2 $p$thly payments, annuities, loans and lifetime distributions

A.1. Find, on the basis of an effective interest rate of 4% per unit time, the values of

$$a^{(4)}_{\overline{67}|},\quad s^{(12)}_{\overline{18}|},\quad \ddot{a}^{(4)}_{\overline{16.5}|},\quad \ddot{s}^{(12)}_{\overline{15.25}|},\quad {}_{4.25|}a^{(4)}_{\overline{3.75}|},\quad (Is)_{\overline{4}|}.$$

Describe the meaning of each of the symbols.

A.2. (i) Show that

$$_{n|m}q_x = \int_n^{n+m} {}_tp_x\mu_{x+t}dt.$$

[*Here* $_{n|m}q_x$ *means the probability of death of a life currently aged* x *between times* $x+n$ *and* $x+n+m$.]

(ii) Show that at age x if $0 \le a < b \le 1$ then

$$_{b-a}q_{x+a} = 1 - \frac{_bp_x}{_ap_x}.$$

A.3. Gompertz's Law has $\mu^{(1)}_x = Bc^x$.

(i) Show that the corresponding survival function is given by $_tp^{(1)}_x = g^{c^x(c^t-1)}$ where $\log g = -B/\log c$.

The force of mortality for table 2 has twice the force of mortality for table 1.

(ii) Show that the probability of survival for n years under table 2 is the square of that under table 1.

(iii) Suppose that table 1 follows the Gompertz Law. Show that the probability of survival for n years for a life aged x under table 2 is the same as that under table 1 for a life aged $x+a$, for some $a > 0$. Find a. Comment on the result.

A.4. For $m \in \mathbb{N}$, the prefix $_{m|}$ before an annuity symbol indicates that the sequence of payments concerned is *deferred* by an amount of time m. For example, the discounted present value (in the constant interest-rate model) of a *deferred annuity*, with unit payments per unit time payable from $m+1$ to $m+n$, is denoted by $_{m|}a_{\overline{n}|}$.

(a) Express $_{m|}a_{\overline{n}|}$ in terms of the ordinary annuity symbols introduced in the lectures.

(b) The case $m = -1$ corresponds to an annuity-due and is denoted by $\ddot{a}_{\overline{n}|}$. Express $\ddot{a}_{\overline{n}|}$ as simply as you can (i) in terms of $a_{\overline{n}|}$ and (ii) in terms of $a_{\overline{n-1}|}$

(c) The discounted present value of an *increasing annuity* with payments j at time $j = 1,\dots,n$ is denoted by $(Ia)_{\overline{n}|}$. Express $(Ia)_{\overline{n}|}$ in terms of ordinary annuity symbols.

(d) Consider a security redeemable at par, with term n and $p$thly coupon payments at nominal rate j. Express the accumulated (time n) and discounted (time 0) values in the constant i model in terms of annuity symbols.

B.1. A ten-year loan for £10,000 has a fixed rate of 5% for the first 3 years. Thereafter the rate is 8% for the rest of the term. Repayments are annual. Find the amounts (i) of the first 3 repayments and (ii) of the remaining 7 payments. Show that the APR of the loan is 6.4%.

B.2. Show that in any constant i interest rate model, the cash-flows $c_\infty(s) = \delta$, $0 \leq s \leq 1$, and $c_p = (k/p, i^{(p)}/p)_{k=1,\ldots,p}$ are equivalent (have the same value) for all $p \in \mathbb{N}$.

B.3. A loan of £30,000 is to be repaid by a level annuity payable monthly in arrears for 25 years, and calculated on the basis of an (effective) interest rate of 12% pa. Calculate the intitial monthly repayments.

After ten years of repayments the borrower asks to:

(a) pay off the loan which is outstanding. Calculate the lump sum which would be required to pay off the outstanding loan.

(b) extend the loan by a further five years (i.e. to 30 years in total), and with repayments changed from monthly to quarterly in arrears. Calculate the revised level of quarterly repayments.

(c) reduce the loan period by five years (i.e. to 20 years in total) and for repayments to be biannually (i.e. once every two years) in arrears. Calculate the revised level biannual repayments.

[*Hint: calculate annual repayment levels first, then combine each two consecutive payments into an equivalent single payment.*]

B.4. An insurance company issues an annuity of £10,000 p.a. payable monthly in arrears for 25 years. The cost of the annuity is calculated using an effective rate of 10% p.a.

(a) Calculate the interest component of the first instalment of the sixth year.

(b) Calculate the total interest paid in the first 5 years.

B.5. A couple buying a house require a £150,000 mortgage. They know that they will have to sell the house (and redeem the mortgage) in three years' time. They have the choice of two mortgages with monthly payments and 25-year term.

Mortgage A is at effective rate $i_A = 10.25\%$ p.a. This mortgage stipulates, however, that if you pay off the mortgage any time before the fifth anniversary, you will have to pay a penalty equal to 2.4% of the outstanding debt at the time of repayment.

Mortgage B is at $i_B = 10.7\%$ but can be paid off at any time without penalty.

Given that the buyers will have to repay the mortgage after three years and that they can save money at $j = 6\%$, which mortgage should they choose?

Calculate the yields (assuming redemption after three years).

B.6. Let μ_x be the force of mortality, and ℓ_x the corresponding life table.

(i) Show that $\mu_{x+0.5} \approx -\log p_x$ and $\mu_x \approx -0.5\,(\log p_x + \log p_{x-1})$.

(ii) Show that if deaths occurring in the year of age $(x, x+1)$ are uniformly distributed that year, then

$$ {}_{b-a}q_{x+a} = \frac{(b-a)q_x}{1-aq_x}. $$

(iii) If $\ell_x = 100(100-x)^{1/2}$, find μ_{84} exactly.

C.1. (a) An annuity-certain is payable annually in advance for n years. The first payment of the annuity is 1. Thereafter the amount of each payment is $(1+r)$ times that of the preceding payment.

Show that, on the basis of an interest rate of i per annum, the present value of the annuity is $\ddot{a}_{\overline{n}|j}$ where $j = (i-r)/(1+r)$.

(b) Suppose instead that the annuity is payable annually in arrear. Is its present value (at rate i) now equal to $a_{\overline{n}|j}$?

(c) In return for a single premium of £10,000 an investor will receive an annuity payable annually in arrear for 20 years. The annuity payments increase from year to year at the (compound) rate of 5% per annum.

Given that the initial amount of the annuity is determined on the basis of an interest rate of 9% per annum, find the amount of the first payment.

C.2. (i) Makeham's Law has $\mu_x^{(2)} = A + Bc^x$. Show that ${}_tp_x^{(2)} = s^t\,{}_tp_x^{(1)}$ where $s = e^{-A}$.

(ii) If $\mu_x = A\log x$, find an expression for ℓ_x/ℓ_0.

(iii) If it is assumed that A1967-70 table follows Makeham's Law, use ℓ_{30}, ℓ_{40}, ℓ_{50} and ℓ_{60} to find A, B and c.

($\ell_{30} = 33839$, $\ell_{40} = 33542$, and $\ell_{50} = 32670$, $\ell_{60} = 30040$).