

(i) Show that the yield obtained by an investor who purchased 200 units on 1 April in each year from 1999 to 2004 inclusive, and who sold his holding on 1 April 2005, is approximately 10.60%.

(ii) Show that the yield obtained by a person who invested £500 in the fund on 1 April each year from 1999 to 2004 inclusive, and who sold back his holding to the fund managers on 1 April 2005, is approximately 10.67% (You should assume that investors may purchase fractional parts of units.)

(b) Suppose that, in order to allow for expenses, the fund's managers sell units 2% above the published middle price and buy back units 2% below the middle price. On this basis find revised answers to (i) and (ii) of (a).

B.6 A 20-year unit-linked assurance contract for a life then aged 35 exactly was underwritten 15 years ago. Premiums, allocation percentage and benefits are, as follows:

- Premiums are payable annually in advance at the rate of £5,000.00 per annum.
- 75% of the first, 95% of subsequent premium payments are invested in a unit fund.
- In the case of survival for 20 years, the value of the accumulated fund units is paid to the policy holder.
- In the case of an earlier death, the maximum of the fund value and £150,000 is paid to the estate of the policy holder at the end of the year of death.

The policy basis was, as follows.

- The mortality assumption is according to the A1967-70 table, health check passed.
- The insurer has expenses of £750.00 initially, £100.00 per annum in advance of subsequent years, and £200.00 when the survival or death benefit is paid.
- The unit price is expected to grow at 5% per annum effective, after the deduction of annual management charges payable to the fund manager to cover his expenses, which are fixed at the rate of 1% of the fund value.
- The insurer's cash reserves are expected to attract interest at 3% per annum effective.

Profit vector and profit signature for the insurer are given by

k	1	2	3	4	5	6	7	8	9	10
$(\text{PRO})_k$	A	C	14.24	F	-4.77	-15.18	-25.75	-36.06	-45.56	-53.56
$(\text{PS})_k$	B	D	E	5.12	-4.75	-15.11	-25.60	-35.78	-45.11	-52.93

k	11	12	13	14	15	16	17	18	19	20
$(\text{PRO})_k$	-59.26	-61.65	-59.54	-51.46	-35.70	-10.19	27.52	80.35	G	I
$(\text{PS})_k$	-58.42	-60.62	-58.37	-50.28	-34.75	-9.88	26.54	77.08	H	J

(a) At what time, if any, is the value of the unit fund expected to exceed £150,000?

(b) (i) Calculate A, B, C, D, E and F.

(ii) Use the entries of the above profit tables to calculate ${}_{17}p_{[35]}$.

(iii) Calculate G, H, I and J.

(iv) Explain the pattern of positive and negative entries.

(c) (i) At $i_d = 4\%$, the net present value of the profit signature is $-£237.32$ for years 4-18. Calculate the net present value of the profit signature for all years 1-20.

(ii) At $i_d = 4\%$, the expected net present value of the premium payments for years 4-18 is £50,627.10. Calculate the total net present value of premium payments.

(iii) Deduce that the profit margin is 0.4%.

(d) Suppose that the life survives. Show that the discounted net present value of all profits per unit net present value of all premium payments is 3.5%. Note that this would be the actual profit margin for a bank on an investment contract (without mortality benefits) that matures after 20 years.

(e) In practice, 3.5% may be a reasonable profit margin for a life policy, while 0.4% is too low. On the other hand, 0.4% may be a reasonable profit margin for an investment contract, while 3.5% is too high. Discuss at least two risks that affect the actual profits of a life insurer (having underwritten 1,000 similar policies) and compare with the risks of a bank (entering 1,000 similar investment contracts).

C.1. Show algebraically that the product of

- the reserve after t years for an annual premium n-year pure endowment issued to a life aged x

and

- the annual premium for pure endowment of like amount issued at the same age but maturing in t years

is constant for all values of t. (Ignore expenses.)

Try to find an argument for this by general reasoning also.

C.2 (a) Consider two lives x and y. Show that

$$f_{T_{\overline{xy}}}(t) = f_{T_x}(t) + f_{T_y}(t) - f_{T_{xy}}(t).$$

Deduce that $\overline{A}_{\overline{xy}} = \overline{A}_x + \overline{A}_y - \overline{A}_{xy}$.

(b) Consider three lives x, y and z. Show that the probability density function of the time of the second death can be expressed as any of the following

(i) $f_{T_{xy}}(t)F_{T_z}(t) + f_{T_{xz}}(t)F_{T_y}(t) + f_{T_{yz}}(t)F_{T_x}(t)$,

(ii) $f_{T_{\overline{xy}}}(t)\overline{F}_{T_z}(t) + f_{T_{\overline{xz}}}(t)\overline{F}_{T_y}(t) + f_{T_{\overline{yz}}}(t)\overline{F}_{T_x}(t)$,

(iii) $f_{T_x}(t) + f_{T_y}(t) + f_{T_z}(t) - f_{T_{xyz}}(t) - f_{T_{\overline{xyz}}}(t)$.

C.3 Use a spreadsheet or R to profit test a 40-year temporary life assurance for a life aged 25.

- Mortality: Gompertz with hazard rate $\mu(t) = Bc^t$ for $B = 10^{-5}$ and $c = 1.13$,
- interest on reserves at constant rate $i = 4\%$,
- premium payments of £400.00 per annum, payable in advance,
- benefit payment of £100,000.00 at the end of the year of death,
- charges of £10.00 per annum in advance, taken from premium payments.

The aim of this exercise is to show that when discounting at $i_d = 5\%$, the profit margin of this assurance contract is 25.6%. Here are some intermediate steps:

(a) Show that the one-year death probabilities under the Gompertz distribution are

$$q_x = 1 - \exp(-ac^x), \qquad \text{where } a = B(c-1)/\ln(c),$$

and that the 40-year survival probability of a life aged 25 is 0.795.

(b) Show that the net reserve at time $k = n-1, \dots, 0$ of an n-year temporary life assurance issued to a life aged x satisfies the recursive relation

$$({}_kV_x + P_{x:\overline{n}|})(1+i) = q_{x+k} + p_{x+k}\ {}_{k+1}V_x, \qquad \text{with the convention } {}_nV_x = 0$$

and that taking into account the gross premium payments and expenses, the reserves required at times 39 and 6 for the contract of this question are £2,127.95 and £275.36, respectively. Show also that no reserve is required for the first five years.

(c) Show that the profit vector of the first six years is given by

$$(383.02, 380.00, 376.59, 372.73, 368.38, 88.62).$$

(d) Find the profit signature and show that the net present value at $i_d = 5\%$ is £1,806.82.

(e) Calculate the expected discounted (at $i_d = 5\%$) premium payments and show that the profit margin is 25.6%.