SB4
Actuarial Science

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Aims

This unit has been designed as an introduction to financial mathematics, to introduce the concepts of risk, and to explain the foundations of insurance. The course extends the concepts of expected present value to encompass expected utility theory. It provides an introduction to the syllabus of the professional exams of the Institute & Faculty of Actuaries.

Synopsis

Fundamental nature of actuarial work. Use of generalised cash flow model to describe financial transactions. Time value of money using the concepts of compound interest and discounting. Interest rate models. Present values and accumulated values of a stream of equal or unequal payments using specified rates of interest. Interest rates in terms of different time periods. Equation of value, rate of reunion of a cash flow, existence criteria.

Single decrement model. Present values and accumulated values of a stream of payments taking into account the probability of the payments being made according to a single decrement model. Annuity functions and assurance functions for a single decrement model. Risk and premium calculation.

Liabilities under a simple assurance contract or annuity contract.

Theories of value, St Petersburg Paradox, statement of expected Utility Theory (EUT) and Subjective Expected Utility (SEU) representation theorems.

Risk aversion, the Arrow-Pratt approximation, comparative risk aversion, classical utility functions.

First and second order stochastic dominance, the Rothschild-Stiglitz Proposition. Mossin's Theorem, static portfolio choice.

Consumption and saving. Felicity function and prudence. Time consistency. Desynchronisation.

Acknowledgements

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Jethro Green, Oxford, 2018

Reading

- J. J. McCutcheon and W. F. Scott: *An Introduction to the Mathematics of Finance.* Heinemann (1986)
- S. Garrett: *An Introduction to the Mathematics of Finance: A Deterministic Approach.* Butterworth-Heinemann (2013). This is the second edition of McCutcheon and Scott.
- L. Eeckhoudt, C. Gollier and H. Schlesinger: *Economic and Financial Decisions under Risk.* Princeton University Press (2005), especially Chapters 1 – 6.
- C. Gollier: *The Economics of Risk and Time.* MIT Press (2001), especially Chapters 1 – 4, 15 – 16.
- P. Zima and R. P. Brown: *Mathematics of Finance.* McGraw-Hill Ryerson (1993)
- N. L. Bowers *et al*: *Actuarial Mathematics, 2nd edition* , Society of Actuaries (1997)
- H. Gerber: *Life Insurance Mathematics, 3rd edition* , Springer (1997)
- J. Danthine and J. Donaldson: *Intermediate Financial Theory.* 2nd edition, Academic Press Advanced Finance (2005)
- M. Milevsky: *The Calculus of Retirement Income*, Cambridge University Press (2006), especially Chapter 12
- R. Vaitilingham, *The Financial Times Guide to Using the Financial Pages.* Financial Times/Prentice Hall (2010). A useful reference book to explain jargon, and more reliable than wikipedia.
- *CM1: Actuarial Mathematics Core Reading.* Faculty & Institute of Actuaries, www.actuaries.org.uk

Contents

Introduction

Reading: http://www.actuaries.org.uk

The actuarial profession

By actuarial science, we mean the application of mathematical and statistical methods to assess risk, particularly in insurance and finance, but also and other industries and contexts. Actuaries are professionals who are qualified in this field.

In the UK, the Institute & Faculty of Actuaries oversees a qualification based on a series of exams, including mathematics, statistics, economics and finance, but also risk management, reporting and communication skills. This programme takes normally at least three or four years after a mathematical university degree, and there is also an experience requirement. The Institute & Faculty also aims to maintain and promote ethical and professional standards and promote the public good. Other actuarial bodies exist around the world, but the UK profession maintains an international perspective, and includes members working in many countries.

The Institute and Faculty of Actuaries was formed in 2010, from the merger of the Institute of Actuaries (established in 1848) and the Faculty of Actuaries (establish in Scotland in 1856). The roots of actuarial science go back further: the first life table was created by Sir Edmund Halley in 1693, and it was in life insurance that actuaries first worked. This was later extended to general insurance (health, home, property etc). As typically large amounts of money, reserves, have to be maintained, this naturally extended to investment strategies including the assessment of risk in financial markets.

Actuaries today still work in pensions and insurance, but also in investment, banking and wider fields requiring risk assessment skills. The discipline has adopted the development of probability and financial theory as it developed in the 20th century; computing power and new communication technologies have pushed back the frontier of possible applications of actuarial science. The profession has a key role to play in current debates, in both traditional fields such as the causes and consequences of changes in life expectancy, and wider fields, such as responses to climate change.

This lecture course aims to set out the foundations of financial mathematics and economics, with some applications and notation that have particular relevance to actuarial practice.

Lecture 1

Cashflow modelling and compound interest

Reading: Garrett, chapter 1, and sections 2.1 - 2.4, 4.1

Quite a few problems that we deal with in this course can be approached in an intuitive way. However, the mathematical and more powerful approach to problem solving is to set up a mathematical model in which the problem can be formalised and generalised. The concept of cash-flows seen in the last lecture is one part of such a model. In this lecture, we shall construct another part, the compound interest model in which interest on capital investments, loans etc. can be computed. This model will play a crucial role throughout the course.

In any mathematical model, reality is only partially represented. An important part of mathematical modelling is the discussion of model assumptions and the interpretation of the results of the model.

1.1 The generalised cash-flow model

The cash-flow model systematically captures payments either between different parties or, as we shall focus on, in an inflow/outflow way from the perspective of one party. This can be done at different levels of detail, depending on the purpose of an investigation, the complexity of the situation, the availability of reliable data etc.

Example 1. Look at the transactions on a bank statement for September 2011.

Date	Description	Money out	Money in
01-09-11	Gas-Elec-Bill	£21.37	
04-09-11	Withdrawal	£100.00	
15-09-11	Telephone-Bill	£14.72	
16-09-11	Mortgage Payment	£396.12	
28-09-11	Withdrawal	£150.00	
30-09-11	Salary		£1,022.54

Extracting the mathematical structure of this example we define elementary cash-flows.

Definition 2. A *cash-flow* is a vector $(t_j, c_j)_{1\le j\le m}$ of *times* $t_j \in \mathbb{R}$ and *amounts* $c_j \in \mathbb{R}$. Positive amounts $c_j > 0$ are called *inflows*. If $c_j < 0$, then $|c_j|$ is called an *outflow*.

Example 3. The cash-flow of Example 1 is mathematically given by

j	t_j	c_j
1	1	−21.37
2	4	−100.00
3	15	−14.72

j	t_j	c_j
4	16	−396.12
5	28	−150.00
6	30	1,022.54

Often, the situation is not as clear as this, and there may be uncertainty about the time/amount of a payment. This can be modelled stochastically.

Definition 4. A *random cash-flow* is a *random* vector $(T_j, C_j)_{1\le j\le M}$ of *times* $T_j \in \mathbb{R}$ and *amounts* $C_j \in \mathbb{R}$ with a possibly random length $M \in \mathbb{N}$.

Sometimes, the random structure is simple and the times or the amounts are deterministic, or even the only randomness is that a well specified payment may fail to happen with a certain probability.

Example 5. Future transactions on a bank account (say for November 2011)

j	T_j	C_j	Description
1	1	−21.37	Gas-Elec-Bill
2	T_2	C_2	Withdrawal?
3	15	C_3	Telephone-Bill

j	T_j	C_j	Description
4	16	−396.12	Mortgage payment
5	T_5	C_5	Withdrawal?
6	30	1,022.54	Salary

Here we assume a fixed Gas-Elec-Bill but a varying telephone bill. Mortgage payment and salary are certain. Any withdrawals may take place. For a full specification of the random cash-flow we would have to give the (joint!) distribution of the random variables.

This example shows that simple situations are not always easy to model. It is an important part of an actuary's work to simplify reality into tractable models. Sometimes, it is worth dropping or generalising the time specification and just list approximate or qualitative ('big', 'small', etc.) amounts of income and outgo. cash-flows can be represented in various ways as the following important examples illustrate.

1.2 Examples

Example 6 (Zero-coupon bond)**.** Usually short-term investments with interest paid at the end of the term, e.g. invest £99 for ninety days for a payoff of £100.

j	t_j	c_j
1	0	−99
2	90	100

Example 7 (Government bonds, fixed-interest securities)**.** Usually long-term investments with annual or semi-annual *coupon payments* (interest), e.g. invest £10,000 for ten years at 5% per annum. The government uses these securities to borrow money from investors, for its own spending, investment or refinance. It is common to treat the government bonds of developed economies as 'risk free', i.e. we are treating timing and amounts of payments as certain, but this is an assumption we could relax (see below).

$-£10,000$	$+£500$	$+£500$	$+£500$	- - - - -	$+£500$	$+£10,500$
0	1	2	3		9	10

Example 8 (Corporate bonds)**.** The underlying cash-flow looks the same as for government bonds, but typically they are not as secure. *Credit rating agencies* assess the insolvency risk. If a company goes bankrupt, invested money is often lost. One may therefore wish to add probabilities to the cash-flow in the above figure. Typically, the interest rate in corporate bonds is higher to allow for this extra *risk of default* that the investor takes.

Example 9 (Equities)**.** Shares in the ownership of a company that entitle to regular *dividend payments* of amounts depending on the profit and strategy of the company. Equities can be bought and sold on *stock markets* (via a stock broker) at fluctuating market prices. In the above diagram (including payment probabilities) the inflow amounts are not fixed, the term at the discretion of the shareholder and sales proceeds are not fixed. There are *advanced stochastic models* for stock price evolution. A wealth of derivative products is also available, e.g. forward contracts, options to sell or buy shares. We will discuss some aspects of derivatives, otherwise refer to alternative Oxford courses on Financial Derivatives.

Example 10 (Index-linked securities)**.** Inflation-adjusted securities: coupons and redemption payment increase in line with inflation, by tracking an *inflation index*. The UK and US government both issue index-linked securities, but these are less common in the private sector.

Example 11 (Annuity-certain)**.** Long term investments that provide a series of regular annual (semi-annual or monthly) payments for an initial lump sum, e.g.

$-£10,000$	$+£1,400$	$+£1,400$	$+£1,400$	- - - - - - -	$+£1,400$	$+£1,400$
0	1	2	3		9	10

Here the term is $n = 10$ years. *Perpetuities* provide regular payment forever ($n = \infty$).

Example 12 (Loans)**.** The cash flows may be modelled as equivalent to the bond (interest-only loan) or annuity-certain (repayment loan), but the rights of the parties may be different in some cases. A company may seek to borrow money either by issuing bonds, or by seeking a loan - for example from a bank. There are costs associated with the issuance, but the bond investor may be able to redeem or sell the bond early, making it attractive to the lender and hence reducing the cost to the borrower. A bank or other lender of a loan often has to obey stricter rules, to protect the borrower.

Example 13 (Life annuity)**.** Life annuities are like annuities-certain, but do not terminate at a fixed time but when the beneficiary dies. Risks due to age, health, profession etc. when entering the annuity contract determine the payment level. They are a basic form of a pension. Several modifications exist (minimal term, maximal term, etc.).

Example 14 (Life assurance)**.** Pays a lump sum on death for monthly or annual premiums that depend on age and health of the policy holder when the policy is underwritten. The sum assured may be decreasing in accordance with an outstanding mortgage.

Example 15 (Property insurance)**.** A class of general insurance (others are health, building, motor etc.). In return for regular premium payments, an insurance company replaces or refunds any stolen or damaged items included on the policy.

1.3 Simple versus compound interest

We are familiar with the concept of interest in everyday banking: the bank pays interest on positive balances on current accounts and savings accounts (not much, but some), and it charges interest on loans and overdrawn current accounts. Reasons for this include that

- people/institutions borrowing money are willing to pay a fee (in the future) for the use of this money now,
- there is price inflation in that £100 lose purchasing power between the beginning and the end of a loan as prices increase,
- there is often a risk that the borrower may not be able to repay the loan.

To develop a mathematical framework, consider an "interest rate h" per unit time, under which an investment of C at time 0 will receive interest Ch by time 1, giving total value $C(1+h)$:

$$C \quad \longrightarrow \quad C(1+h),$$

e.g. for $h = 4\%$ we get $C \longrightarrow 1.04C$.

There are two natural ways to extend this to general times t:

Definition 16 (Simple interest)**.** Invest C, receive $C(1+th)$ after t years. The *simple* interest on C at rate h for time t is Cth.

Definition 17 (Compound interest)**.** Invest C, receive $C(1+i)^t$ after t years. The *compound* interest on C at rate i for time t is $C((1+i)^t - 1)$.

For integer $t = n$, this is as if a bank balance was updated at the end of each year

$$C \longrightarrow C(1+i) \longrightarrow (C(1+i))(1+i) = C(1+i)^2 \longrightarrow (C(1+i)^{n-1})(1+i) = C(1+i)^n.$$

Example 18. Given an interest rate of $i = 6\%$ per annum (p.a.), investing $C = £1,000$ for $t = 2$ years yields

$$I_{\text{simp}} = C2i = £120.00 \quad \text{and} \quad I_{\text{comp}} = C\left((1+i)^2 - 1\right) = C(2i + i^2) = £123.60,$$

where we can interpret Ci^2 as interest on interest, i.e. interest for the second year paid at rate i on the interet Ci for the first year.

Compound interest behaves well under term-splitting: for $t = s + r$

$$C \longrightarrow C(1+i)^s \longrightarrow (C(1+i)^s)\,(1+i)^r = C(1+i)^t,$$

i.e. investing C at rate i first for s years and then the resulting $C(1+i)^s$ for a further r years gives the same as directly investing C for $t = s + r$ years. Under simple interest

$$C \longrightarrow C(1+hs) \longrightarrow C(1+hs)(1+hr) = C(1+ht+srh^2) > C(1+ht),$$

(in the case $C > 0$, $r > 0$, $s > 0$). The difference $Csrh^2 = (Chs)hr$ is interest on the interest Chs that was already paid at time s for the first s years.

Now we investigate what is the greatest return we can achieve by term-splitting under simple interest? First, denote by $S_t(C) = C(1+th)$ the accumulated value under simple interest at rate h for time t. We have seen that $S_r \circ S_s(C) > S_{r+s}(C)$.

Proposition 19. Fix $t > 0$ and h. Then

$$\sup_{n\in\mathbb{N}, r_1,\ldots,r_n\in\mathbb{R}_+ : r_1+\cdots+r_n=t} S_{r_n} \circ S_{r_{n-1}} \circ \cdots \circ S_{r_1}(C) = \lim_{n\to\infty} S_{t/n} \circ \cdots \circ S_{t/n}(C) = e^{th}C.$$

Proof: For the second equality we first note that

$$S_{t/n} \circ \cdots \circ S_{t/n}(C) = \left(1 + \frac{t}{n}h\right)^n C \to e^{th}C,$$

because

$$\log((1+th/n)^n) = n\log(1+th/n) = n(th/n + O(1/n^2)) \to th.$$

For the first equality,

$$e^{rh} = 1 + rh + \frac{r^2h^2}{2} + \cdots \geq 1 + rh$$

so if $r_1 + \cdots + r_n = t$, then

$$e^{th}C = e^{r_1h}e^{r_2h}\cdots e^{r_nh}C \geq (1+r_1h)(1+r_2h)\cdots(1+r_nh)C = S_{r_n} \circ S_{r_{n-1}} \circ \cdots \circ S_{r_1}(C).$$

□

So, the optimal achievable is $C \longrightarrow Ce^{th}$. If $e^{th} = (1+i)^t$, i.e. $e^h = 1+i$ or $h = \log(1+i)$, we recover the compound interest case.

From now on, we will always consider compound interest.

Definition 20. Given an *effective interest rate i per unit time* and an *initial capital C* at time 0, the *accumulated value at time t* under the compound interest model (with constant rate) is given by

$$C(1+i)^t = Ce^{\delta t},$$

where

$$\delta = \log(1+i) = \frac{\partial}{\partial t}(1+i)^t\bigg|_{t=0},$$

is called the *force of interest.*

The second expression for the force of interest means that it is the "instantaneous rate of growth per unit capital per unit time".

1.4 Nominal and effective rates

The effective annual rate is i such that $C \longrightarrow C(1+i)$ after one year. We have already seen the force of interest $\delta = \log(1+i)$ as a way to describe the same interest rate model. In practice, rates are often quoted in other ways still.

Definition 21. A *nominal rate h convertible pthly* (or compounded p times per year) means that an accumulated value $C(1+h/p)$ is achieved after time $1/p$.

By compounding, the accumulated value at time 1 is $C(1+h/p)^p$, and at time t is $C(1+h/p)^{pt}$. This again describes the same model of accumulated values if $(1+h/p)^p = 1+i$, i.e. if $h = p((1+i)^{1/p} - 1$. Actuarial notation for the nominal rate convertible $p$thly associated with effective rate i is $i^{(p)} = p((1+i)^{1/p} - 1)$.

Example 22. An annual rate of 8% convertible quarterly, i.e. $i^{(4)} = 8\%$ means that $i^{(4)}/4 = 2\%$ is credited each 3 months (and compounded) giving an annual effecive rate $i = (1+i^{(4)}/4)^4 - 1 \approx 8.24\%$.

The most common frequencies are for $p = 2$ (half-yearly, semi-annually), $p = 4$ (quarterly), $p = 12$ (monthly), $p = 52$ (weekly), although the latter used to be approximated using

$$\lim_{p\to\infty} i^{(p)} = \lim_{p\to\infty} \frac{(1+i)^{1/p} - 1}{1/p} = \frac{\partial}{\partial t}(1+i)^t\bigg|_{t=0} = \log(1+i) = \delta;$$

the force of interest δ can be called the "nominal rate of interest convertible continuously".

Example 23. Here are two genuine and one artificial options for a savings account.

(1) 3.25% p.a. effective ($i_1 = 3.25\%$)

(2) 3.20% p.a. nominal convertible monthly ($i_2^{(12)} = 3.20\%$)

(3) 3.20% p.a. nominal "convertible continuously" ($\delta_3 = 3.20\%$)

After one year, an initial capital of £10,000 accumulates to

(1) $10,000 \times (1 + 3.25\%) = 10,325.00 = R_1$,

(2) $10,000 \times (1 + 3.20\%/12)^{12} = 10,324.74 = R_2$,

(3) $10,000 \times e^{3.20\%} = 10,325.18 = R_3$.

Although interest may be credited to the account differently, an investment into (j) just consists of deposit and withdrawal, so the associated cash-flow is $((0, -10000), (1, R_j))$, and we can use R_j to decide between the options. We can also compare $i_2 \approx 3.2474\%$ and $i_3 \approx 3.2518\%$ or calculate δ_1 and δ_2 to compare with δ_3 etc.

Interest rates always refer to some time unit. The standard choice is one year, but it sometimes eases calculations to choose six months, one month or one day. All definitions we have made reflect the assumption that the interest rate does not vary with the initial capital C nor with the term t. We refer to this model of accumulated values as the constant-i model, or the constant-δ model.

1.5 Discount factors and discount rates

Before we more fully apply the constant-i model to cash-flows in Lecture 3, let us discuss the notion of discount. We are used to discounts when shopping, usually a percentage reduction in price, time being implicit. Actuaries use the notion of an *effective rate of discount d per time unit* to represent a reduction of C to $C(1-d)$ if payment takes place a time unit early.

This is consistent with the constant-i model, if the payment of $C(1-d)$ accumulates to $C = (C(1-d))(1+i)$ after one time unit, i.e. if

$$(1-d)(1+i) = 1 \iff d = 1 - \frac{1}{1+i}.$$

A more prominent role will be played by the *discount factor* $v = 1 - d$, which answers the question

How much will we have to invest now to have 1 at time 1?

Definition 24. In the constant-i model, we refer to $v = 1/(1+i)$ as the associated *discount factor* and to $d = 1 - v$ as the associated effective annual rate of discount.

Example 25. How much do we have to invest now to have 1 at time t? If we invest C, this accumulates to $C(1+i)^t$ after t years, hence we have to invest $C = 1/(1+i)^t = v^t$.

Lecture 2

Valuing cash-flows

Reading: Garrett, chapter 2 (especially sections 2.5 - 2.8)

In the previous lecture we set up the cash-flow model by assigning time-t values to cash-flows. We then established the constant-i interest rate model and saw how a past deposit accumulates and a future payment can be discounted. In this lecture, we combine these concepts and introduce general (deterministic) time-dependent interest models, and continuous cash-flows that model many small payments as infinitesimal payment streams.

2.1 Accumulating and discounting in the constant-i model

Given a cash-flow $c = (c_j, t_j)_{1 \le j \le m}$ of payments c_j at time t_j and a time t with $t \ge t_j$ for all j, we can write the joint accumulated value of all payments by time t according to the constant-i model as

$$\mathrm{AVal}_t(c) = \sum_{j=1}^{m} c_j (1+i)^{t-t_j} = \sum_{j=1}^{m} c_j e^{\delta(t-t_j)},$$

because each payment c_j at time t_j earns compound interest for $t - t_j$ time units. Note that some c_j may be negative, so the accumulated value could become negative. We assume implicitly that the same interest rate applies to positive and negative balances.

Similarly, given a cash-flow $c = (c_j, t_j)_{1 \le j \le m}$ of payments c_j at time t_j and a time $t \le t_j$ for all j, we can write the joint *discounted value* at time t of all payments as

$$\mathrm{DVal}_t(c) = \sum_{j=1}^{m} c_j v^{t_j - t} = \sum_{j=1}^{m} c_j (1+i)^{-(t_j - t)} = \sum_{j=1}^{m} c_j e^{-\delta(t_j - t)}.$$

This discounted value is the amount we invest at time t to be able to spend c_j at time t_j for all j.

2.2 Time-dependent interest rates

So far, we have assumed that interest rates are constant over time. Suppose, we now let $i = i(k)$ vary with time $k \in \mathbb{N}$. We define the accumulated value at time n for an investment of C at time 0 as

$$C(1 + i(1))(1 + i(2)) \cdots (1 + i(n-1)) \cdots (1 + i(n)).$$

Example 26. A savings account pays interest at $i(1) = 2\%$ in the first year and $i(2) = 5\%$ in the second year, with interest from the first year reinvested. Then the account balance evolves as $1,000 \longrightarrow 1,000(1 + i(1)) = 1,020 \longrightarrow 1,000(1 + i(1))(1 + i(2)) = 1,071$.

When varying interest rates between non-integer times, it is often nicer to specify the force of interest $\delta(t)$ which we saw to have a local meaning as the infinitesimal rate of capital growth under compound interest:

$$C \longrightarrow C \exp\left(\int_0^t \delta(s)ds\right) = R(t).$$

Note that now (under some right-continuity assumptions)

$$\left.\frac{\partial}{\partial t}R(t)\right|_{t=0} = \delta(0) \qquad \text{and more generally} \qquad \frac{\partial}{\partial t}R(t) = \delta(t)R(t),$$

so that the interpretation of $\delta(t)$ as local rate of capital growth at time t still applies.

Example 27. If $\delta(\cdot)$ is piecewise constant, say constant δ_j on $(t_{j-1}, t_j]$, $j = 1, \ldots, n$, then

$$C \longrightarrow Ce^{\delta_1 r_1} e^{\delta_2 r_2} \cdots e^{\delta_n r_n}, \qquad \text{where } r_j = t_j - t_{j-1}.$$

Definition 28. Given a time-dependent force of interest $\delta(t)$, $t \in \mathbb{R}_+$, we define the *accumulated value* at time $t \geq 0$ of an initial capital $C \in \mathbb{R}$ *under a force of interest* $\delta(\cdot)$ as

$$R(t) = C \exp\left(\int_0^t \delta(s)ds\right).$$

Also, we may refer to $I(t) = R(t) - C$ as the *interest from time* 0 *to time* t *under* $\delta(\cdot)$.

2.3 Accumulation factors

Given a time-dependent interest model $\delta(\cdot)$, let us define *accumulation factors from* s *to* t

$$A(s,t) = \exp\left(\int_s^t \delta(r)dr\right), \qquad s < t. \tag{1}$$

Just as $C \longrightarrow R(t) = CA(0,t)$ for an investment of C at time 0 for a term t, we use $A(s,t)$ as a factor to turn an investment of C at time s into its accumulated value $CA(s,t)$ at time t. This behaves well under term-splitting, since

$$C \longrightarrow CA(0,s) \longrightarrow (CA(0,s))A(s,t) = C\exp\left(\int_0^s \delta(r)dr\right)\exp\left(\int_s^t \delta(r)dr\right) = CA(0,t).$$

More generally, note the *consistency property* $A(r,s)A(s,t) = A(r,t)$, and conversely:

Proposition 29. Suppose, $A\colon \{(s,t) : s \le t\} \to (0,\infty)$ satisfies the consistency property and $t \mapsto A(s,t)$ is differentiable for all s, then there is a function $\delta(\cdot)$ such that (1) holds.

Proof: Since consistency for $r = s = t$ implies $A(t,t) = 1$, we can define as (right-hand) derivative

$$\delta(t) = \lim_{h\downarrow 0} \frac{A(t,t+h) - A(t,t)}{h} = \lim_{h\downarrow 0} \frac{A(0,t+h) - A(0,t)}{hA(0,t)},$$

where we also applied consistency. With $g(t) = A(0,t)$ and $f(t) = \log(A(0,t))$

$$\delta(t) = \frac{g'(t)}{g(t)} = f'(t) \quad \Rightarrow \quad \log(A(0,t)) = f(t) = \int_0^t \delta(s)ds.$$

Since consistency implies $A(s,t) = A(0,t)/A(0,s)$, we obtain (1). $\square$

We included the apparently unrealistic $A(s,t) < 1$ (accumulated value less than the initial capital) that leads to negative $\delta(\cdot)$. This can be useful for some applications where $\delta(\cdot)$ is not pre-specified, but connected to investment performance where prices can go down as well as up, or to inflation/deflation. Similarly, we allow any $i \in (-1,\infty)$, so that the associated 1-year accumulation factor $1+i$ is positive, but possibly less than 1.

2.4 Time value of money

We have discussed accumulated and discounted values in the constant-i model. In the time-varying $\delta(\cdot)$ model with accumulation factors $A(s,t) = \exp(\int_s^t \delta(r)dr)$, we obtain

$$\mathrm{AVal}_t(c) = \sum_{j=1}^m c_j A(t_j,t) \quad \text{if all } t_j \le t, \qquad \mathrm{DVal}_t(c) = \sum_{j=1}^m c_j V(t,t_j) \quad \text{if all } t_j > t,$$

where $V(s,t) = 1/A(s,t) = \exp(-\int_s^t \delta(r)dr)$ is the discount factor from time t back to time $s \le t$. With $v(t) = V(0,t)$, we get $V(s,t) = v(t)/v(s)$. Notation $v(t)$ is useful, as it is often the *present value*, i.e. the discounted value at time 0, that is of interest, and we then have

$$\mathrm{DVal}_0(c) = \sum_{j=1}^m c_j v(t_j), \quad \text{if all } t_j > 0,$$

where each payment is discounted by $v(t_j)$. Each future payment has a different present value. Note that the formulas for AVal_t and DVal_t are identical, if we express $A(s,t)$ and $V(s,t)$ in terms of $\delta(\cdot)$.

Definition 30. The *time-t value* of a cash-flow c is defined as

$$\mathrm{Val}_t(c) = \mathrm{AVal}_t(c_{\le t}) + \mathrm{DVal}_t(c_{>t}),$$

where $c_{\le t}$ and $c_{>t}$ denote restrictions of c to payments at times $t_j \le t$ resp. $t_j > t$.

Proposition 31. For all $s \le t$ we have $\mathrm{Val}_t(c) = \mathrm{Val}_s(c)A(s,t) = \mathrm{Val}_s(c)\dfrac{v(s)}{v(t)}$.

The proof is straightforward and left as an exercise. Note in particular, that if $\mathrm{Val}_t(c) = 0$ for some t, then $\mathrm{Val}_t(c) = 0$ for all t.

Remark 32.
1. A sum of money without time specification is meaningless.
2. Do not add or directly compare values at different times.
3. If values of two cash-flows are equal at one time, they are equal at all times.

2.5 Continuous cash-flows

If many small payments are spread evenly over time, it is natural to model them by a continuous stream of payment.

Definition 33. A *continuous cash-flow is a function $c\colon \mathbb{R} \to \mathbb{R}$. The total* net *inflow between times s and t is*

$$\int_s^t c(r)dr,$$

and this may combine periods of inflow and outflow.

As before, we can consider random c. We can also mix continuous and discrete parts. Note that the net inflow "adds" values at different times ignoring the time-value of money. More useful than the net inflow are accumulated and discounted values

$$\mathrm{AVal}_t(c) = \int_0^t c(s)A(s,t)ds \quad \text{and} \quad \mathrm{DVal}_t(c) = \int_t^\infty c(s)V(t,s)ds = \frac{1}{v(t)}\int_t^\infty c(s)v(s)ds.$$

Everything said in the previous section applies in an analogous way.

2.6 Example: withdrawal of interest as a cash-flow

Consider a savings account that does not credit interest to the savings account itself (where it is further compounded), but triggers a cash-flow of interest payments.

1. In a $\delta(\cdot)$-model, $1 \longrightarrow 1 + I = \exp(\int_0^1 \delta(ds))$. Consider the interest cash-flow $(1, -I)$. Then $\mathrm{Val}_1((0,1),(1,-I)) = 1$ is again the capital, at time 1.

2. In the constant-i model, recall the nominal rate $i^{(p)} = p((1+i)^{1/p} - 1)$. Interest on an initial capital 1 up to time $1/p$ is $i^{(p)}/p$. After one or indeed k such $p$thly interest payments of $i^{(p)}/p$, we have $\mathrm{Val}_{k/p}((0,1),(1/p,-i^{(p)}/p),\ldots,(k/p,-i^{(p)}/p)) = 1$.

3. In a $\delta(\cdot)$-model, continuous cash-flow $c(s) = \delta(s)$ has $\mathrm{Val}_t((0,1),-c_{\le t})=1$ for all t.

We leave as an exercise to check these directly from the definitions.

Note, in particular, that accumulation of interest itself does not correspond to events in the cash-flow. Cash-flows describe external influences on the account. Although interest is not credited continuously or at every withdrawal in practice, our mathematical model does assign a balance=value that changes continuously between instances of external cash-flow. We include the effect of interest in a cash-flow by withdrawal.

Lecture 3

Annuities and fixed-interest securities

Reading: Garrett, chapters 3 and 4, and section 7.1

In this chapter we introduce actuarial notation for discounted and accumulated values of regular payment streams, so-called *annuity symbols*. These are useful not only in the pricing of annuity products, but wherever regular payment streams occur. Our main example here will be fixed-interest securities.

3.1 Annuity symbols

Annuity-certain. An annuity-certain of term n entitles the holder to a cash-flow

$$c = ((1, X), (2, X), \ldots, (n-1, X), (n, X)).$$

Take $X = 1$ for convenience. In the constant-i model, its Net Present Value is

$$a_{\overline{n|}} = a_{\overline{n|}i} = \mathrm{NPV}(i) = \mathrm{Val}_0(c) = \sum_{k=1}^{n} v^k = v\frac{1-v^n}{1-v} = \frac{1-v^n}{i}.$$

The symbols $a_{\overline{n|}}$ and $a_{\overline{n|}i}$ are annuity symbols, pronounced “a angle n (at i)”.

The accumulated value at end of term is

$$s_{\overline{n|}} = s_{\overline{n|}i} = \mathrm{Val}_n(c) = v^{-n}\mathrm{Val}_0(c) = \frac{(1+i)^n - 1}{i}.$$

$p$thly payable annuities. A $p$thly payable annuity spreads (nominal) payment of 1 per unit time equally into p payments of $1/p$, leading to a cash-flow

$$c_p = ((1/p, 1/p), (2/p, 1/p), \ldots, (n - 1/p, 1/p), (n, 1/p))$$

with

$$a_{\overline{n}|}^{(p)} = \mathrm{Val}_0(c_p) = \frac{1}{p}\sum_{k=1}^{np} v^{k/p} = v^{1/p}\frac{1-v^n}{1-v^{1/p}} = \frac{1-v^n}{i^{(p)}},$$

where $i^p = p((1+i)^{1/p} - 1)$ is the nominal rate of interest convertible $p$thly associated with i. This calculation hence the symbol is meaningful for n any integer multiple of $1/p$.

We saw in Section 3.6 that, (now expressed in our new notation)

$$ia_{\overline{n}|} = \mathrm{Val}_0((1,i),(2,i),\ldots,(n,i)) = \mathrm{Val}_0((1/p, i^{(p)}/p),(2/p, i^{(p)}/p),\ldots,(n, i^{(p)}/p)) = i^{(p)}a_{\overline{n}|}^{(p)},$$

since both cash-flows correspond to the income up to time n on 1 unit invested at time 0, at effective rate i.

The accumulated value of c_p at the end of term is

$$s_{\overline{n}|}^{(p)} = \mathrm{Val}_n(c_p) = v^{-n}\mathrm{Val}_0(c_p) = \frac{(1+i)^n - 1}{i^{(p)}}.$$

Perpetuities. As $n \to \infty$, we obtain perpetuities that pay forever

$$a_{\overline{\infty}|} = \mathrm{Val}_0((1,1),(2,1),(3,1),\ldots) = \sum_{k=1}^{\infty} v^k = \frac{v}{1-v} = \frac{1}{i}.$$

Continuously payable annuities. As $p \to \infty$, the cash-flow c_p "tends to" the continuous cash-flow $c(s) = 1$, $0 \le s \le n$, with

$$\overline{a}_{\overline{n}|} = \mathrm{Val}_0(c) = \int_0^n c(s)v^s ds = \int_0^n v^s ds = \int_0^n e^{-\delta s} ds = \frac{1-e^{-\delta n}}{\delta} = \frac{1-v^n}{\delta}.$$

Or $\overline{a}_{\overline{n}|} = \mathrm{Val}_0(c) = \lim_{p\to\infty} a_{\overline{n}|}^{(p)} = \lim_{p\to\infty}\frac{1-v^n}{i^{(p)}} = \frac{1-v^n}{\delta}$. Similarly $\overline{s}_{\overline{n}|} = \mathrm{Val}_n(c) = v^{-n}\overline{a}_{\overline{n}|}$.

Annuity-due. This simply means that the first payment is *now*

$$((0,1),\ldots,(n-1,1))$$

with

$$\ddot{a}_{\overline{n}|} = \mathrm{Val}_0((0,1),(1,1),\ldots,(n-1,1)) = \sum_{k=0}^{n-1} v^k = \frac{1-v^n}{1-v} = \frac{1-v^n}{d}$$

and

$$\ddot{s}_{\overline{n}|} = \mathrm{Val}_n((0,1),(1,1),\ldots,(n-1,1)) = v^{-n}\ddot{a}_{\overline{n}|} = \frac{(1+i)^n - 1}{d}.$$

Similarly

$$\ddot{a}_{\overline{n}|}^{(p)} = \mathrm{Val}_0((0,1/p),(1/p,1/p),\ldots,(n-1/p,1/p))$$

and

$$\ddot{s}_{\overline{n}|} = \mathrm{Val}_n((0,1/p),(1/p,1/p),\ldots,(n-1/p,1/p)).$$

Also $\ddot{a}_{\overline{\infty}|}$, $\ddot{a}_{\overline{\infty}|}^{(p)}$, etc.

Deferred and increasing annuities. For $m \in \mathbb{N}$, the prefix ${}_{m|}$ before an annuity symbol indicates that the sequence of payments concerned is *deferred* by an amount of time m. For example, the discounted present value (in the constant interest-rate model) of a *deferred annuity*, with unit payments per unit time payable from $m+1$ to $m+n$, is denoted by ${}_{m|}a_{\overline{n}|}$. The corresponding symbols are

$${}_{m|}a_{\overline{n}|}, \quad {}_{m|}a^{(p)}_{\overline{n}|}, \quad {}_{m|}\ddot{a}_{\overline{n}|}, \quad {}_{m|}\ddot{a}^{(p)}_{\overline{n}|}, \quad {}_{m|}\overline{a}_{\overline{n}|},$$

The discounted present value of an *increasing annuity* with payments j at time $j = 1, \ldots, n$ is denoted by $(Ia)_{\overline{n}|}$. The corresponding symbols are

$$(Ia)_{\overline{n}|}, \quad (I\ddot{a})_{\overline{n}|}, \quad (I\overline{a})_{\overline{n}|}, \quad (\overline{I}\overline{a})_{\overline{n}|}, \quad {}_{m|}(Ia)_{\overline{n}|} \text{ etc.}$$

3.2 Fixed-interest securities

Simple fixed-interest securities. A simple fixed-interest security entitles the holder to a cash-flow

$$c = ((1, Nj), (2, Nj), \ldots, (n-1, Nj), (n, Nj + N)),$$

where j is the *coupon rate*, N is the *nominal amount* and n is the *term*. The value in the constant-j model is

$$\mathrm{NPV}(j) = Nja_{\overline{n}|j} + N(1+j)^{-n} = Nj\frac{1-(1+j)^{-n}}{j} + N(1+j)^{-n} = N.$$

This is not a surprise: compare with point 2. of Section 3.6. The value in the constant-i model is

$$\mathrm{NPV}(i) = Nja_{\overline{n}|i} + N(1+i)^{-n} = Nj\frac{1-(1+i)^{-n}}{i} + N(1+i)^{-n} = Nj/i + Nv^n(1-j/i).$$

More general fixed-interest securities. There are fixed-interest securities with $p$thly payable coupons at a *nominal* coupon rate j and with a redemption price of R per unit nominal

$$c = ((1/p, Nj/p), (2/p, Nj/p), \ldots, (n-1/p, Nj/p), (n, Nj/p + NR)),$$

where we say that the security is redeemable *at (resp. above or below) par* if $R = 1$ (resp. $R > 1$ or $R < 1$). We compute

$$\mathrm{NPV}(i) = Nja^{(p)}_{\overline{n}|} + NR(1+i)^{-n}.$$

If $\mathrm{NPV}(i) = N$ (resp. $> N$ or $< N$), we say that the security is valued or traded at (resp. above or below) par. Redemption at par is standard. If redemption is not at par, this is usually expressed as e.g. "redemption at 120%" meaning $R = 1.2$. If redemption is not at par, we can calculate the coupon rate per unit redemption money as $j' = j/R$; with $N' = NR$, the cash-flow of a $p$thly payable security of nominal amount N' with coupon rate j' redeemable at par is identical.

Interest payments are always calculated from the nominal amount. Redemption at par is the standard. In practice, a security is a piece of paper (with coupon strips to cash in the interest) that can change owner (sometimes under some restrictions).

Fixed-interest securities as investments. Fixed-interest securities are issued by Governments and are also called Government bonds as opposed to corporate bonds, which are issued by companies. Corporate bonds are less secure than Government bonds since (in either case, actually) bankruptcy can stop the payment stream. Since Government typically issues large quantities of bonds, they form a very liquid/marketable form of investment that is actively traded on bond markets.

Government bonds are either issued at a fixed price or *by tender*, in which case the highest bidders get the bonds at a set issue date. Government bonds usually have a term of several years. There are also shorter-term Government bills which have no coupons, so they are just offered at a discount on their nominal value.

Example 34. Consider a fixed-interest security of $N = 100$ nominal, coupon rate $j = 3\%$ payable annually and redeemable at par after a term of $n = 2$. If the security is currently trading below par, with a purchase price of $P = £97$, the investment has a cash-flow

$$c = ((0, -97), (1, 3), (2, 103))$$

and we can calculate the yield by solving the equation of value

$$-97 + 3(1+i)^{-1} + 103(1+i)^{-2} = 0 \iff 97(1+i)^2 - 3(1+i) - 103 = 0$$

to obtain $1 + i = 1.04604$, i.e. $i = 4.604\%$ (the second solution of the quadradic is $1+i = -1.01512$, i.e. $i = -2.01512$, which is not in $(-1, \infty)$; note that we knew already by Proposition 40 that there can only be one admissible solution).

Here, we could solve the quadratic equation explicitly; for fixed-interest securities of longer term, it is useful to note that the yield is composed of two effects, first the coupons payable at rate j/R per unit redemption money (or at rate jN/P per unit purchasing price) and then any capital gain/loss $RN - P$ spread over n years. Here, these rough considerations give

$$j/R+(R-P/N)/n = 4.5\%, \quad \text{or more precisely } (j/R+jN/P)/2+(R-P/N)/n = 4.546\%,$$

and often even rougher considerations give us an idea of the order of magnitude of a yield that we can then use as good initial guesses for a numerical approximation.

Lecture 4

The yield of a cash-flow

Reading: Garrett, chapters 3 and 7, especially sections 3.2, 7.3, 7.6

Given a cash-flow representing an investment, its yield is the constant interest rate that makes the cash-flow a fair deal. Yields allow to assess and compare the performance of possibly quite different investment opportunities as well as mortgages and loans.

4.1 Definition of the yield of a cash-flow

In that follows, it does not make much difference whether a cash-flow c is discrete, continuous or mixed, whether the time horizon of c is finite or infinite (like e.g. for perpetuities). However, to keep statements and technical arguments simple, we assume:

The time horizon of c is finite and payment rates of c are bounded. (H)

Since we will compare values of cash-flows under different interest rates, we need to adapt our notation to reflect this:

$$\mathrm{NPV}(i) = i\text{-}\mathrm{Val}_0(c)$$

denotes the Net Present Value of c discounted in the constant-i interest model, i.e. the value of the cash-flow c at time 0, discounted using discount factors $v(t) = v^t = (1+i)^{-t}$.

Lemma 35. Given a cash-flow c satisfying hypothesis (H), the function $i \mapsto \mathrm{NPV}(i)$ is continuous on $(-1, \infty)$.

Proof: In the discrete case $c = ((t_1, c_1), \ldots, (t_n, c_n))$, we have $\mathrm{NPV}(i) = \sum_{k=1}^{n} c_k(1+i)^{-t_k}$, and this is clearly continuous in i for all $i > -1$. For a continuous-time cash-flow $c(s)$, $0 \leq s \leq t$ (and mixed cash-flows) we use the uniform continuity of $i \mapsto (1+i)^{-s}$ on compact intervals $s \in [0, t]$ for continuity to be maintained after integration

$$\mathrm{NPV}(i) = \int_0^t c(s)(1+i)^{-s} ds.$$

□

Corollary 36. Under hypothesis (H), $i \mapsto i\text{-Val}_t(i)$ is continuous on $(-1, \infty)$ for any t.

Often the situation is such that an investment deal is profitable ($\text{NPV}(i) > 0$) if the interest rate i is below a certain level, but not above, or vice versa. By the intermediate value theorem, this threshold is a zero of $i \mapsto \text{NPV}(i)$, and we define

Definition 37. Given a cash-flow c, if $i \mapsto \text{NPV}(i)$ has a unique root on $(-1, \infty)$, we define the *yield* $y(c)$ to be this root. If $i \mapsto \text{NPV}(i)$ does not have a root in $(-1, \infty)$ or the root is not unique, we say that the yield is not well-defined.

The yield is also known as the "internal rate of return" or also just "rate of return". We can say that the yield is the fixed interest rate at which c is a "fair deal". The equation $\text{NPV}(i) = 0$ is called *yield equation*.

Example 38. Suppose that for an initial investment of £1,000 you obtain a payment of £400 after one year and 770 after two years. What is the yield of this deal? Clearly $c = ((0, -1000), (1, 400), (2, 770)$. By definition, we are looking for roots $i \in (-1, \infty)$ of

$$\begin{aligned} \text{NPV}(i) = -1,000 + 400(1+i)^{-1} + 800(1+i)^{-2} = 0 \\ \iff \qquad 1,000(i+1)^2 - 400(i+1) - 770 = 0 \end{aligned}$$

The solutions to this quadratic equation are $i_1 = -1.7$ and $i_2 = 0.1$. Since only the second zero lies in $(-1, \infty)$, the yield is $y(c) = 0.1$, i.e. 10%.

Sometimes, it is onvenient to solve for $v = (1+i)^{-1}$, here $1,000 - 400v - 770v^2 = 0$ etc. Note that $i \in (-1, \infty) \iff v \in (0, \infty)$.

Example 39. Consider the security of Example 7 in Lecture 1. The yield equation $\text{NPV}(i) = 0$ can be written as

$$10,000 = 500 \sum_{k=1}^{10} (1+i)^{-k} + 10,000(1+i)^{-10}.$$

We will introduce some short-hand actuarial notation in Lecture 5. Note, however, that we already know a root of this equation, because the cash-flow is the same as for a bank account with capital $\pounds 10,000$ and a cash-flow of annual interest payments of $\pounds 500$, i.e. at 5%, so $i = 5\%$ solves the yield equation. We will now see in much higher generality that there is usually only one solution to the yield equation for investment opportunities.

4.2 General results ensuring the existence of yields

Since the yield does not always exist, it is useful to have sufficient existence criteria.

Proposition 40. If c has in- and outflows and all inflows of c precede all outflows of c (or vice versa), then the yield $y(c)$ exists.

Remark 41. This includes the vast majority of projects that we will meet in this course. Essentially, investment projects have outflows first, and inflows afterwards, while loan schemes (from the borrower's perspective) have inflows first and outflows afterwards.

Proof: By assumption, there is T such that all inflows are strictly before T and all outflows are strictly after T. Then the accumulated value

$$p_i = i\text{-Val}_T(c_{<T})$$

is positive strictly increasing in i with $p_{-1} = 0$ and the discounted value $p_\infty = \infty$ (by assumption there are inflows) and

$$n_i = i\text{-Val}_T(c_{>T})$$

is negative strictly increasing with $n_{-1} = -\infty$ (by assumption there are outflows) and $n_\infty = 0$. Therefore

$$b_i = p_i + n_i = i\text{-Val}_T(c)$$

is strictly increasing from $-\infty$ to ∞, continuous by Corollary 36; its unique root i_0 is also the unique root of $i \mapsto \text{NPV}(i) = (1+i)^{-T}\,(i\text{-Val}_T(c))$ by Corollary 31.

For the "vice versa" part, replace c by $-c$ and use $\text{Val}_0(c) = -\text{Val}_0(-c)$ etc. □

Corollary 42. If all inflows precede all outflows, then

$$y(c) > i \iff \text{NPV}(i) < 0.$$

If all outflows precede all inflows, then

$$y(c) > i \iff \text{NPV}(i) > 0.$$

Proof: In the first setting assume $y(c) > 0$, we know $b_{y(c)} = 0$ and $i \mapsto b_i$ increases with i, so $i < y(c) \iff b_i < 0$, but $b_i = i\text{-Val}_T(c) = (1+i)^T\text{NPV}(i)$.

The second setting is analogous (substitute $-c$ for c). □

As a useful example, consider $i = 0$, when $\text{NPV}(0)$ is the sum of undiscounted payments.

Example 39 (continued) By Proposition 40, the yield exists and equals $y(c) = 5\%$.

4.3 Example: APR of a loan

A yield that is widely quoted in practice, is the Annual Percentage Rate (APR) of a loan. This is straightforward if the loan agreement is based on a constant interest rate i. Particularly for mortgages (loans to buy a house), it is common, however, to have an initial period of lower interest rates and lower monthly payments followed by a period of higher interest rates and higher payments. The APR then gives a useful summary value:

Definition 43. Given a cash-flow c representing a loan agreement (with inflows preceding outflows), the yield $y(c)$ rounded down to next lower 0.1% is called the *Annual Percentage Rate (APR)* of the loan.

Example 44. Consider a mortgage of £85,000 with interest rates of 2.99% in year 1, 4.19% in year 2 and 5.95% for the remainder of a 20-year term. A Product Fee of £100 is added to the loan amount, and a Funds Transfer Fee is deducted from the Net Amount provided to the borrower. We will discuss in Lecture 6 how this leads to a cash flow of

$$c = ((0, 84975), (1, -5715), (2, -6339), (3, -7271), (4, -7271), \dots, (20, -7271)),$$

and how the annual payments are further transformed into equivalent monthly payments. Let us here calculate the APR, which exists by Proposition 40. Consider

$$f(i) = 84,975 - 5,715(1+i)^{-1} - 6,339(1+i)^{-2} - 7,271 \sum_{k=3}^{20} (1+i)^{-k}.$$

Solving the geometric progression, or otherwise, we find the root iteratively by evaluation

$$f(5\%) = -3,310, \quad f(5.5\%) = 396, \quad f(5.4\%) = -326, \quad f(5.45\%) = 36.$$

From the last two, we see that $y(c) \approx 5.4\%$, actually $y(c) = 5.44503\ldots\%$. We can see that APR=5.4% already from the middle two evaluations, since we always round down, by definition of the APR.

4.4 Numerical calculation of yields

Suppose we know the yield exists, e.g. by Proposition 40. Remember that $f(i) = \text{NPV}(i)$ is continuous and (usually) takes values of different signs at the boundaries of $(-1, \infty)$.

Interval splitting allows to trace the root of $f\colon (l_0, r_0) = (-1, \infty)$, make successive guesses $i_n \in (l_n, r_n)$, calculate $f(i_n)$ and define

$$(l_{n+1}, r_{n+1}) := (i_n, r_n) \quad \text{or} \quad (l_{n+1}, r_{n+1}) = (l_n, i_n)$$

such that the values at the boundaries $f(l_{n+1})$ and $f(r_{n+1})$ are still of different signs. Stop when the desired accuracy is reached.

The challenge is to make good guesses. Bisection

$$i_n = (l_n + r_n)/2$$

(once $r_n < \infty$) is the ad hoc way, *linear interpolation*

$$i_n = l_n \frac{f(r_n)}{f(r_n) - f(l_n)} + r_n \frac{-f(l_n)}{f(r_n) - f(l_n)}$$

an efficient improvement. There are more efficient variations of this method using some kind of convexity property of f, but that is beyond the scope of this course.

Actually, the iterations are for computers to carry out. For assignment and examination questions, you should make good guesses of l_0 and r_0 and carry out one linear interpolation, then claiming an *approximate* yield.

Example 44 (continued) Good guesses are $r_0 = 6\%$ and $l_0 = 5\%$, since $i = 5.95\%$ is mostly used. [Better, but a priori less obvious guess would be $r_0 = 5.5\%$.] Then

$$\left.\begin{array}{l} f(5\%) = -3,310.48 \\ f(6\%) = 3874.60 \end{array}\right\} \Rightarrow y(c) \approx 5\%\frac{f(6\%)}{f(6\%) - f(5\%)} + 6\%\frac{-f(5\%)}{f(6\%) - f(5\%)} = 5.46\%.$$

4.5 Random cashflows

In this section we start to explore random cashflows, with examples based on common financial securities. Our expected present value will be given by the following equation:

$$\mathbb{E}\left(\mathrm{NPV}(i)\right) = \mathbb{E}\left[\sum_{m=1}^{n} C_m(1+i)^{-T_m}\right]$$

Note that we can take the expectation inside the summation (the expectation of a sum is the sum of the expectations. If the times of C are deterministic $T_m = t_m$ and only the amounts C_m random, the expected present value is

$$A = \sum_{m=1}^{n} \mathbb{E}[C_m](1+i)^{-t_m}$$

and depends only on the mean amounts, since the deterministic $(1+i)^{-t_m}$ can be taken out of the expectation. Such a situation arises for share dividends.

Example 45 (Discounted Dividend Model)**.** Consider a share which has just paid a dividend of d_0. Suppose that each year, the dividend increases by an independent random factor $1 + G_m$, $m \geq 1$, with $G \in (-1, \infty)$ and $\mathbb{E}(G_m) = g$. Then the mth dividend will be

$$D_m = d_0\times(1+G_1)\times\cdots\times(1+G_m), \qquad \text{with } \mathbb{E}(D_m) = d_0\times\mathbb{E}(1+G_1)\times\cdots\times\mathbb{E}(1+G_m) = d_0(1+g)^m$$

What is the net price for this share? We assume that annual dividends continue indefinitely, so the random cash-flow is $C = ((1, D_1), (2, D_2), \ldots) = ((m, D_m), m \geq 1)$ with

$$\begin{aligned} \mathbb{E}(\mathrm{NPV}(i)) &= \sum_{m\geq 1} \mathbb{E}(D_m)(1+i)^{-m} \\ &= \sum_{m\geq 1} d_0(1+g)^m(1+i)^{-m} \\ &= \frac{d_0(1+g)}{1+i}\,\frac{1}{1-(1+g)(1+i)^{-1}} \\ &= \frac{d_0(1+g)}{i-g} \end{aligned}$$

provided that $g < i$, for the geometric series to converge.

Random cashflows with Bernoulli random variables Consider the special case where $C_m = c_m$ with probability p_m, and $C_m = 0$ with probability $(1 - p_m)$. We can say $C_m = B_m c_m$, where B_m is a Bernoulli random variable with parameter p_m, i.e.

$$B_m = \begin{cases} 1 & \text{with probability } p_m, \\ 0 & \text{with probability } 1 - p_m. \end{cases}$$

For the random cash-flow $C = ((t_1, B_1 c_1), \ldots, (t_n, B_n c_n))$, we have

$$A = \sum_{m=1}^{n} \mathbb{E}(B_m c_m (1+i)^{-t_m}) = \sum_{m=1}^{n} c_m (1+i)^{-t_m} \mathbb{E}(B_m) = \sum_{m=1}^{n} p_m c_m (1+i)^{-t_m}.$$

Note that we have not required the B_m to be independent (nor assumed anything at all about their dependence structure).

Example 46. A corporate bond is of this form, with $B_m = 1$ if $T > m$ for a default time T. A life annuity is also of this form, again with $B_m = 1$ if $T > m$ for a lifetime T.

Random times for cashflows We can also consider the special case where the amounts of a random cash-flow are fixed $C_j = c_j$ and only the times T_j are random. In this case, the net premium is

$$A = \sum_{m=1}^{n} c_m \mathbb{E}((1+i)^{-T_m}) = \sum_{m=1}^{n} c_m \mathbb{E}(e^{-\delta T_m}),$$

where $\delta = \log(1+i)$. These expectations are generating functions of T_m.

Example 47. A 'whole of life' insurance payment is of this form (with $n = 1$), where a single payment is made at the time of death.

In order to get expected values for the annuity or insurance, we would need a model of a future lifetime. We turn to this in the next lecture.

4.6 Expected yield

For deterministic cash-flows that can be interpreted as investment deals (or loan schemes), we defined the yield as an intrinsic rate of return. For a random cash-flow, this notion gives a random yield which is usually difficult to use in practice. Instead, we define:

Definition 48. Let C be a random cash-flow. The *expected yield* of C is the interest rate $i \in (-1, \infty)$, if it exists and is unique such that

$$\mathbb{E}(\mathrm{NPV}(i)) = 0,$$

where $\mathrm{NVP}(i) = i\text{-}\mathrm{Val}_0(C)$ denotes the net present value of C at time 0 discounted at interest rate i.

This corresponds to the yield of the "average cash-flow". Note that this terminology may be misleading – this is *not* the expectation of the yield of C, even if that were to exist.

Example 49. An investment of £$500,000$ provides

- a continuous income stream of £$50,000$ per year, starting at an unknown time S and ending in 6 years' time;
- a payment of unknown size A in 6 years' time.

What is the expected yield under the following assumptions?

- S is uniformly distributed between $[2\text{years}, 3\text{years}]$ (time from now);
- the mean of A is £$700,000$.

We use units of £$10,000$ and 1 year. The time-0 value at rate y is

$$-50+\int_{s=S}^{6} 5(1+y)^{-s}ds+(1+y)^{-6}A.$$

The expected time-0 value is

$$\begin{aligned}
&-50+\int_{s=2}^{6} \mathbb{P}(S<s)5(1+y)^{-s}ds+(1+y)^{-6}\mathbb{E}(A)\\
&\quad=-50+\int_{s=2}^{3}(s-2)5(1+y)^{-s}ds+\int_{s=3}^{6}5(1+y)^{-s}ds+(1+y)^{-6}70=:f(y).
\end{aligned}$$

Set $f(y)=0$ and find $f(10.45\%)=0.1016$ and $f(10.55\%)=-0.1502$. So the expected yield is 10.5% to 1d.p. (note that we only need to use the *mean* of A).

As a consequence of Proposition 52, we note:

Corollary 50. If $T\sim\text{geom}(p)$ and c is a cash-flow at integer times with yield $y(c)$, then the expected yield of $C=c_{<T}$ is $p(1+y(c))-1$.

Proof: Note that $\text{NPV}_c(y(c))=0$, and by Proposition 52, we have $\mathbb{E}(\text{NPV}_C(i))=0$, if $y(c)=(1+i-p)/p$, i.e. $1+i=p(1+y(c))$, as required. Also, this is the unique solution to the expected yield equation as otherwise the relationship $1+i=p(1+k)$ would give more solutions to the yield equation. □

4.7 Gross redemption yield

Definition 51. The *gross redemption yield* is the yield on a bond, assuming that it is held to maturity, and that all interest and principal payments are made as scheduled.

Sometimes you will see a reference to the 'yield' on a bond, which actually means the gross redemption yield, i.e. a yield which by convention, ignores the credit risk.

Let's consider again the corporate bond of example 46, where we reflect credit risk with the simplest model of a random default time T at which all payments stop: the actual cash-flow is $C = c_{<T}$.

$$\mathbb{E}\left[\sum_{m=1}^{n} B_m c_m (1+i)^{-t_m}\right] = \sum_{m=1}^{n} p_m c_m (1+i)^{-t_m}.$$

More specifically, suppose there is an annual default probability of $1-p$ (conditionally given that default has not yet happened), in the sense that m years without default happen with probability p^m, so that T has a geometric distribution geom(p), i.e.

$$\mathbb{P}(T = m+1) = p^m(1-p), \qquad m \geq 0,$$

and a payment at time m will happen if and only if $T > m$, i.e. with probability $\mathbb{P}(T > m) = p^m$. A potential payment (t_m, c_m) is then modelled by a random cash-flow $(t_m, B_m c_m)$, where $\mathbb{P}(B_m = 1) = p^m$ and $\mathbb{P}(B_m = 0) = 1 - p^m$, so that also $\mathbb{E}(B_m) = p^m$. For an interest rate i, the expected discounted value of the cash-flow $C = ((t_1, B_1 c_1), \ldots, (t_n, B_n c_n))$ is

$$\mathbb{E}\left(\mathrm{NPV}_C(i)\right) = \mathbb{E}\left(\sum_{m=1}^{n} B_m c_m (1+i)^{-t_m}\right) = \sum_{m=1}^{n} \mathbb{E}(B_m) c_m (1+i)^{-t_m} = \sum_{m=1}^{n} p^m c_m (1+i)^{-m}.$$

If we take k such that $p(1+i)^{-1} = (1+k)^{-1}$, we get

$$\mathbb{E}\left(\mathrm{NPV}_C(i)\right) = \sum_{m=1}^{n} c_m (1+k)^{-m} = \mathrm{NPV}_c(k).$$

We have proved the following result:

Proposition 52. Let c be a discrete cash-flow with integer payment times and $T \sim$ geom(p), i.e. $\mathbb{P}(T = m) = p^{m-1}(1-p)$, $m \geq 1$. Let $C = c_{<T}$. Then for any $i > -1$,

$$\mathbb{E}\left(\mathrm{NPV}_C(i)\right) = \mathrm{NPV}_c(k),$$

where $k = (1 + i - p)/p$.

Note that here if for some i, say i^* we have $\mathbb{E}\left(\mathrm{NPV}_C(i^*)\right) = 0$ then i^* is the *expected yield* and k^* is then the *gross redemption yield.*

Market prices and expected present value The probability of future default is not observable in the market, although it could be estimated based on historical data. If we are given the yield k of a corporate bond (which is only achieved if default does not occur!) and the yield i of a comparable default-free bond, there is an implied annual default probability of $1 - p = 1 - (1+i)/(1+k) = (k-i)/(1+k)$.

In many cases, this seems to give an implausibly high probability of default - why? If we substitute an estimated probability of default, then we can rephrase the question: why do some bonds appear to trade at a price *lower* than their expected present value? In the later part of the course we will demonstrate other factors that can affect the price of the risky bond.

Lecture 5

Modelling future lifetimes

Reading: Gerber Sections 2.1, 2.2, 2.4, 3.1, 3.2, 4.1

In this lecture we introduce and apply actuarial notation for lifetime distributions.

5.1 Introduction to life insurance

The lectures that follow are motivated by the following problems.

1. An individual aged x would like to buy a *life annuity* (e.g. a pension) that pays him a fixed amount N p.a. for the rest of his life. How can a life insurer determine a fair price for this product?

2. An individual aged x would like to buy a *whole life insurance* that pays a fixed amount S to his dependants upon his death. How can a life insurer determine a fair single or annual premium for this product?

3. Other related products include *pure endowments* that pay an amount S at time n provided an individual is still alive, an *endowment assurance* that pays an amount S either upon an individual's death or at time n whichever is earlier, and a *term assurance* that pays an amount S upon an individual's death only if death occurs before time n.

The answer to these questions will depend on the chosen model of the future lifetime T_x of the individual.

5.2 Lives aged x

For a continuously distributed random lifetime T, we write $F_T(t) = \mathbb{P}(T \le t)$ for the cumulative distribution function, $f_T(t) = F'_T(t)$ for the probability density function, $\overline{F}_T(t) = \mathbb{P}(T > t)$ for the survival function and $\omega_T = \inf\{t \ge 0 : \overline{F}_T(t) = 0\}$ for the maximal possible lifetime.

Definition 53. The function

$$\mu_T(t) = \frac{f_T(t)}{\overline{F}_T(t)}, \qquad t \geq 0,$$

specifies the *force of mortality* or *hazard rate* at (time) t.

Proposition 54. We have $\frac{1}{\varepsilon}\mathbb{P}(T \leq t+\varepsilon | T > t) \to \mu_T(t)$ as $\varepsilon \to 0$.

Proof: We use the definitions of conditional probabilities and differentiation:

$$\begin{aligned}\frac{1}{\varepsilon}\mathbb{P}(T \leq t+\varepsilon|T>t) &= \frac{1}{\varepsilon}\frac{\mathbb{P}(T>t, T\leq t+\varepsilon)}{\mathbb{P}(T>t)} = \frac{1}{\varepsilon}\frac{\mathbb{P}(T\leq t+\varepsilon)-\mathbb{P}(T\leq t)}{\mathbb{P}(T>t)} \\ &= \frac{1}{\overline{F}_T(t)}\frac{F_T(t+\varepsilon)-F_T(t)}{\varepsilon} \longrightarrow \frac{F_T'(t)}{\overline{F}_T(t)} = \frac{f_T(t)}{\overline{F}_T(t)} = \mu_T(t).\end{aligned}$$

□

So, we can say informally that $\mathbb{P}(T \in (t, t+dt)|T > t) \approx \mu_T(t)dt$, i.e. $\mu_T(t)$ represents for each $t \geq 0$ the current "rate of death" given survival up to t.

Example 55. The exponential distribution of rate μ is given by

$$F_T(t) = 1 - e^{-\mu t}, \quad \overline{F}_T(t) = e^{-\mu t}, \quad f_T(t) = \mu e^{-\mu t}, \quad \mu_T(t) = \frac{\mu e^{-\mu t}}{e^{-\mu t}} = \mu \quad \text{constant}.$$

Lemma 56. We have $\overline{F}_T(t) = \exp\left(-\int_0^t \mu_T(s)ds\right)$.

Proof: First note that $\overline{F}_T(0) = \mathbb{P}(T > 0) = 1$. Also

$$\frac{d}{dt}\log \overline{F}_T(t) = \frac{\overline{F}_T'(t)}{\overline{F}_T(t)} = \frac{-f_T(t)}{\overline{F}_T(t)} = -\mu_T(t).$$

So

$$\log \overline{F}_T(t) = \log \overline{F}_T(0) + \int_0^t \frac{d}{ds}\log \overline{F}_T(s)ds = 0 + \int_0^t -\mu_T(s)ds.$$

□

Suppose now that T models the future lifetime of a new-born person. In life insurance applications, we are often interested in the future lifetime of a person aged x, or more precisely the *residual lifetime* $T - x$ given $\{T > x\}$, i.e. given survival to age x. For life annuities this determines the random number of annuity payments that are payable. For a life assurance contract, this models the time of payment of the sum assured. In practice, insurance companies perform medical tests and/or collect employment/geographical/medical data that allow more accurate modelling. However, let us here assume that no such other information is available. Then we have, for each $x \in [0, \omega)$,

$$\mathbb{P}(T - x > y | T > x) = \frac{\mathbb{P}(T > x + y)}{\mathbb{P}(T > x)} = \frac{\overline{F}(x + y)}{\overline{F}(x)}, \qquad y \geq 0,$$

by the definition of conditional probabilities $\mathbb{P}(A|B) = \mathbb{P}(A \cap B)/\mathbb{P}(B)$.

It is natural to directly model the residual lifetime T_x of an individual (a *life*) aged x as

$$\overline{F}_x(y) = \mathbb{P}(T_x > y) = \frac{\overline{F}(x + y)}{\overline{F}(x)} = \exp\left(-\int_x^{x+y} \mu(s)ds\right) = \exp\left(-\int_0^y \mu(x + t)dt\right).$$

We can read off $\mu_x(t) = \mu_{T_x}(t) = \mu(x + t)$, i.e. the force of mortality is still the same, just shifted by x to reflect the fact that the individual aged x and dying at time T_x is aged $x + T_x$ at death. We can also express cumulative distribution functions

$$F_x(y) = 1 - \overline{F}_x(y) = \frac{\overline{F}(x) - \overline{F}(x + y)}{\overline{F}(x)} = \frac{F(x + y) - F(x)}{1 - F(x)},$$

and probability density functions

$$f_x(y) = \begin{cases} F_x'(y) = \frac{f(x+y)}{1-F(x)} = \frac{f(x+y)}{\overline{F}(x)}, & 0 \leq y \leq \omega - x, \\ 0, & \text{otherwise.} \end{cases}$$

There is also actuarial lifetime notation, as follows

$${}_tq_x = F_x(t), \quad {}_tp_x = 1 - {}_tq_x = \overline{F}_x(t), \qquad q_x = {}_1q_x, \quad p_x = {}_1p_x, \qquad \mu_x = \mu(x) = \mu_x(0).$$

In this notation, we have the following *consistency condition* on lifetime distributions for different ages:

Proposition 57. For all $x \geq 0$, $s \geq 0$ and $t \geq 0$, we have

$${}_{s+t}p_x = {}_sp_x \times {}_tp_{x+s}.$$

By general reasoning, the probability that a life aged x survives for $s + t$ years is the same as the probability that it first survives for s years and then, aged $x + s$, survives for another t years.

Proof: Formally, we calculate the right-hand side

$$\begin{aligned} {}_sp_x \times {}_tp_{x+s} &= \mathbb{P}(T_x > s)\mathbb{P}(T_{x+s} > t) = \frac{\mathbb{P}(T > x + s)}{\mathbb{P}(T > x)} \frac{\mathbb{P}(T > x + s + t)}{\mathbb{P}(T > x + s)} = \mathbb{P}(T_x > s + t) \\ &= {}_{s+t}p_x. \end{aligned}$$

□

We can also express other formulas, which we have already established, in actuarial notation:

$$f_x(t) = {}_tp_x\mu_{x+t}, \qquad {}_tp_x = \exp\left(-\int_0^t \mu_{x+s}ds\right), \qquad {}_tq_x = \int_0^t {}_sp_x\mu_{x+s}ds.$$

The first one says that to die at t, life x must survive for time t and then die instantaneously.

5.3 Curtate lifetimes

In practice, many cash-flows pay at discrete times, often at the end of each month. Let us begin here by discretising continuous lifetimes to integer-valued lifetimes. This is often done in practice, with interpolation being used for finer models.

Definition 58. Given a continuous lifetime random variable T_x, the random variable $K_x = [T_x]$, where $[\cdot]$ denotes the integer part, is called the associated *curtate lifetime.*

Of course, one can also model curtate lifetimes directly. Note that, for continuously distributed T_x

$$\mathbb{P}(K_x = k) = \mathbb{P}(k \leq T_x < k+1) = \mathbb{P}(k < T_x \leq k+1) = {}_kp_x \times q_{x+k}.$$

Also, by Proposition 57,

$$\mathbb{P}(K_x \geq k) = {}_kp_x = \prod_{j=0}^{k-1} p_{x+j},$$

i.e. K_x can be thought of as the number of successes before the first failure in a sequence of independent Bernoulli trials with varying success probabilities p_{x+j}, $j \geq 0$. Here, success is the survival of a year, while failure is death during the year.

Proposition 59. We have $\mathbb{E}(K_x) = \sum_{k=1}^{[\omega-x]} {}_kp_x$.

Proof: By definition of the expectation of a discrete random variable,

$$\begin{aligned}
\sum_{k=1}^{[\omega-x]} {}_kp_x = \sum_{k=1}^{[\omega-x]} \mathbb{P}(K_x \geq k) &= \sum_{k=1}^{[\omega-x]} \sum_{m=k}^{[\omega-x]} \mathbb{P}(K_x = m) \\
&= \sum_{m=1}^{[\omega-x]} \sum_{k=1}^{m} \mathbb{P}(K_x = m) = \sum_{m=0}^{[\omega-x]} m\mathbb{P}(K_x = m) = \mathbb{E}(K_x).
\end{aligned}$$

□

Example 60. If T is exponentially distributed with parameter $\mu \in (0, \infty)$, then $K = [T]$ is geometrically distributed:

$$\mathbb{P}(K = k) = \mathbb{P}(k \leq T < k+1) = e^{-k\mu} - e^{-(k+1)\mu} = (e^{-\mu})^k(1 - e^{-\mu}), \qquad k \geq 0.$$

We identify the parameter of the geometric distribution as $e^{-\mu}$. Note also that here $p_x = e^{-\mu}$ and $q_x = 1 - e^{-\mu}$ for all x.

In general, we get

$$\mathbb{P}(K_x \geq k) = \prod_{j=0}^{k-1} p_{x+j} = {}_kp_x = \exp\left(-\int_0^k \mu(x+t)dt\right) = \prod_{j=0}^{k-1} \exp\left(-\int_{x+j}^{x+j+1} \mu(s)ds\right),$$

so that we read off

$$p_x = \exp\left(-\int_x^{x+1} \mu(s)ds\right), \qquad x \geq 0.$$

In practice, μ is often assumed constant between integer points (denoted $\mu_{x+0.5}$) or continuous piecewise linear between integer points.

5.4 Examples and Actuarial Notation

Let us now return to our motivating problem.

Example 61 (Whole life insurance)**.** Let K_x be a curtate future lifetime. A whole life insurance pays one unit at the end of the year of death, i.e. at time $K_x + 1$. In the model of a constant force of interest δ, the random discounted value at time 0 is $Z = e^{-\delta(K_x+1)}$, so the expected present value for this random cash-flow is

$$A_x = \mathbb{E}(Z) = \mathbb{E}(e^{-\delta(K_x+1)}) = \sum_{k=1}^{\infty} e^{-\delta k}\mathbb{P}(K_x = k-1) = \sum_{m=0}^{\infty}(1+i)^{-m-1}{}_m p_x q_{x+m},$$

where $i = e^{\delta} - 1$.

Example 62 (Term assurance, pure endowment and endowment)**.** The expected present value of a term insurance is denoted by

$$A^1_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^{k+1}{}_k p_x q_{x+k}.$$

The superscript 1 above the x indicates that 1 is only paid in case of death within the period of n years.

The expected present value of a pure endowment is denoted by $A_{x:\overset{1}{\overline{n}|}} = v^n{}_n p_x$. Here the superscript 1 indicates that 1 is only paid in case of survival of the period of n years.

The expected present value of an endowment is denoted by $A_{x:\overline{n}|} = A^1_{x:\overline{n}|} + A_{x:\overset{1}{\overline{n}|}}$, where we could have put a 1 above both x and n, but this is omitted being the default, like in previous symbols.

Example 63 (Life annuities)**.** Given a constant i interest model, the expected present value of an ordinary (respectively temporary) life annuity for a life aged x is given by

$$a_x = \sum_{k=1}^{\infty} v^k{}_k p_x \quad \text{respectively} \quad a_{x:\overline{n}|} = \sum_{k=1}^{n} v^k{}_k p_x.$$

For an ordinary (respectively temporary) life annuity-due, an additional certain payment at time 0 is made (and any payment at time n omitted):

$$\ddot{a}_x = 1 + a_x \quad \text{respectively} \quad \ddot{a}_{x:\overline{n}|} = 1 + a_{x:\overline{n-1}|}.$$

5.5 Simple laws of mortality

As we have seen, the theory is nicest for exponentially distributed lifetimes. However, the exponential distribution is not actually a good distribution to model human lifetimes:

- One reason that we have seen is that the curtate lifetime is geometric, i.e. each year given survival up to then, there is the same probability of dying in the next year. In practice, you would expect that this probability increases for higher ages.
- There is clearly significantly positive probability to survive up to age 70 and zero probability to survive to age 140, and yet the exponential distribution suggests that

  $$\mathbb{P}(T > 140) = e^{-140\mu} = (e^{-70\mu})^2 = (\mathbb{P}(T > 70))^2.$$

  Specifically, if we think there is at least 50% chance of a newborn to survive to age 70, there would be at least 25% chance to survive to age 140; if we think that the average lifetime is more than 70, then $\mu < 1/70$, so $\mathbb{P}(T > 140) > e^{-2} > 10\%$.
- More formally, the exponential distribution has the lack of memory property, which here says that the distribution of T_x is still exponential with the same parameter, independent of x. This would mean that there is no ageing.

These observations give some ideas for more realistic models. Generally, we would favour models with an eventually increasing force of mortality (in reliability theory such distributions are called IFR distributions – increasing failure rate).

1. Gompertz' Law: $\mu(t) = Bc^t$ for some $B > 0$ and $c > 1$.
2. Makeham's Law: $\mu(t) = A + Bc^t$ for some $A \geq 0$, $B > 0$ and $c > 1$ (or $c > 0$ to include DFR – decreasing failure rate cases).
3. Weibull: $\mu(t) = kt^\beta$ for some $k > 0$ and $\beta > 0$.

Makeham's Law actually gives a reasonable fit for ages 30-70.

5.6 The life-table

Suppose we have a population of newborn individuals (or individuals aged $\alpha > 0$, some lowest age in the table). Denote the size of the population by ℓ_α. Then let us observe each year the number ℓ_x of individuals still alive, until the age when the last individual dies reaching $\ell_\omega = 0$. Then out of ℓ_x individuals, ℓ_{x+1} survived age x, and the proportion ℓ_{x+1}/ℓ_x can be seen as the probability for each individual to survive. So, if we set

$$p_x = \ell_{x+1}/\ell_x \qquad \text{for all } \alpha \leq x \leq \omega - 1.$$

we specify a curtate lifetime distribution. The function $x \mapsto \ell_x$ is usually called the *life-table* in the strict sense.

Note that also vice versa, we can specify a life-table with any given curtate lifetime distribution by choosing $\ell_0 = 100,000$, say, and setting $\ell_x = \ell_0 \times {}_xp_0$. In this case, we should think of ℓ_x as the *expected* number of individuals alive at age x. Strictly speaking, we should distinguish p_x and its estimate $\widehat{p}_x = \ell_{x+1}/\ell_x$, but actuarial practice does not tend to develop this idea.

5.6.1 Interpolation for non-integer ages $x+u$, $x \in \mathbb{N}$, $u \in (0,1)$

If a life table describes a distribution for Curtate Lifetimes, it is sometimes necessary to extend this to deal with payments in continuous time. There are two popular models.

Model 1 Assume that the force of mortality μ_t is constant between each $(x, x+1)$, $x \in \mathbb{N}$. This implies

$$p_x = \exp\left(-\int_x^{x+1} \mu_t dt\right) = \exp\left(-\mu_{x+0.5}\right) \qquad \Rightarrow \qquad \mu_{x+\frac{1}{2}} = -\ln(p_x).$$

Also, for $0 \le u \le 1$,

$$\mathbb{P}(T > x+u | T > x) = \exp\left\{-\int_x^{x+u} \mu_t dt\right\} = \exp\left\{-u\mu_{x+0.5}\right\} = (1-q_x)^u,$$

and with notation $T = K + S$, where $K = [T]$ is the integer part and $S = \{T\} = T - [T] = T - K$ the fractional part of T, this means that

$$\mathbb{P}(S \le u | K = x) = \frac{\mathbb{P}(x \le T \le x+u)/\mathbb{P}(T > x)}{\mathbb{P}(x \le T < x+1)/\mathbb{P}(T > x)} = \frac{1 - \exp\left\{-u\mu_{x+0.5}\right\}}{1 - \exp\left\{-\mu_{x+0.5}\right\}}, \qquad 0 \le u \le 1,$$

a distribution that is in fact the exponential distribution with parameter $\mu_{x+0.5}$, truncated at $\omega = 1$: for exponentially distributed E

$$\mathbb{P}(E \le u | E \le 1) = \frac{\mathbb{P}(E \le u)}{\mathbb{P}(E \le 1)} = \frac{1 - \exp\left\{-u\mu_{x+0.5}\right\}}{1 - \exp\left\{-\mu_{x+0.5}\right\}}, \qquad 0 \le u \le 1.$$

Since the parameter depends on x, S is not independent of K here.

Model 2 Assume that S and K *are* independent, and that S has a uniform distribution on $[0,1]$. This is convenient, for example, when calculating variances of continuous lifetimes. Mathematically speaking, these models are not compatible: in Model 2, we have, instead, for $0 \le u \le 1$,

$$\begin{aligned}
\bar{F}_{T_x}(u) &= \mathbb{P}(T > x+u | T > x) \\
&= \mathbb{P}(K \ge x+1 | K \ge x) + \mathbb{P}(S > u | K = x)\mathbb{P}(K = x | T > x) \\
&= (1-q_x) + (1-u)q_x = 1 - uq_x.
\end{aligned}$$

We then calculate the force of mortality at $x+u$ as

$$\mu_{x+u} = -\frac{\bar{F}'_{T_x}(u)}{\bar{F}_{T_x}(u)} = \frac{q_x}{1-uq_x}$$

and this is increasing in u. Note that μ is discontinuous at (some if not all) integer times.[1]

If one of the two assumptions is satisfied, the above formulae allow to reconstruct the full distribution of a lifetime T from the entries $(q_x)_{x\in\mathbb{N}}$ of a life-table: from the definition of conditional probabilities

$$\mathbb{P}(S \le t-[t]|K=[t]) = \frac{\mathbb{P}(K=[t], S\le t-[t])}{\mathbb{P}(K=[t])} = \begin{cases} \dfrac{1-e^{-(t-[t])\mu_{x+0.5}}}{1-e^{-\mu_{x+0.5}}} & \text{in Model 1,} \\ t-[t] & \text{in Model 2,} \end{cases}$$

we deduce that

$$\mathbb{P}(T\le t) = \mathbb{P}(K \le [t]-1) + \mathbb{P}(K=[t])\mathbb{P}(S\le t-[t]|K=[t]),$$

and we have already expressed the distribution of K in terms of $(q_x)_{x\in\mathbb{N}}$.

5.7 Practical concerns

- Mortality depends on individual characteristics (wealth, lifestyle, genetic factors). Even in dealing with large populations, once we try to allow for these factors, there may be less data with which to calibrate life tables, especially at high ages. Models include scaling or shifting already existing tables as a function of the characteristics.
- We need to estimate future mortality, which may not be the same as current or past mortality. To allow for mortality improvement, most life tables are now two-dimensional: a 60-year old in 2060 is likely to have a different mortality from a 60-year old in 2010. A two-dimensional life-table should be indexed separately by calendar year of birth and age, or current calendar year and age.
- Prediction of future mortality is a major actuarial problem. Extrapolating substantially into the future is subject to considerable uncertainty, given the difficulty of predicting medical advances or the future pattern of disease.

[1] The only exception is very artificial, as we require $q_{x+1} = q_x/(1-q_x)$, and in order for this to not exceed 1 at some point, we need $q_0 = \alpha = 1/n$, and then $q_k = \frac{\alpha}{1-k\alpha}$, $k=1,\ldots,n-1$, with $\omega = n$ maximal age. Usually, one accepts discontinuities.

Lecture 6

Evaluation of life insurance products

Reading: Gerber Sections 3.2 - 3.6, 4.2

6.1 Life assurances

Recall that the fair single premium for a whole life assurance is

$$A_x = \mathbb{E}(v^{K_x+1}) = \sum_{k=0}^{\infty} v^{k+1} {}_kp_x q_{x+k},$$

where $v = e^{-\delta} = (1+i)^{-1}$ is the discount factor in the constant-i model. Note also that higher moments of the present value are easily calculated. Specifically, the second moment

$${}^2A_x = \mathbb{E}(v^{2(K_x+1)}),$$

associated with discount factor v^2, is the same as A_x calculated at a rate of interest $i' = (1+i)^2 - 1$, and hence

$$\mathrm{Var}(v^{K_x+1}) = \mathbb{E}(v^{2(K_x+1)}) - \left(\mathbb{E}(v^{K_x+1})\right)^2 = {}^2A_x - (A_x)^2.$$

Remember that the variance as expected *squared* deviation from the mean is a quadratic quantity and mean and variance of a whole life assurance of sum assured S are

$$\mathbb{E}(Sv^{K_x+1}) = SA_x \qquad \text{and} \qquad \mathrm{Var}(Sv^{K_x+1}) = S^2\mathrm{Var}(v^{K_x+1}) = S^2({}^2A_x - (A_x)^2).$$

Similarly for a term assurance,

$$\mathbb{E}\left(Sv^{K_x+1}1_{\{K_x<n\}}\right) = SA^1_{x:\overline{n}|} \quad \text{and} \quad \mathrm{Var}\left(Sv^{K_x+1}1_{\{K_x<n\}}\right) = S^2\left({}^2A^1_{x:\overline{n}|} - (A^1_{x:\overline{n}|})^2\right)$$

for a pure endowment

$$\mathbb{E}\left(Sv^n1_{\{K_x\geq n\}}\right) = SA_{x:\overline{n}|}^{\ \ 1} = Sv^n{}_np_x \quad \text{and} \quad \mathrm{Var}(Sv^n1_{\{K_x\geq n\}}) = S^2\left({}^2A_{x:\overline{n}|}^{\ \ 1} - (A_{x:\overline{n}|}^{\ \ 1})^2\right)$$

and for an endowment assurance

$$\mathbb{E}(Sv^{\min(K_x+1,n)}) = SA_{x:\overline{n}|} \quad \text{and} \quad \mathrm{Var}(Sv^{\min(K_x+1,n)}) = S^2\left({}^2A_{x:\overline{n}|} - (A_{x:\overline{n}|})^2\right)$$

Note that

$$S^2\left({}^2A_{x:\overline{n}|} - (A_{x:\overline{n}|})^2\right) \neq S^2\left({}^2A^1_{x:\overline{n}|} - (A^1_{x:\overline{n}|})^2\right) + S^2\left({}^2A_{x:\overset{1}{\overline{n}|}} - (A_{x:\overset{1}{\overline{n}|}})^2\right),$$

because term assurance and pure endowment are *not* independent, quite the contrary, the product of their discounted values always vanishes, so that their covariance $-S^2A^1_{x:\overline{n}|}A_{x:\overset{1}{\overline{n}|}}$ is maximally negative. In other notation, from the variance formula for sums of dependent random variables,

$$\begin{aligned}
&\mathrm{Var}(Sv^{\min(K_x+1,n)}) = \mathrm{Var}(Sv^{K_x+1}1_{\{K_x<n\}} + Sv^n1_{\{K_x\geq n\}}) \\
&\quad = \mathrm{Var}(Sv^{K_x+1}1_{\{K_x<n\}}) + \mathrm{Var}(Sv^n1_{\{K_x\geq n\}}) + 2\mathrm{Cov}(Sv^{K_x+1}1_{\{K_x<n\}}, Sv^n1_{\{K_x\geq n\}}) \\
&\quad = \mathrm{Var}(Sv^{K_x+1}1_{\{K_x<n\}}) + \mathrm{Var}(Sv^n1_{\{K_x\geq n\}}) - 2\mathbb{E}(Sv^{K_x+1}1_{\{K_x<n\}})\mathbb{E}(Sv^n1_{\{K_x\geq n\}}) \\
&\quad = S^2\left({}^2A^1_{x:\overline{n}|} - (A^1_{x:\overline{n}|})^2\right) + S^2\left({}^2A_{x:\overset{1}{\overline{n}|}} - (A_{x:\overset{1}{\overline{n}|}})^2\right) - S^2A^1_{x:\overline{n}|}A_{x:\overset{1}{\overline{n}|}}.
\end{aligned}$$

6.2 Life annuities and premium conversion relations

Recall present values of whole-life annuities, temporary annuities and their due versions

$$a_x = \sum_{k=1}^{\infty} v^k{}_kp_x, \quad a_{x:\overline{n}|} = \sum_{k=1}^{n} v^k{}_kp_x, \qquad \ddot{a}_x = 1 + a_x \quad \text{and} \quad \ddot{a}_{x:\overline{n}|} = 1 + a_{x:\overline{n-1}|}.$$

Also note the simple relationships (that are easily proved algebraically)

$$a_x = vp_x\ddot{a}_{x+1} \qquad \text{and} \qquad a_{x:\overline{n}|} = vp_x\ddot{a}_{x+1:\overline{n}|}.$$

By general reasoning, they can be justified by saying that the expected discounted value of regular payments in arrears for up to n years contingent on a life x is the same as the expected discounted value of up to n payments in advance contingent on a life $x+1$, discounted by a further year, and given survival of x for one year (which happens with probability p_x).

To calculate variances of discounted life annuity values, we use premium conversion relations:

Proposition 64. $A_x = 1 - d\ddot{a}_x$ and $A_{x:\overline{n}|} = 1 - d\ddot{a}_{x:\overline{n}|}$, where $d = 1 - v$.

Proof: The quickest proof is based on the formula $\ddot{a}_{\overline{n}|} = (1-v^n)/d$ from last term

$$\ddot{a}_x = \mathbb{E}(\ddot{a}_{\overline{K_x+1}|} = \mathbb{E}\left(\frac{1-v^{K_x+1}}{d}\right) = \frac{1-\mathbb{E}(v^{K_x+1})}{d} = \frac{1-A_x}{d}.$$

The other formula is similar, with K_x+1 replaced by $\min(K_x+1,n)$. □

Now for a whole life annuity,

$$\mathrm{Var}(a_{\overline{K_x}|}) = \mathrm{Var}\left(\frac{1-v^{K_x}}{i}\right) = \mathrm{Var}\left(\frac{v^{K_x+1}}{d}\right) = \frac{1}{d^2}\mathrm{Var}(v^{K_x+1}) = \frac{1}{d^2}\left({}^2A_x - (A_x)^2\right).$$

For a whole life annuity-due,

$$\mathrm{Var}(\ddot{a}_{\overline{K_x+1}|}) = \mathrm{Var}(1 + a_{\overline{K_x}|}) = \frac{1}{d^2}\left({}^2A_x - (A_x)^2\right).$$

Similarly,

$$\begin{aligned}\mathrm{Var}(a_{\overline{\min(K_x,n)}|}) &= \mathrm{Var}\left(\frac{1-v^{\min(K_x,n)}}{i}\right) = \mathrm{Var}\left(\frac{v^{\min(K_x+1,n+1)}}{d}\right) \\ &= \frac{1}{d^2}\left({}^2A_{x:\overline{n+1}|} - (A_{x:\overline{n+1}|})^2\right)\end{aligned}$$

and

$$\mathrm{Var}(\ddot{a}_{\overline{\min(K_x+1,n)}|}) = \mathrm{Var}\left(1 + a_{\overline{\min(K_x,n-1)}|}\right) = \frac{1}{d^2}\left({}^2A_{x:\overline{n}|} - (A_{x:\overline{n}|})^2\right).$$

6.3 Continuous life assurance functions

A whole of life assurance with payment exactly at date of death has expected present value

$$\overline{A}_x = \mathbb{E}(v^{T_x}) = \int_0^\infty v^t f_x(t)dt = \int_0^\infty v^t {}_tp_x \mu_{x+t} dt.$$

An annuity payable continuously until the time of death has expected present value

$$\overline{a}_x = \mathbb{E}\left(\frac{1-v^{T_x}}{\delta}\right) = \int_0^\infty v^t {}_tp_x dt.$$

Note also the premium conversion relation $\overline{A}_x = 1 - \delta\overline{a}_x$.

For a term assurance with payment exactly at the time of death, we obtain

$$\overline{A}^1_{x:\overline{n}|} = \mathbb{E}(v^{T_x}1_{\{T_x \le n\}}) = \int_0^n v^t {}_tp_x \mu_{x+t} dt.$$

Similarly, variances can be expressed, as before, e.g.

$$\mathrm{Var}(v^{T_x}1_{\{T_x \le n\}}) = {}^2\overline{A}^1_{x:\overline{n}|} - (\overline{A}^1_{x:\overline{n}|})^2.$$

6.4 More general types of life insurance

In principle, we can find appropriate premiums for any cash-flow of benefits that depend on T_x, by just taking expected discounted values. An example of this was on Assignment

4, where an increasing whole life assurance was considered that pays $K_x + 1$ at time $K_x + 1$. The purpose of the exercise was to establish the premium conversion relation

$$(IA)_x = \ddot{a}_x - d(I\ddot{a})_x$$

that relates the premium $(IA)_x$ to the increasing life annuity-due that pays $k + 1$ at time k for $0 \le k \le K_x$. It is natural to combine such an assurance with a regular savings plan and pay annual premiums. The principle that the total expected discounted premium payments coincide with the total expected discounted benefits yield a level annual premium $(IP)_x$ that satisfies

$$(IP)_x\ddot{a}_x = (IA)_x = \ddot{a}_x - d(I\ddot{a})_x \quad \Rightarrow \quad (IP)_x = 1 - d\frac{(I\ddot{a})_x}{\ddot{a}_x}.$$

Similarly, there are decreasing life assurances. A regular decreasing life assurance is useful to secure mortgage payments. The (simplest) standard case is where a payment of $n-K_x$ is due at time $K_x + 1$ provided $K_x < n$. This is a term assurance. We denote its single premium by

$$(DA)^1_{x:\overline{n}|} = \sum_{k=0}^{n-1}(n-k)v^{k+1}{}_kp_xq_{x+k}$$

and note that

$$(DA)^1_{x:\overline{n}|} = A^1_{x:\overline{n}|} + A^1_{x:\overline{n-1}|} + \cdots + A^1_{x:\overline{1}|} = nA^1_{x:\overline{n}|} - (IA)^1_{x:\overline{n}|},$$

where $(IA)^1_{x:\overline{n}|}$ denotes the present value of an increasing term-assurance.

Lecture 7

Premiums

Reading: Gerber Sections 5.1, 5.3, 6.2, 6.5, 10.1, 10.2

In this lecture we incorporate expenses into premium calculations. On average, such expenses are to cover the insurer's administration cost, contain some risk loading and a profit margin for the insurer. We assume here that expenses are incurred for each policy separately. In practice, actual expenses per policy also vary with the total number of policies underwritten. There are also strategic variations due to market forces.

7.1 Different types of premiums

Consider the future benefits payable under an insurance contract, modelled by a random cash-flow C. Recall that typically payment for the benefits are either made by a single lump sum premium payment at the time the contract is effected (a *single-premium contract*) or by a regular annual (or monthly) premium payments of a level amount for a specified term (a *regular premium contract*). Note that we will be assuming that all premiums are paid *in advance*, so the first payment is always due at the time the policy is effected.

Definition 65.
- The *net premium* (or pure premium) is the premium amount required to meet the expected benefits under a contract, given mortality and interest assumptions.
- The *office premium* (or gross premium) is the premium required to meet all the costs under an insurance contract, usually including expected benefit cost, expenses and profit margin. This is the premium which the policy holder pays.

In this terminology, the *net premium for a single premium contract* is the expected cost of benefits $\mathbb{E}(\mathrm{Val}_0(C))$. E.g., the net premium for a single premium whole life assurance policy of sum assured 1 issued to a life aged x is A_x.

In general, recall the principle that the expected present value of *net* premium payment equals the expected present value of benefit payments. For office premiums, and later premium reserves, it is more natural to write this from the insurer's perspective as

expected present value of net premium income = expected present value of benefit outgo.

Then, we can say similarly

$$\begin{aligned}\text{expected present value of office premium income}\\ &= \text{expected present value of benefit outgo}\\ &\quad + \text{expected present value of outgo on expenses}\\ &\quad + \text{expected present value of required profit loading.}\end{aligned}$$

Definition 66. A *(policy) basis* is a set of assumptions regarding future mortality, investment returns, expenses etc.

The basis used for calculating premiums will usually be more cautious than the best estimate for a number of reasons, including to allow for a contingency margin (the insurer does not want to go bust) and to allow for uncertainty in the estimates themselves.

7.2 Net premiums

We will use the following notation for the regular net premium payable annually throughout the duration of the contract:

$$\begin{aligned} P_{x:\overline{n}|} &\quad \text{for an endowment assurance}\\ P^1_{x:\overline{n}|} &\quad \text{for a term assurance}\\ P_x &\quad \text{for a whole life assurance.}\end{aligned}$$

In each case, the understanding is that we apply a second principle which stipulates that the premium payments end upon death, making the premium payment cash-flow a random cash-flow. We also introduce

$${}_nP_x \quad \text{as regular net premium payable for a maximum of } n \text{ years.}$$

To calculate net premiums, recall that premium payments form a life annuity (temporary or whole-life), so we obtain net annual premiums from the first principle of equal expected discounted values for premiums and benefits, e.g.

$$\ddot{a}_{x:\overline{n}|}P_{x:\overline{n}|} = A_{x:\overline{n}|} \qquad \Rightarrow \qquad P_{x:\overline{n}|} = \frac{A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}.$$

Similarly, $P^1_{x:\overline{n}|} = \dfrac{A^1_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}$, $P_x = \dfrac{A_x}{\ddot{a}_x}$, ${}_nP_x = \dfrac{A_x}{\ddot{a}_{x:\overline{n}|}}$.

7.3 Office premiums

For office premiums, the *basis* for their calculation is crucial. We are already used to making assumptions about an interest rate model and about mortality. Expenses can be set in a variety of ways, and often it is a combination of several expenses that are charged differently. It is of little value to categorize expenses by producing a list of possibilities, because whatever their form, they describe nothing else than a cash-flow, sometimes involving the premium to be determined. Finding the premium is solving an equation of value, which is usually a linear equation in the unknown. We give an example.

Example 67. Calculate the premium for a whole-life assurance for a sum assured of £10,000 to a life aged 40, where we have

Expenses:	£100 to set up the policy,
	30% of the first premium as a commission,
	1.5% of subsequent premiums as renewal commission,
	£10 per annum maintenance expenses (after first year).

If we denote the gross premium by P, then the equation of value that sets expected discounted premium payments equal to expected discounted benefits plus expected discounted expenses is

$$P\ddot{a}_{40} = 10,000A_{40} + 100 + 0.3P + 0.015Pa_{40} + 10a_{40}.$$

Therefore, we obtain

$$\text{gross premium } P = \frac{10,000A_{40} + 100 + 10a_{40}}{\ddot{a}_{40} - 0.3 - 0.015a_{40}}.$$

In particular, we see that this exceeds the net premium

$$\frac{10,000A_{40}}{\ddot{a}_{40}}.$$

7.4 Prospective policy values

Consider the benefit and premium payments under a life insurance contract. Given a policy basis and given survival to time t, we can specify the expected present value of the contract (for the insured) at a time t during the term of the contract as

$$\begin{aligned}\text{Prospective policy value} \quad = \quad & \text{expected time-}t\text{ value of future benefits} \\ & - \text{expected time-}t\text{ value of future premiums.}\end{aligned}$$

We call *net premium policy value* the prospective policy value when no allowance is made for future expenses and where the premium used in the calculation is a notional premium, using the policy value basis. For the net premium policy values of the standard products at time t we write

$${}_tV_{x:\overline{n}|}, \quad {}_tV^1_{x:\overline{n}|}, \quad {}_tV_x, \quad {}_t\overline{V}_{x:\overline{n}|}, \quad {}_t\overline{V}_x, \quad \text{etc.}$$

Example 68 (n-year endowment assurance)**.** The contract has term $\max\{K_x+1,n\}$. We assume annual level premiums. When we calculate the net premium policy value at time $k=1,\ldots,n-1$, this is for a life aged $x+k$, i.e. a life aged x at time 0 that survived to time k. The residual term of the policy is up to $n-k$ years, and premium payments are still at rate $P_{x:\overline{n}|}$. Therefore,

$$_kV_{x:\overline{n}|}=A_{x-k:\overline{n-k}|}-P_{x:\overline{n}|}\ddot{a}_{x+k:\overline{n-k}|}$$

But from earlier calculations of premiums and associated premium conversion relations, we have

$$P_{x:\overline{n}|}=\frac{A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}\quad\text{and}\quad A_{y:\overline{m}|}=1-d\ddot{a}_{y:\overline{m}|},$$

so that the value of the endowment assurance contract at time k is

$$\begin{aligned}_kV_{x:\overline{n}|}&=A_{x+k:\overline{n-k}|}-A_{x:\overline{n}|}\frac{\ddot{a}_{x+k:\overline{n-k}|}}{\ddot{a}_{x:\overline{n}|}}\\&=1-d\ddot{a}_{x+k:\overline{n-k}|}-(1-d\ddot{a}_{x:\overline{n}|})\frac{\ddot{a}_{x+k:\overline{n-k}|}}{\ddot{a}_{x:\overline{n}|}}=1-\frac{\ddot{a}_{x+k:\overline{n-k}|}}{\ddot{a}_{x:\overline{n}|}},\end{aligned}$$

where we recall that this quantity refers to a surviving life, while the prospective value for a non-surviving life is zero, since the contract will have ended.

As an aside, for a life aged x at time 0 that did not survive to time k, there are no future premium or benefit payments, so the prospective value of such an (expired) policy at such time k is zero. The insurer may have made a loss on this individual policy, such loss is paid for by parts of premiums under other policy contracts (in the same portfolio).

Example 69 (Whole-life policies)**.** Similarly, for whole-life policies with payment at the end of the year of death, for $k=1,2,\ldots,$

$$_kV_x=A_{x+k}-P_x\ddot{a}_{x+k}=A_{x+k}-\frac{A_x}{\ddot{a}_x}\ddot{a}_{x+k}=(1-d\ddot{a}_{x+k})-(1-d\ddot{a}_x)\frac{\ddot{a}_{x+k}}{\ddot{a}_x}=1-\frac{\ddot{a}_{x+k}}{\ddot{a}_x},$$

or with payment at death for any real $t\geq 0$.

$$_t\overline{V}_x=\overline{A}_{x+t}-\overline{P}_x\overline{a}_{x+t}=\overline{A}_{x+t}-\frac{\overline{A}_x}{\overline{a}_x}\overline{a}_{x+t}=(1-\delta\overline{a}_{x+t})-(1-\delta\overline{a}_x)\frac{\overline{a}_{x+t}}{\overline{a}_x}=1-\frac{\overline{a}_{x+t}}{\overline{a}_x}.$$

While for whole-life and endowment policies the prospective policy values are increasing (because there will be a benefit payment at death, and death is more likely to happen soon, as the policyholder ages), the behaviour is quite different for temporary assurances (because it is also getting more and more likely that no benefit payment is made):

Example 70 (Term assurance policy)**.** Consider a 40-year policy issued to a life aged 25 subject to A1967/70 mortality. For a sum assured of £100,000 and $i=4\%$, the net premium of this policy can best be worked out by a computer: $£100,000P_{25:\overline{40}|}=£310.53$. The prospective policy values $_kV^1_{25:\overline{40}|}=A^1_{25+k,\overline{40-k}|}-P^1_{25:\overline{40}|}\ddot{a}_{25+k:\overline{40-k}|}$ per unit sum assured give policy values as in Figure 7.1, plotted against age, rising up to age 53, then falling.

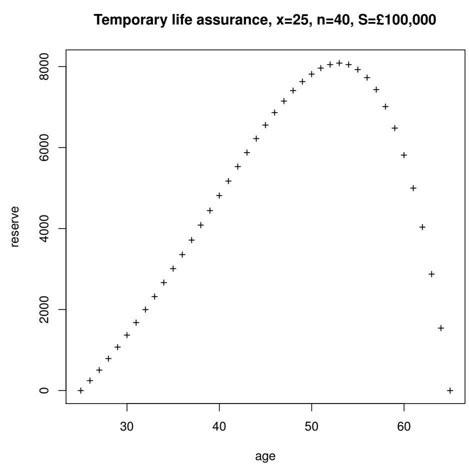


Figure 7.1: Prospective policy values for a temporary assurance.

Lecture 8

Reserves and risk

Reading: Gerber Sections 6.1, 6.3, 6.11

In many long-term life insurance contracts the cost of benefits is increasing over the term but premiums are level (or single). Therefore, the insurer needs to set aside part of early premium payments to fund a shortfall in later years of contracts. In this lecture we calculate such reserves.

8.1 Reserves and random policy values

Recall

$$\begin{aligned}\text{Prospective policy value} &= \text{expected time-}t\text{ value of future benefits}\\ &\quad - \text{expected time-}t\text{ value of future premiums.}\end{aligned}$$

If this prospective policy value is positive, the life office needs a *reserve* for that policy, i.e. an amount of funds held by the life office at time t in respect of that policy. Apart possibly from an initial reserve that the insurer provides for solvency reasons, such a reserve typically consists of parts of earlier premium payments.

If the reserve exactly matches the prospective policy value and if experience is exactly as expected in the policy basis then reserve plus future premiums will exactly meet future liabilities. Note, however, that the mortality assumptions in the policy basis usually build a stochastic model that for a single policy will produce some spread around expected values. If life offices hold reserves for portfolios of policies, where premiums for each policy are set to match expected values, randomness will mean that some policies will generate surplus that is needed to pay for the shortfall of other policies.

When calculating present values of insurance policies and annuity contracts in the first place, it was convenient to work with expectations of present values of random cashflows depending on a lifetime random variable T_x or $K_x = [T_x]$. We can formalise prospective policy values in terms of an underlying stochastic lifetime model, and a constant-i (or constant-δ) interest model. A life insurance contract issued to a life aged x gives rise to a random cash-flow $C = C^B - C^P$ of benefit inflows C^B and premium outflows $-C^P$. The associated prospective policy value is

$$\mathbb{E}(L_t|T_x > t), \qquad \text{where } L_t = \mathrm{Val}_t(C^B_{|(t,\infty)} - C^P_{|[t,\infty)}),$$

where the subtlety of restricting to times (t,∞) and $[t,\infty)$, respectively, arises naturally (and was implicit in calculations for Examples 68 and 69), because premiums are paid in advance and benefits in arrears in the discrete model, so for $t = k \in \mathbb{N}$, a premium payment at time k is in advance (e.g. for the year $(k, k+1]$), while a benefit payment at time k is in arrears (e.g. for death in $[k-1,k)$). Note, in particular, that if death occurs during $[k-1,k)$, then $L_k = 0$ and $L_{k-1} = v - P_x$, where $v = (1+i)^{-1}$.

Proposition 71 (Recursive calculation of policy values)**.** For a whole-life assurance, we have

$$({}_kV_x + P_x)(1+i) = q_{x+k} + p_{x+k}\ {}_{k+1}V_x.$$

By general reasoning, the value of the policy at time k plus the annual premium for year k payable in advance, all accumulated to time $k+1$ will give the death benefit of 1 for death in year $k+1$, i.e., given survival to time k, a payment with expected value q_{x+k} and, for survival, the policy value ${}_{k+1}V_x$ at time $k+1$, a value with expectation $p_{x+k}\ {}_{k+1}V_x$.

Proof: The most explicit actuarial proof exploits the relationships between both assurance and annuity values for consecutive ages (which are obtained by partitioning according to one-year death and survival)

$$A_{x+k} = vq_{x+k} + vp_{x+k}A_{x+k+1} \quad \text{and} \quad \ddot{a}_{x+k} = 1 + vp_{x+k}\ddot{a}_{x+k+1}.$$

Now we obtain

$$\begin{aligned} {}_kV_x + P_x &= A_{x+k} - P_x\ddot{a}_{x+k} + P_x \\ &= vq_{x+k} + vp_{x+k}A_{x+k+1} - (1 + vp_{x+k}\ddot{a}_{x+k+1})P_x + P_x \\ &= v(q_{x+k} + p_{x+k}(A_{x+k+1} - P_x\ddot{a}_{x+k+1}) \\ &= v(q_{x+k} + p_{x+k}\ {}_{k+1}V_x). \end{aligned}$$

□

An alternative proof can be obtained by exploiting the premium conversion relationship, which reduces the recursive formula to $\ddot{a}_{x+k} = 1 + vp_{x+k}\ddot{a}_{x+k+1}$. However, such a proof does not use insight into the cash-flows underlying the insurance policy.

A probabilistic proof can be obtained using the underlying stochastic model: by definition, ${}_kV_x = \mathbb{E}(L_k|T_x > k)$, but we can split the cash-flow underlying L_k as

$$C^B_{|(k,\infty)} - C^P_{|[k,\infty)} = ((0, -P_x), (1, 1_{\{k \le T_k < k+1\}}), C^B_{|(k+1,\infty)} - C^P_{|[k+1,\infty)}).$$

Taking Val_k and $\mathbb{E}(\,\cdot\,|T_x > k)$ then using $\mathbb{E}(1_{\{k \le T_x < k+1\}}|T_x > k) = \mathbb{P}(T_x < k+1|T_k > k) = q_{x+k}$, we get

$${}_kV_x = -P_x + vq_{x+k} + v\mathbb{E}(L_{k+1}|T_x > k) = -P_x + vq_{x+k} + vp_{x+k}\ {}_{k+1}V_x,$$

where the last equality uses $\mathbb{E}(X|A) = \mathbb{P}(B|A)\mathbb{E}(X|A \cap B) + \mathbb{P}(B^c|A)\mathbb{E}(X|A \cap B^c)$, the partition theorem, where here $X = L_{k+1} = 0$ on $B^c = \{T_x < k+1\}$.

There are similar recursive formulas for the other types of life insurance contracts.

8.2 Risk pooling

There are many contradictory and ambiguous definitions of risk; for our purposes, when we talk of 'a risk', we mean an uncertain, variable, yet quantifiable, financial outcome.

The insurance industry takes on certain risks against premium payment, e.g. as fire insurance or temporary life assurance. Mathematically, we can capture many of these risks in stochastic models as events with small probability and large cost.

Consider a random variable X representing an expenditure without insurance. In order for a policyholder to eliminate this risk, they could transfer this to an insurance company in return for a deterministic premium payment P.

Why would an insurer want to take on the risk? First of all, we expect $P > \mathbb{E}(X)$. The policyholder is willing to pay more than the net premium to eliminate the risk, and this gives the insurer a positive expected return (ignoring costs). However, writing the policy might expose the insurer to an unacceptably high level of risk. This can be overcome by taking on many risks. Let us now formalise this pooling effect.

An insurer takes on many risks $X_1, \dots, X_n$, say. In the simplest case, these can be assumed to be independent and identically distributed. This is a realistic assumption for a portfolio of identical life assurance policies issued to a homogeneous population. This is not a reasonable assumption for a flood insurance, because if one property is flooded due to extreme weather then it is likely other properties will be similarly flooded.

The random total claim amount $S = X_1 + \cdots + X_n$ must be met by premium payments $A = P_1 + \cdots + P_n$, say. We set premiums so that the probability of a loss for the insurer does not exceed $\varepsilon > 0$.

Proposition 72. Given a random variable X_1 with mean μ and variance σ^2, representing the benefits from an insurance policy, we have

$$\mathbb{P}\left(X_1 \geq \mu + \frac{\sigma}{\sqrt{\varepsilon}}\right) \leq \varepsilon,$$

and $A_1(\varepsilon) = \mu + \sigma/\sqrt{\varepsilon}$ is the premium to be charged to achieve a loss probability below ε.

Given independent and identically distributed $X_1, \dots, X_n$ from n independent policies, we obtain

$$\mathbb{P}\left(\sum_{j=1}^{n} X_j \geq n\left(\mu + \frac{\sigma}{\sqrt{n\varepsilon}}\right)\right) \leq \varepsilon,$$

i.e. $A_n(\varepsilon) = \mu + \sigma/\sqrt{n\varepsilon}$ suffices if the risk of n policies is pooled.

Proof. The statements follow as consequences of Tchebychev's inequality:

$$\begin{aligned}
\mathbb{P}\left(\sum_{j=1}^{n} X_j \geq n\left(\mu + \frac{\sigma}{\sqrt{n\varepsilon}}\right)\right) &\leq \mathbb{P}\left(\left|\frac{1}{n}\sum_{j=1}^{n} X_j - \mu\right| \geq \frac{\sigma}{\sqrt{n\varepsilon}}\right) \\
&\leq \frac{\mathrm{Var}\left(\frac{1}{n}\sum_{j=1}^{n} X_j\right)}{\left(\frac{\sigma}{\sqrt{n\varepsilon}}\right)^2} = \frac{\sigma^2}{n\frac{\sigma^2}{n\varepsilon}} = \varepsilon.
\end{aligned}$$

□

The estimates used in this proposition are rather weak, and the premiums suggested require some modifications in practice, but adding a multiple of the standard deviation is one important method. This is because the variance, and hence the standard deviation, can often be easily calculated. For large n, so-called safety loadings $A_n(\varepsilon)-\mu$ proportional to $n^{-1/2}$ are of the right order, e.g. for normally distributed risks, or in general by the Central Limit Theorem for large n, when

$$\mathbb{P}\left(\frac{X_1 + \cdots + X_n - n\mu}{\sqrt{n\sigma^2}} \geq c\right) \approx \mathbb{P}(Z > c), \qquad \text{with } Z \text{ standard normally distributed.}$$

The important observation in these results is that the premiums $A_n(\varepsilon)$ decrease with n. This means, that the more policies an insurer can sell the smaller gets the (relative) risk, allowing him to reduce the premium. The proposition indicates this for identical policies, but in fact, this is a general rule about risks with sufficient independence and no unduly large risks.

Lecture 9

Theories of Value

Reading: Eeckhoudt-Gollier-Schlesinger section 1.1

So far, the course has dealt with situations where financial decisions, such as a choice between investment projects or setting the price of an insurance policy, have been based on the expected present value of the contract. In the remainder of the course we consider the limitations of expected present value as a theory of value, and introduce a new concept, Expected Utility Theory, which will provide a framework for evaluating investors' financial decisions in the face of uncertainty, such as choosing an investment vehicle or purchasing insurance.

9.1 The St Petersburg Paradox

In 1738 Daniel Bernoulli wrote a paper in St Petersburg to show how the value placed on a risky contract, or lottery, could depend upon *subjective* factors, not just an objective mathematical expectation. We would now call this subjectivity the 'attitude to risk' or 'risk appetite' of the parties to the contract

Consider a game in which a fair coin is tossed repeatedly until a head appears. If a head appears on the first toss, the payout is one ducat. The payout doubles with each toss, so if the head first appears on the kth toss, the payout is 2^{k-1}. The expected present value is therefore:

$$\sum_{k=1}^{\infty} \mathbb{P}(K = k)2^{k-1} = \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k 2^{k-1} = \infty$$

We are interested in creating a model of (rational) decision-making. How much should someone be prepared to pay to take part in such a game? Expected Present Value theory appears to imply that a rational person would be prepared to stake any (finite) wealth on the game - can that be right?

9.1.1 Sempronius and his ships

Bernoulli gave another example in the same paper:

> Sempronius owns goods at home worth a total of 4000 ducats and in addition possesses 8000 ducats worth of commodities in foreign countries from where they can only be transported by sea. However, our daily experience teaches us that of ten ships one perishes.

Let us describe Sempronius's wealth as a lottery $\tilde{x}$, which takes on the value of 4,000 ducats with probability $\frac{1}{10}$ (if his ship sinks), or 12,000 ducats with probability $\frac{9}{10}$. Now let us instead assume he splits his overseas wealth into two equal portions, and transports them on two ships. Assuming their probabilities of sinking are independent, we now have a new lottery:

$$\tilde{y} : \left(4{,}000, \tfrac{1}{100}; 8{,}000, \tfrac{18}{100}; 12{,}000, \tfrac{81}{100}\right)$$

Now we can see that

$$\begin{aligned} \mathbb{E}[\tilde{x}] &= \tfrac{1}{10}4{,}000 + \tfrac{9}{10}12{,}000 = 11{,}200 \\ \mathbb{E}[\tilde{y}] &= \tfrac{1}{100}4{,}000 + \tfrac{18}{100}8{,}000 + \tfrac{81}{100}12{,}000 = 11{,}200 \\ \text{hence} \quad \mathbb{E}[\tilde{x}] &= \mathbb{E}[\tilde{y}] \end{aligned}$$

We can see that, although the expected present values of both lotteries are equal, our intuition is that $\tilde{y}$ is preferable, because it 'spreads the risk'. Just as in the St Petersburg paradox, we can see that we need a more sophisticated theory of value to aid decisions under uncertainty than expected present value.

9.2 Expected Utility Theory

The solution suggested by Bernoulli and developed by modern risk theory is that what matters to an individual is not their wealth, but rather the 'utility' they derive from the wealth. We create a 'utility function' u of wealth level x to define this relationship, written as $u(x)$. The theory is that a (rational) investor's choices can be explained, or predicted, on the basis that she chooses to optimise not expected present value, but expected utility. Let us assume that $u(x) = \sqrt{x}$ to show how this could resolve our two previous problems. First, the St Petersburg paradox:

$$\begin{aligned} \mathbb{E}[u(x)] &= \sum_{k=1}^{\infty} \mathbb{P}(K = k)\sqrt{2^{k-1}} \\ &= \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k 2^{(k-1)/2} \\ &= \frac{1}{\sqrt{2}} \sum_{k=1}^{\infty} \left(\frac{1}{\sqrt{2}}\right)^k \\ &= 1 + \frac{\sqrt{2}}{2} \end{aligned}$$

Now let's assume we have a (degenerate) lottery with only one outcome, certain payout e. The expected utility of this lottery is $u(e) = \sqrt{e}$. So our investor with this utility function should be indifferent between the game above and this wealth e iff $\mathbb{E}[u(x)] = u(e)$ i.e. iff $1 + \frac{\sqrt{2}}{2} = \sqrt{e}$ or $e = 1.5 + \sqrt{2} \approx 2.91$. This means that the individual would be willing to pay up to 2.91 ducats to play the game - at higher prices, she just isn't willing to risk her stake, even though the reward is theoretically unlimited.

Now let's tackle Sempronius and his ships:

$$
\begin{aligned}
&\mathbb{E}[u(\tilde{x})] = \tfrac{1}{10}\sqrt{4{,}000} + \tfrac{9}{10}\sqrt{12{,}000} = 104.91 \\
&\mathbb{E}[u(\tilde{y})] = \tfrac{1}{100}\sqrt{4{,}000} + \tfrac{18}{100}\sqrt{8{,}000} + \tfrac{81}{100}\sqrt{12{,}000} = 105.46 \\
\text{so}\quad &\mathbb{E}[u(\tilde{x})] < \mathbb{E}[u(\tilde{y})]
\end{aligned}
$$

Hence we would conclude that $\tilde{y}$ is preferred to $\tilde{x}$ because it has a higher expected utility; this supports our intuition that we would prefer to spread our risk by splitting the wealth across two ships.

Some economists might interpret 'utility' as an objective measure of happiness or wellbeing - but this interpretation is not necessary. We do not need to believe in some objective 'utility measure'; the utility function can be defined purely in terms of the preferences of an individual. We will see this in the next section.

9.3 Expected Utility Theorem

We will now introduce the notation and formalise the axioms that will allow us to state the Expected Utility Theorem.

- $X = \{x_s\}_{s=1,\ldots,N}$ is a set of outcomes - typically these outcomes will be monetary outcomes, but could be multidimensional (e.g. an additional variable could represent health). X is often assumed to be finite to make proofs simpler, although the theorem holds more generally.

- $p_s \geq 0$ is the objective probability of occurrence of x_s where $\sum_{s=1}^{N} p_s = 1$

- A (simple) lottery L is described by a vector $(x_1, p_1; x_2, p_2; \ldots; x_N, p_N)$

- A compound lottery is a lottery whose outcomes are simple lotteries. E.g. consider a lottery L' which yields lottery L^a with probability α and lottery L^b with probability $(1-\alpha)$. We write:

$$
\begin{aligned}
L' &= \alpha L^a + (1-\alpha)L^b \\
&= \big(x_1,\, \alpha p_1^a + (1-\alpha)p_1^b;\, \ldots;\, x_N,\, \alpha p_N^a + (1-\alpha)p_N^b\big)
\end{aligned}
$$

  The simple lottery is its *reduced form*.

Consequentialism. We assume that only the lottery over final outcomes matters for decision making. How the lottery is 'framed' doesn't matter, only the probabilities of the outcomes.

Under consequentialism the agent has preference relation $\succcurlyeq$ on the set of simple lotteries $\mathcal{L}$ on outcomes X.

$$\mathcal{L} = \{(x_1, p_1; x_2, p_2; \ldots; x_N, p_N) \in \mathbb{R}_+^N | p_1 + \cdots + p_N = 1|\}$$

$\succ$ is the asymmetric part of the relation 'strictly preferred to', and $\sim$ is the symmetric part 'indifferent between'.

Completeness. The relation $\succcurlyeq$ is *complete* if for any pair $L^a, L^b \in \mathcal{L}$, either $L^a \succcurlyeq L^b$ or $L^b \succcurlyeq L^a$.

Transitivity. The relation $\succcurlyeq$ is *transitive* if for any $L^a, L^b, L^c \in \mathcal{L}$, such that $L^a \succcurlyeq L^b$ and $L^b \succcurlyeq L^c$, then $L^a \succcurlyeq L^c$.

Axiom 1 (Rationality). The relation $\succcurlyeq$ is complete and transitive.

Axiom 2 (Continuity). The preference relation $\succcurlyeq$ on the space of lotteries $\mathcal{L}$ is such that for all $L^a, L^b, L^c \in \mathcal{L}$ such that $L^a \succcurlyeq L^b \succcurlyeq L^c$ there exists a scalar $\alpha \in [0, 1]$ such that $L^b \sim \alpha L^a + (1 - \alpha)L^c$.

Axiom 3 (Independence). The preference relation $\succcurlyeq$ on the space of lotteries $\mathcal{L}$ is such that for all $L^a, L^b, L^c \in \mathcal{L}$ and for all $\alpha \in [0, 1]$,

$$L^a \succcurlyeq L^b \iff \alpha L^a + (1 - \alpha)L^c \succcurlyeq \alpha L^b + (1 - \alpha)L^c$$

We are now ready to state the Expected Utility Theorem.

Theorem (Expected Utility). Axioms 1 - 3 hold iff the preference relation $\succcurlyeq$ of the space of simple lotteries $\mathcal{L}$ has an expected utility representation. That is, there exists a scalar u_s associated to each outcome $x_s, s = 1, \ldots, N$, in such a manner that for any two lotteries $L^a = (x_1, p_1^a; \ldots; x_N, p_N^a)$ and $L^b = (x_1, p_1^b; \ldots; x_N, p_N^b)$, we have

$$L^a \succcurlyeq L^b \iff \sum_{s=1}^{N} p_s^a u_s \geq \sum_{s=1}^{N} p_s^b u_s$$

In other words a lottery can be evaluated by its expected utility

$$\mathbb{E}[u(x)] = \sum_{s=1}^{N} p(x_s)u(x_s)$$

The proof of the Expected Utility Theorem is beyond the scope of this course. If you wish to attempt it as an exercise, some authors add a fourth axiom (which can be derived from the three above) to simplify the proof. Axiom 4 states that there exist best and worst lotteries, denoted as $\overline{L}$ and $\underline{L}$ respectively, such that $\forall L \in \mathcal{L}$, $\underline{L} \preccurlyeq L \preccurlyeq \overline{L}$.

9.3.1 Interpretation of the Expected Utility Theorem

What the Expected Utility Theorem is saying is that, provided individual's preferences conform to the axioms we have stated, those preference can be represented by a utility function. It is *not* the case that a decision-maker prefers one risky choice over another *because* it yields the higher expected utility. Rather, it yields higher expected utility *because* it is preferred to the other.

We can see now that what is important is the ranking provided by the expected utility function. For example, if $\mathbb{E}[u(\tilde{x})] = 2\mathbb{E}[u(\tilde{y})]$ it is wrong to think that our investor is "twice as satisfied" with $\tilde{x}$ as $\tilde{y}$.

Cardinality of Expected Utility. The utility function is cardinal; an increasing linear transformation of u, $v(\cdot) = au(\cdot) + b, a > 0$, will not change the ranking of lotteries.

Proof. If $\tilde{w}_1 \succcurlyeq \tilde{w}_2$,we have

$$\begin{aligned}
\mathbb{E}[v(\tilde{w}_1)] &= \mathbb{E}[au(\tilde{w}_1) + b\,] \\
&= a\mathbb{E}[u(\tilde{w}_1)] + b \\
&\geq a\mathbb{E}[u(\tilde{w}_2)] + b \\
&= \mathbb{E}[v(\tilde{w}_2)]
\end{aligned}$$

□

Where preferences are representable with a utility function, it will be convenient to suggest a form of the function, and use it to *infer* choices of our decision-maker.

9.4 Subjective Expected Utility (SEU)

Savage (1954) extended the expected utility theorem to situations where the probabilities are not objectively known (i.e. most real-life situations). He showed that:

> Under <a set of axioms> a decision maker behaves *as though* there exists a subjective probability measure P and real-valued utility function u such that the decision maker ranks various distributions of consequences ω by their subjective expected utility $\int u(\omega_s)dF_P(s)$.

Savage's theorem doesn't assume the existence of a probability measure. Instead it derives it from preferences. This provides a justification for expected utility analysis even when probabilities are not objectively known.

Lecture 10

The Shape of the Utility Function

Reading: Eeckhoudt-Gollier-Schlesinger sections 1.2 – 1.7

We saw in the previous lecture how a rational decision-maker's preferences can be represented through a utility function. We note that this function is 'subjective' in the sense that it is specific to each individual. Different investors may have different utility functions. However, there are some basic properties of the function we can expect given rational behaviour. We'll first look at the special case where lottery outcomes are in one dimension, and the outcome space X is measured in pounds for final wealth (i.e. we will assume the individual consumes all their goods at the end of one period).

- A lottery L is represented by a cumulative distribution function $F : \mathbb{R} \to [0, 1]$, where $F(x)$ is the probability of receiving less than or equal to x pounds.

We will often assume the following:

Non-Satiation. More wealth is always preferred to less: an incremental increase in wealth has positive value, hence $u' > 0$.

Decreasing Marginal Utility. The marginal utility of wealth is decreasing with wealth, (we could say that each additional unit of wealth has incrementally less value, the richer we get, the less we value a single additional ducat,) hence $u'' < 0$.

Bernouilli argued that the function u should be concave in wealth. We can see that this is related to decreasing marginal utility:

Concavity. A function f is *concave* iff for all $\lambda \in [0, 1]$ and all pairs (a, b) in the domain of f the following condition holds:

$$\lambda f(a) + (1 - \lambda)f(b) \leq f(\lambda a + (1 - \lambda)b)$$

Proposition 73. If function f is twice differentiable and has domain in $\mathbb{R}$ it is concave iff $f'' \leq 0$.

The proof is beyond the scope of this course.

10.1 Risk Aversion

We saw in a previous lecture how the function $u(w) = \sqrt{w}$ would support our intuition over Sempronius's ships. And we can see it satisfies the conditions above, since $u'(w) > 0$ and $u''(w) < 0$, and is also concave. In the next section, we show how this whole family of functions will do the same. First, we want to define our problem more precisely than the example of the ships:

Risk Aversion. An agent is *risk averse* if she dislikes all zero-mean risks at all wealth levels:

$$\forall w, \forall \tilde{x} \text{ with } \mathbb{E}[\tilde{x}] = 0, \mathbb{E}[u(w + \tilde{x})] \leq u(w)$$

This is equivalent to one written in terms of total wealth $\tilde{z} = w + \tilde{x}$:

$$\mathbb{E}[u(\tilde{z})] \leq u(\mathbb{E}[\tilde{z}])$$

This is telling us that the agent will reject a fair bet. Our working assumption is that risk aversion is a fair representation of human behaviour, at least in the context of the insurance and investment decisions we are most interested in. However, we should remember that we might meet agents who prefer to accept the fair bet; these will be called *risk loving*. Agents who are indifferent between the fair bet and its expectation will be described as *risk neutral*. These possibilities are not exhaustive, because there might also be agents with more complex preferences, who are (say) risk loving for part of the domain and risk averse in other parts.

Proposition 74. An agent with utility function u is risk averse iff u is concave.

Proof. We prove for u twice differentiable (i.e. $u'' \leq 0$) by Proposition 73. A second-order Taylor expansion of $u(w + x)$ around $w + \mathbb{E}[\tilde{x}]$ yields, for any x:

$$u(w + x) = u(w + \mathbb{E}[\tilde{x}]) + (x - \mathbb{E}[\tilde{x}])u'(w + \mathbb{E}[\tilde{x}]) + \frac{1}{2}(x - \mathbb{E}[\tilde{x}])^2 u''(\xi(x))$$

for some $\xi(x)$, $x \leq \xi(x) \leq \mathbb{E}[\tilde{x}]$. Since this equation must hold for all x, we can allow x to take the values of our lottery $\tilde{x}$ and take the expectation:

$$\mathbb{E}[u(w + \tilde{x})] = \mathbb{E}[u(w + \mathbb{E}[\tilde{x}])] + \mathbb{E}[(\tilde{x} - \mathbb{E}[\tilde{x}])]u'(w + \mathbb{E}[\tilde{x}]) + \frac{1}{2}\mathbb{E}\left[(\tilde{x} - \mathbb{E}[\tilde{x}])^2 u''(\xi(x))\right]$$

Now the second term on the right hand side is zero, since $\mathbb{E}[(\tilde{x} - \mathbb{E}[\tilde{x}])] = 0$. The third term is nonpositive since it is the expectation of a square times the nonpostive second derivative (since it is concave). So it follows that:

$$\mathbb{E}[u(w + \tilde{x})] \leq \mathbb{E}[u(w + \mathbb{E}[\tilde{x}])]$$

We can drop the expectation on the right hand side since this is already certain, and replace with final wealth $\tilde{z} = w + \tilde{x}$ to give:

$$\mathbb{E}[u(\tilde{z})] \leq u(\mathbb{E}[\tilde{z}])$$

So the expected utility of the lottery is *less than or equal to* the utility of the expectation, and so the agent chooses the certain amount over the risky lottery. This shows that concavity (of a twice differentiable function) implies risk aversion.

For the opposite direction we prove by contradiction. Suppose that u is not concave. Then there must exist some w and some $\delta > 0$ for which $u''(x)$ is positive in the interval $[w-\delta, w+\delta]$. Choosing $\tilde{x}$ with support entirely in $[w-\delta, w+\delta]$, and applying the same Taylor expansion as above will show that $\mathbb{E}[u(w+\tilde{x})] > u(w+\mathbb{E}[\tilde{x}])$ which would mean the agent accepts the zero-mean lottery, and hence contradicts risk aversion. $\square$

Note that the above proposition is just Jensen's inequality rewritten:

Jensen's Inequality. $\mathbb{E}[f(\tilde{x})] \leq f(\mathbb{E}[\tilde{x}])$ for any real-valued random variable $\tilde{x}$ iff f is concave.

10.2 Mapping the Attitude to Risk

Risk aversion says nothing about lotteries with a non-zero mean. This leaves open the possibility that our agent will be willing to accept a risk for a suitable reward. Some agents will take a lot of inducement to accept even small risks, whilst others might be prepared to gamble if the odds are just slightly in their favour. It makes sense to think of these agents as *more* or *less* risk averse than each other. In this lecture we explore how we might compare the risk aversion of different decision-makers. The following definitions will be very useful:

The (Arrow-Pratt) coefficient of absolute risk aversion. This is dependent on the level of wealth w, and is given by

$$A(w) = -\frac{u''(w)}{u'(w)} = -\frac{\partial \ln u'(w)}{\partial w}$$

The (Arrow-Pratt) coefficient of relative risk aversion. Again, this depends on the current level of wealth w and is given by:

$$R(w) = -w\frac{u''(w)}{u'(w)}$$

Risk Premium. The *risk premium* of a zero-mean risk $\tilde{x}$ for an agent with initial wealth w and utility function u is the $\pi(w, u, \tilde{x})$ that satisfies:

$$\mathbb{E}[u(w+\tilde{x})] = u(w-\pi)$$

Certainty Equivalent. The *certainty equivalent* of a risk $\tilde{x}$ for an agent with initial wealth w and utility function u is the $e(w, u, \tilde{x})$ that satisfies:

$$\mathbb{E}[u(w+\tilde{x})] = u(w+e)$$

Note that π is only defined for *zero-mean* risks. Where $\mathbb{E}[\tilde{x}] = 0$ we will typically refer to the risk premium π, and where $\mathbb{E}[\tilde{x}] \neq 0$ we will refer to the certainty equivalent e.

The risk premium and certainty equivalent are measured in the same units as wealth.

The Arrow-Pratt approximation. If the utility function is differentiable, the risk premium of a pure risk $\tilde{y}$ with variance σ^2, is approximately $\frac{1}{2}\sigma^2 A(w)$ where $A(w) = -\frac{u''(w)}{u'(w)}$ is the coefficient of absolute risk aversion.

Proof. Let $\tilde{y} = k\tilde{x}$ for some pure risk $\tilde{x}$ such that $\mathbb{E}[\tilde{x}] = 0$ and some constant k. Denote the risk premium of $\tilde{y}$, $\pi(w, u, \tilde{y})$, as $g(k)$. Then we take a second order Taylor expansion of $g(k)$ around $k = 0$.

$$\pi(w, u, \tilde{y}) = g(k) = g(0) + kg'(0) + \frac{1}{2}k^2 g''(0) + O(k^3)$$

From the definition of the risk premium we have:

$$\mathbb{E}[u(w + k\tilde{x})] = u\left(w - g(k)\right)$$

We can see that if $k = 0$ then $g(0) = 0$

Differentiating with respect to k gives:

$$\mathbb{E}[\tilde{x}u'(w + k\tilde{x})] = -g'(k)u'(w - g(k))$$

Setting $k = 0$ again this simplifies to

$$\mathbb{E}[\tilde{x}u'(w)] = -g'(0)u'(w)$$

The left hand side is $u'(w)\mathbb{E}[\tilde{x}] = 0$ since $\mathbb{E}[\tilde{x}] = 0$ and hence $g'(0) = 0$.

Differentiating again with respect to k gives:

$$\mathbb{E}[\tilde{x}^2 u''(w + k\tilde{x})] = \left(g'(k)\right)^2 u''(w - g(k)) - g''(k)u'(w - g(k))$$

Again, we set $k = 0$ and use $g(0) = g'(0) = 0$ to get

$$\mathbb{E}[\tilde{x}^2 u''(w)] = -g''(0)u'(w)$$

Using $\mathbb{E}[\tilde{x}^2] = \sigma_x^2$ and rearranging we have:

$$g''(0) = -\sigma_x^2 \frac{u''(w)}{u'(w)}$$

Substituting our values of $g(0)$, $g'(0)$ and $g''(0)$ back into the second order Taylor expansion gives

$$\begin{aligned}\pi(w,u,\tilde{y}) = g(k) &= g(0) + kg'(0) + \frac{1}{2}k^2 g''(0) + O(k^3)\\ &= 0 + k \times 0 - \frac{1}{2}k^2\sigma_x^2 \frac{u''(w)}{u'(w)} + O(k^3)\\ &\simeq \frac{1}{2}\sigma^2 A(w)\end{aligned}$$

since $\sigma^2 = Var(\tilde{y}) = Var(k\tilde{x}) = k^2\sigma_x^2$ and $A(w) = -\frac{u''(w)}{u'(w)}$ □

Proposition 75. If the utility function is differentiable, the risk premium tends to zero as the square of the size of the risk.

Proof. Consider the pure risk $\tilde{y} = k\tilde{x}$, $\mathbb{E}[\tilde{x}] = 0$, and interpret k as the size of the risk. We have shown that for small k the risk premium is given by

$$\pi(w,u,\tilde{y}) \simeq \frac{1}{2}k^2\sigma_x^2 A(w)$$

which tends to zero with k^2. □

10.3 Comparative Risk Aversion

We can see how the risk premium, certainty equivalent or coefficients of risk aversion could be used to compare two different investors. However, we should also see that, in general, each of these measures will depend upon the level of wealth. For the same risk $\tilde{x}$, investor P might have a higher risk premium than investor Q at low levels of wealth, but the situation might be reversed at high levels of wealth. To say that one agent is more risk averse than another, we need a more strict definition.

Comparative risk aversion. Suppose that agents P and Q have the same initial wealth w, which is arbitrary. Agent P is *more risk-averse* than agent Q if any risk that is undesirable for Q is also undesirable for P. This must be true independent of their common initial wealth w.

Proposition 76. Suppose that u_P and u_Q are twice differentiable. The following conditions are equivalent:

1. Agent P is more risk-averse than agent Q; namely P rejects all lotteries that Q rejects.

2. The risk premium of any risk is larger for agent P than agent Q.

3. u_P is a concave transformation of u_Q.

4. A_P is uniformly larger that A_Q, so that $A_P(w) \geq A_Q(w)$, $\forall w \in \mathbb{R}$ where A_i is the coefficient of absolute risk aversion for agent i.

Proof. $1 \Rightarrow 4$ This is equivalent to $\neg 4 \Rightarrow \neg 1$, so let us assume that $A_P(w) < A_Q(w)$ for some w. We then then create a lottery $\tilde{y}$ with mean μ and variance σ^2 such that agent Q is only just prepared to reject it.

$$u_Q(w) = \mathbb{E}[u_Q(w + \tilde{y})]$$

Now we take a Taylor expansion of this

$$u_Q(w) = \mathbb{E}[u_Q(w) + \tilde{y}\, u'_Q(w) + \tfrac{1}{2}\tilde{y}^2\, u''_Q(w) + O(\tilde{y}^3)]$$

And so taking the expectation of the individual terms we get

$$0 = u'_Q(w)\mu + \tfrac{1}{2}u''_Q(w)(\sigma^2 + \mu^2) + \mathbb{E}[O(\tilde{y}^3)]$$

We divide through by $u'_Q(w)$ and rearrange to get

$$\mu \approx \tfrac{1}{2}A_Q(w)(\sigma^2 + \mu^2)$$

Now we want to ask ourselves if P will accept or reject the lottery. So we take the same expansion of $u_P(w + \tilde{y})$

$$u_P(w + \tilde{y}) = u_P(w) + u'_P(w)\mu + \tfrac{1}{2}u''_P(w)(\sigma^2 + \mu^2) + \mathbb{E}[O(\tilde{y}^3)]$$

Now if we ignore the small terms, substitute the lone instance of μ with the previous equation, and rearrange, we get

$$u_P(w + \tilde{y}) = u_P(w) + \tfrac{1}{2}u'_P(w)(\sigma^2 + \mu^2)(A_Q(w) - A_P(w)) \qquad (*)$$

Since we assumed that, at least locally, $A_P(w) < A_Q(w)$, the right hand side of $(*)$ must be greater than $u_P(w)$, and hence P would accept a lottery that Q rejects. Therefore, if P *never* accepts a lottery that Q rejects, he must have a uniformly greater Arrow-Pratt measure of absolute risk aversion.

$3 \Rightarrow 2$ Let $u_P(w) = \phi(u_Q(w))$ Now assume that 3 holds - i.e. that ϕ is a concave function.

$$\begin{aligned}
u_P(w - \pi(w, u_P, \tilde{x})) &= \mathbb{E}[u_P(w + \tilde{x})] \\
&= \mathbb{E}[\phi(u_Q(w + \tilde{x}))] \\
&\leq \phi\left(\mathbb{E}[u_Q(w + \tilde{x})]\right) \quad \text{by Jensen's inequality} \\
&= \phi\left(u_Q(w - \pi(w, u_Q, \tilde{x}))\right) \\
&= u_P(w - \pi(w, u_Q, \tilde{x}))
\end{aligned}$$

This shows that $u_P(w - \pi_P) \leq u_P(w - \pi_Q)$ and hence $\pi_P \geq \pi_Q$ since u is an increasing function.

$4 \Rightarrow 3$ Let $u_P(w) = \phi(u_Q(w))$ once again. Differentiating with respect to w gives:

$$u_P'(w) = u_Q'(w)\phi'(u_Q(w))$$

Differentiating again we get:

$$u_P''(w) = u_Q''(w)\phi'(u_Q(w)) + (u_Q'(w))^2\phi''(u_Q(w))$$

Dividing through by $-u_P'(w)$ gives:

$$A_P(w) = -\frac{u_Q''(w)\phi'(u_Q(w))}{u_P'(w)} - \frac{(u_Q'(w))^2\phi''(u_Q(w))}{u_P'(w)}$$

Substituting in the first derivative gives

$$A_P(w) = A_Q(w) - \frac{(u_Q'(w))^2}{u_P'(w)}\phi''(u_Q(w))$$

The first derivatives of the utility functions are positive. So this shows that if $A_P(w)$ is universally greater than $A_Q(w)$ then the second derivative of the function ϕ must be universally negative - which is equivalent (under the conditions we have chosen) to being a concave function. Note that this formula shows the implication in reverse also - i.e. if ϕ is a concave transformation of u_Q to u_P then the Arrow-Pratt measure of absolute risk aversion must be universally higher for agent P than agent Q.

$2 \Rightarrow 4$ is straightforward - it follows immediately from the Arrow-Pratt approximation. Now all that remains is to show that any one of 2, 3 or 4 implies 1 and our chain will be complete. We choose

$3 \Rightarrow 1$ Let's take an arbitrary wealth level w and lottery $\tilde{x}$ which Q rejects, and assume that 3 holds.

$$u_Q(w) \geq \mathbb{E}[u_Q(w + \tilde{x})]$$

Now since $\phi(u) = u_P(u_Q^{-1}(u))$ and u_P and u_Q are strictly increasing, ϕ is also strictly increasing. So we can take the function on both sides of the equation.

$$\phi(u_Q(w)) \geq \phi(\mathbb{E}[u_Q(w + \tilde{x})])$$

Now we use Jensen's inequality

$$\phi(u_Q(w)) \geq \phi(\mathbb{E}[u_Q(w + \tilde{x})]) \geq \mathbb{E}[\phi(u_Q(w + \tilde{x}))]$$

And then express in terms of u_P

$$u_P(w) \geq \quad \ldots \quad \geq \mathbb{E}[u_P(w + \tilde{x})]$$

Which shows that P will also reject the lottery.

$\square$

10.4 How risk aversion changes with wealth

We have seen how it is plausible to represent the preferences of many decision-makers using a utility function which is strictly increasing and also concave. A third condition we may wish to represent is that, as investors become more wealthy, they are more willing to accept a risk (measured in absolute terms). We formalise this using decreasing absolute risk aversion.

Decreasing absolute risk aversion. Preferences exhibit *decreasing absolute risk aversion (DARA)* if the risk premium associated with any risk is a decreasing function of wealth: $\partial\pi(w,u,\tilde{x})/\partial w \leq 0$ for any w, $\tilde{x}$.

Proposition 77. Suppose that u is three times differentiable. The following conditions are equivalent:

1. The risk premium is a decreasing function of wealth.
2. The absolute risk aversion is decreasing with wealth, meaning that $A(w) = -\frac{u''(w)}{u'(w)}$ is decreasing.

Proof. We prove for a risk averse utility function u. We start with the definition of the risk premium:

$$\mathbb{E}[u(w+\tilde{x})] = u(w-\pi(w,u,\tilde{x}))$$

We differentiate with respect to w to give

$$\mathbb{E}[u'(w+\tilde{x})] = u'(w-\pi(w,u,\tilde{x}))\left(1-\frac{\partial\pi(w,u,\tilde{x})}{\partial w}\right)$$

Rearranging we get

$$\frac{\partial\pi(w,u,\tilde{x})}{\partial w} = 1-\frac{\mathbb{E}[u'(w+\tilde{x})]}{u'(w-\pi(w,u,\tilde{x}))}$$

Since $u' > 0$ this is negative if

$$\mathbb{E}[u'(w+\tilde{x})] \geq u'(w-\pi(w,u,\tilde{x}))$$

We define $v = -u'$, and rewrite the above as

$$\mathbb{E}[v(w+\tilde{x})] \leq v(w-\pi(w,u,\tilde{x}))$$

For this to hold for all w, $\tilde{x}$, it must be that v is more concave than u. We can interpret v as a utility function of another individual. We have shown that for the risk premium to be decreasing in wealth, we must have $v = -u'$ is a concave transformation of u. Using Proposition 76 this will be the case if

$$A_v(w) \geq A_u(w)$$

$$-\frac{\frac{\partial^2(-u'(w))}{\partial w^2}}{\frac{\partial(-u'(w))}{\partial w}} \geq -\frac{u''(w)}{u'(w)}$$

$$-\frac{u'''(w)}{u''(w)} \geq -\frac{u''(w)}{u'(w)}$$

Now differentiating $A_u(w)$ with respect to w gives

$$\begin{aligned} A_u'(w) &= -\frac{u'''(w)}{u'(w)} + \left(\frac{u''(w)}{u'(w)}\right)^2 \\ &= A_u(w)\left(\frac{u'''(w)}{u''(w)} - \frac{u''(w)}{u'(w)}\right) \end{aligned}$$

Since the decision maker is risk averse, $A_u(w) \geq 0$.Therefore the derivative is also negative (i.e $A_u(w)$ is decreasing in wealth) iff $-\frac{u'''(w)}{u''(w)} \geq -\frac{u''(w)}{u'(w)}$. This is the same condition we had for the decreasing risk premium above. □

10.5 Some Classical Utility Functions

We now consider some functional forms for the utility function that are commonly used.

Quadratic utility. $u(w) = aw - \frac{1}{2}w^2, \quad w \leq a$

The main advantage is that the expected utility depends only on the mean and variance of the final wealth distribution, and not any higher moments (sometimes called mean-variance preferences). The main disadvantages are the limited range (since utility decreases above a) and that it satisfies *increasing* absolute risk aversion. $A(w) = \frac{1}{a-w}$ which is increasing in w.

Constant Absolute Risk Aversion (CARA) utility.

$$u(w) = -\frac{e^{-aw}}{a}, \quad a > 0$$

The coefficient of absolute risk aversion $A(w) = a$, $\forall w$. This means it is constant, rather than decreasing, which could be a disadvantage.

Constant Relative Risk Aversion (CRRA) utility.

$$u(w) = \begin{cases} \dfrac{w^{1-\gamma}}{1-\gamma} & \text{for } \gamma > 0, \gamma \neq 1, \\ \\ \ln(w) & \text{for } \gamma = 1 \end{cases}$$

The coefficient of absolute risk aversion is $A(w) = \gamma/w$, which is decreasing in w. The coefficient of relative risk aversion is $R(w) = \gamma$, $\forall w$.

Lecture 11

Changes in Risk

Reading: Eeckhoudt-Gollier-Schlesinger chapter 2

In the previous lectures we saw how risk aversion implies that investors would prefer their certain wealth to that wealth plus any zero-mean risk. We also saw how, given a specific utility function, an individual would choose between two different (zero-mean) risks. In this section, we aim to show how certain characteristics of distributions of two risks would lead to unanimous decisions amongst a group of investors - i.e. the decision would apply to a whole class of utility functions. This is the theory of *stochastic dominance*.

11.1 An Increase in Risk

To begin with, we introduce some terminology. We will be considering random variables $\tilde{x}_1$ and $\tilde{x}_2$ with the same mean, with support in some interval of the real line $[a, b]$.[1]

An *increase in risk* can be defined in (at least) two ways:

Definition (Adding Noise) 1. $\tilde{x}_1$ is obtained from $\tilde{x}_2$ by adding zero-mean noise terms to the possible outcomes of $\tilde{x}_2$.

$$\tilde{x}_1 \sim \tilde{x}_2 + \tilde{\varepsilon} \quad \text{where } \mathbb{E}[\tilde{\varepsilon}|\tilde{x}_2 = x] = 0 \quad \text{for all } x$$ [2]

Definition (Mean Preserving Spread) 2. $\tilde{x}_1$ is a *mean-preserving spread* of $\tilde{x}_2$ if for some interval $I \subset [a, b]$, $\tilde{x}_1$ can be obtained (in distribution) from $\tilde{x}_2$ by taking some probability mass from I and distributing it outside of I. Or, more formally: $\mathbb{E}[\tilde{x}_1] = \mathbb{E}[\tilde{x}_2]$, and there exists an interval I such that $\forall x \in I, f_2(x) \geq f_1(x)$ and $\forall x \notin I, f_2(x) \leq f_1(x)$.

11.2 Second-Order Stochastic Dominance

If $\tilde{x}_1$ is more risky (as formally defined above) than $\tilde{x}_2$, then we want to show that *all* risk averse expected utility maximisers prefer $\tilde{x}_2$ to $\tilde{x}_1$, i.e. that an increase in risk would

[1] We limit our analysis to this interval to avoid problems of convergence of integrals.

[2] '$\sim$' means 'has the same distribution as' in this context.

be rejected by risk averse agents. This is known as *second-order stochastic dominance.*

Definition (Second-Order Stochastic Dominance) 3. $\tilde{x}_2$ second-order stochastically dominates $\tilde{x}_1$ ($\tilde{x}_2$ SSD $\tilde{x}_1$) if all risk averse expected utility maximisers prefer $\tilde{x}_2$ to $\tilde{x}_1$. For all concave functions u, $\mathbb{E}[u(\tilde{x}_1)] \leq \mathbb{E}[u(\tilde{x}_2)]$.

We now begin our proof that an increase in risk as defined by 'adding noise' implies second order stochastic dominance - i.e. $1 \Rightarrow 3$.

Proof.

$$\begin{aligned}
\mathbb{E}[u(\tilde{x}_1)] &= \mathbb{E}[u(\tilde{x}_2 + \tilde{\varepsilon})] \\
&= \mathbb{E}_{\tilde{x}_2}[\mathbb{E}_{\tilde{\varepsilon}}[u(\tilde{x}_2 + \tilde{\varepsilon})|\tilde{x}_2]] \\
&\leq \mathbb{E}_{\tilde{x}_2}[u(\mathbb{E}_{\tilde{\varepsilon}}[(\tilde{x}_2 + \tilde{\varepsilon})|\tilde{x}_2])] \text{ by Jensen's inequality} \\
&= \mathbb{E}_{\tilde{x}_2}[u(\tilde{x}_2 + \mathbb{E}[\tilde{\varepsilon}|\tilde{x}_2])] \\
&= \mathbb{E}[u(\tilde{x}_2)] \quad \text{since } \mathbb{E}[\tilde{\varepsilon}|\tilde{x}_2 = x] = 0 \text{ for all } x
\end{aligned}$$

□

11.3 The Rothschild-Stiglitz Proposition

In the section above, we showed how an increase in risk (as defined by 'adding noise') implies second order stochastic dominance - i.e. the increase would be rejected by all risk-averse decision makers. Using the work of Rothschild and Stiglitz (1970), we can make a much stronger statement.

Rothschild-Stiglitz Propostion. Consider two random variables $\tilde{x}_1$ and $\tilde{x}_2$ with the same mean. The following four statements are equivalent:

1. $\tilde{x}_1$ is obtained from $\tilde{x}_2$ by adding zero-mean noise.

2. $\tilde{x}_1$ is obtained from $\tilde{x}_2$ by a sequence of mean-preserving spreads.

3. $\tilde{x}_2$ second-order stochastically dominates $\tilde{x}_1$.

4. The Rothschild-Stiglitz condition

$$\int_a^\theta F_1(x)\,\mathrm{d}x \geq \int_a^\theta F_2(x)\,\mathrm{d}x \quad \forall\theta \in [a, b]$$

   is satisfied, and with an equality for $\theta = b$.

Proof. We have already proved $1 \Rightarrow 3$. Here we prove $3 \iff 4$ only.

$$\begin{aligned}
\mathbb{E}[u(\tilde{x}_1)] &= \int_a^b u(x) f_1(x)\,\mathrm{d}x \\
&= [u(x)F_1(x)]_{x=a}^{x=b} - \int_a^b u'(x)F_1(x)\,\mathrm{d}x \quad \text{(integrating by parts)} \\
&= u(b) - \int_a^b u'(x)F_1(x)\,\mathrm{d}x \quad \text{since } F_1(b) = 1,\ F_1(a) = 0
\end{aligned}$$

The analogous formula applies for $\tilde{x}_2$ and so

$$\begin{aligned}
\mathbb{E}[u(\tilde{x}_1)] - \mathbb{E}[u(\tilde{x}_2)] &= \int_a^b u'(x)\left(F_2(x) - F_1(x)\right)\,\mathrm{d}x \\
&= -\int_a^b u'(x)S'(x)\,\mathrm{d}x
\end{aligned}$$

where we define $S(x) = \int_a^x (F_1(s) - F_2(s))\,\mathrm{d}s$. We note that $S(a) = S(b) = 0$ and integrate by parts again.

$$\begin{aligned}
\mathbb{E}[u(\tilde{x}_1)] - \mathbb{E}[u(\tilde{x}_2)] &= -\int_a^b u'(x)S'(x)\,\mathrm{d}x \\
&= -\left[u'(x)S(x)\right]_{x=a}^{x=b} + \int_a^b u''(x)S(x)\,\mathrm{d}x \\
&= \int_a^b u''(x)S(x)\,\mathrm{d}x
\end{aligned}$$

This shows that $\tilde{x}_1$ is preferred to $\tilde{x}_2$ for all concave functions if $S(x)$ is negative for all values of x, and conversely $\tilde{x}_2$ is preferred to $\tilde{x}_1$ for all concave functions if $S(x)$ is positive for all values of x.

$$\begin{aligned}
S(x) \geq 0 &\iff \int_a^x F_1(s) - F_2(s)\,\mathrm{d}s \geq 0 \\
&\iff \int_a^x F_1(s)\,\mathrm{d}s - \int_a^x F_2(s)\,\mathrm{d}s \geq 0 \\
&\iff \int_a^x F_1(s)\,\mathrm{d}s \geq \int_a^x F_2(s)\,\mathrm{d}s
\end{aligned}$$

This shows that if the Rothschild-Stiglitz condition holds, then $\forall x\, S(x) \geq 0$, then $\tilde{x}_2$ SSD $\tilde{x}_1$, i.e. $4 \Rightarrow 3$.

To prove $3 \Rightarrow 4$ assume $S(x) < 0$ for some interval $I \subset [a, b]$. Consider a utility function that is linear for all $x \notin I$ and strictly concave for all $x \in I$. Then $\int u''(x)S(x)\,\mathrm{d}x > 0$ and hence $\tilde{x}_1$ is preferred to $\tilde{x}_2$, contradicting the SSD. This completes the proof of $3 \iff 4$. $\square$

In order to prove $2 \Rightarrow 4$ we just need to show that a mean-preserving spread implies the condition that $\forall x\, S(x) \geq 0$. Let $\tilde{x}_1$ be a mean-preserving spread of $\tilde{x}_2$. Let the interval I be bounded by α and β so $a \leq \alpha \leq \beta \leq b$. Using the definition of the mean-preserving spread and the density function we have:

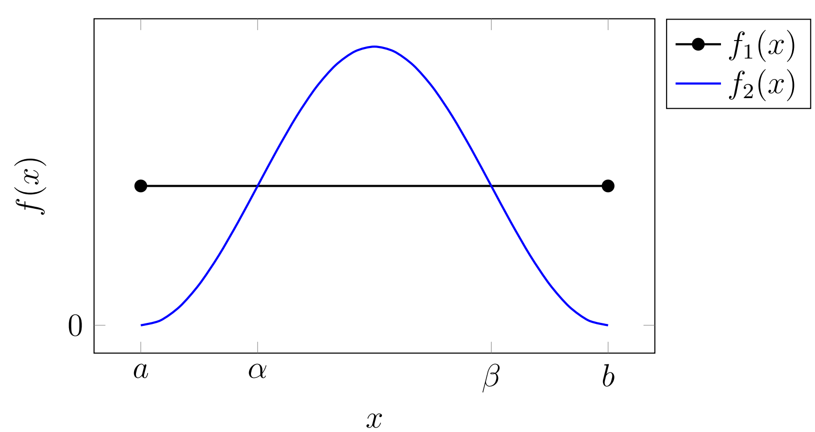
$$S''(x) = f_1(x) - f_2(x) \begin{cases} \geq 0 \text{ for } a < x < \alpha \\ \leq 0 \text{ for } \alpha < x < \beta \\ \geq 0 \text{ for } \beta < x < b \end{cases}$$

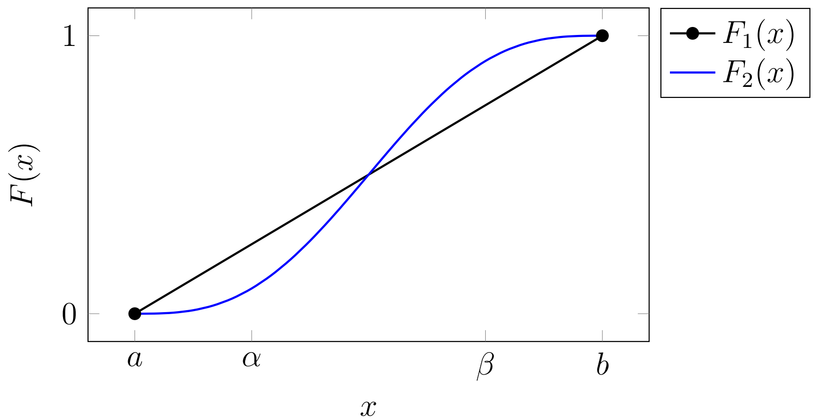
This can only hold (in combination with $S(a) = S(b) = S'(a) = S'(b) = 0$) if $\forall x\, S(x) \geq 0$. This is best explained through a diagram of the density functions and cumulative distribution function.

First, note that $F_1(a) = F_2(a) = 0$ and $F_1(b) = F_2(b) = 1$, simply by the definitions of a and b and the cumulative distribution function. Furthermore, $\int_a^b F_1(x)\mathrm{d}x = \int_a^b F_2(x)\mathrm{d}x$. We can see this by integrating by parts:

$$\begin{aligned}
\int_a^b F_1(x)\mathrm{d}x &= [xF_1(x)]_a^b - \int_a^b x f_1(x)\mathrm{d}x \\
&= (bF_1(b) - aF_1(a)) - \mathbb{E}[\tilde{x}_1] \\
&= (bF_2(b) - aF_2(a)) - \mathbb{E}[\tilde{x}_2] \\
&= [xF_2(x)]_a^b - \int_a^b x f_2(x)\mathrm{d}x \\
&= \int_a^b F_2(x)\mathrm{d}x
\end{aligned}$$

Because of the mean preserving spread, the probability density of $\tilde{x}_1$ is initially greater, then less, then greater again, than that of $\tilde{x}_2$. This means that the slope of the graph of F_1 is first steeper, then shallower, then steeper than F_2. This ensures that there can only be a single crossing-point of the two graphs, since they both start at 0 and finish at 1, and so $\int F_1\, \mathrm{d}x$ must remain ahead of $\int F_2\, \mathrm{d}x$, until they equal at b.





$F_1(x)$
$F_2(x)$
$F(x)$
1
0
a
α
β
b
x

Lecture 12

Risk and Reward

Reading: Eeckhoudt-Gollier-Schlesinger chapter 2

In this lecture we continue to explore the implications of the Rothschild-Stiglitz proposition, in the behaviour of the variance of a distribution in relation to second-order stochastic dominance. We then go on to discuss more general changes in risk which do not preserve the mean.

12.0.1 An increase in risk implies an increase in variance

We saw in the last lecture how the preference of an investor between two risks could depend on higher moments than the mean and variance. Therefore an increase in variance on its own *does not* imply an increase in risk. However, we can show that an increase in risk *does* imply an increase in variance.

Proposition 78. For lotteries $\tilde{x}$ and $\tilde{y}$ such that $\mathbb{E}[\tilde{x}] = \mathbb{E}[\tilde{y}]$, $\tilde{y}$ SSD $\tilde{x} \Rightarrow Var[\tilde{y}] \leq Var[\tilde{x}]$

Proof. From the above proof of the Rothschild-Stiglitz proposition, for any twice differentiable concave u, we have

$$\mathbb{E}[u(\tilde{x})] - \mathbb{E}[u(\tilde{y})] = \int_a^b u''(t)S(t)\,\mathrm{d}t$$

where $S(t) = \int_a^t F_x(s) - F_y(s)\,\mathrm{d}s$

Now if $\tilde{y}$ SSD $\tilde{x}$ then this applies for all concave u, so will apply for $u(x) = bx - \frac{1}{2}x^2$. Substituting this gives

$$\mathbb{E}[bx - \tfrac{1}{2}x^2] - \mathbb{E}[by - \tfrac{1}{2}y^2] = -\tfrac{1}{2}\mathbb{E}[\tilde{x}^2] + \tfrac{1}{2}\mathbb{E}[\tilde{y}^2] = -\int_a^b S(t)\,\mathrm{d}t$$

We rearrange this to give

$$\sigma_x^2 - \sigma_y^2 = 2\int_a^b S(t)\,\mathrm{d}t \geq 0$$

Since $\forall t\, S(t) \geq 0$ (because $\tilde{y}$ SSD $\tilde{x}$) and $\mathbb{E}[\tilde{y}] = \mathbb{E}[\tilde{x}]$. □

12.1 First-Order Stochastic Dominance

In the previous section we relied upon the risk aversion of our agents. If we relax this assumption, and consider all utility maximisers with non-decreasing utility functions, then we obtain more demanding conditions to be sure one lottery is preferred to another. This is known as *first-order stochastic dominance.*

Definition (First-Order Stochastic Dominance). $\tilde{x}_2$ first-order stochastically dominates $\tilde{x}_1$ ($\tilde{x}_2$ FSD $\tilde{x}_1$) if all utility maximisers (with increasing utility functions) prefer $\tilde{x}_2$ to $\tilde{x}_1$. For all non-decreasing functions u, $\mathbb{E}[u(\tilde{x}_1)] \leq \mathbb{E}[u(\tilde{x}_2)]$.

Proposition 79. The following conditions are equivalent.

1. $\tilde{x}_2$ FSD $\tilde{x}_1 :\ \mathbb{E}[u(\tilde{x}_1)] \leq \mathbb{E}[u(\tilde{x}_2)]$ for all non-decreasing functions u
2. $\tilde{x}_1$ is obtained from $\tilde{x}_2$ by a transfer of probability mass from the high wealth states to lower wealth states: $\forall x\, F_1(x) \geq F_2(x)$
3. $\tilde{x}_2$ is obtained from $\tilde{x}_1$ by adding non-negative noise terms to the possible outcomes of $\tilde{x}_1 :\ \tilde{x}_2 \sim \tilde{x}_1 + \tilde{\varepsilon}$, where $\tilde{\varepsilon} \geq 0$ with probability one.

Proof. We prove $1 \iff 2$ only. As for the proof of the Rothschild-Stiglitz proposition, integrating by parts once obtained:

$$\mathbb{E}[u(\tilde{x}_2)] - \mathbb{E}[u(\tilde{x}_1)] = \int_a^b u'(x)\,(F_1(x) - F_2(x))\,\mathrm{d}x$$

For $2 \Rightarrow 1$ we can see that if $\forall x\ F_1(x) \geq F_2(x)$ and $u'(x) \geq 0$ then the expression must be non-negative, and hence $\tilde{x}_2$ is always preferred to $\tilde{x}_1$.

For $1 \Rightarrow 2$ suppose that $F_1(x) < F_2(x)$ for some interval $I \subset [a, b]$. Then consider the utility function that is flat everywhere except in I. The right-hand side must be strictly negative (since $u'(x) = 0$ outside the interval, and so, for this function, $\tilde{x}_1$ is preferred to $\tilde{x}_2$. This means $\neg 2 \Rightarrow \neg 1$ and hence $1 \Rightarrow 2$. □

12.1.1 Risk and Reward

In the previous chapter we introduced the concept of second-order stochastic dominance - a change in risk which preserved the mean. In this section we have introduced first-order stochastic dominance, which can be thought of as an increase in the mean with no effective increase in risk.

In general, we expect some sort of trade-off between risk and reward: an increase in risk (say of an investment) should be associated with a higher expected return. Similarly, when trying to reduce risk, such as by taking out insurance, we usually expect to pay a loading to the premium, which will reduce our expected wealth. These decisions will depend on the form of the investor's utility function.

Lecture 13

Insurance Decisions

Reading: Eeckhoudt-Gollier-Schlesinger chapter 3

13.1 The Justification of Insurance

In this section we explore how Expected Utility Theory can be used to evaluate insurance decisions, and hence the justification for insurance. First, we introduce some notation.

Definition (Indemnity Insurance). An *indemnity insurance contract* is a pair $(P, I(\cdot))$ where the premium $P \geq 0$ is paid to the insurer, and the claim payment is $I(x) \geq 0$ when the loss is x.

Definition (Loading). For a random non-negative loss $\tilde{x}$ and an indemnity insurance contract which pays $I(x)$ in each state x, and costs premium of P, the *loading* λ is defined such that $P = (1+\lambda)\mathbb{E}[I(\tilde{x})]$. The product is termed 'actuarially fair' (or just 'fair') if $\lambda = 0$.

We expect most products to be 'unfair' in the sense that $\lambda > 0$ because an insurer has to meet expenses and payments to providers of capital as well as covering the expected claims.

We will consider the following environment.

- Our decision-maker is strictly non-satiated and risk averse, with twice differentiable utility function u, so that $u'(x) > 0, u''(x) < 0$.
- She has initial wealth of w_0 subject to a risky loss of $\tilde{x}$, so final net wealth is $w_0 - \tilde{x}$. We assume that $\tilde{x}$ is non-degenerate (i.e. it really is risky) and is nonnegative.
- Indemnity insurance can be purchased which pays $I(x) \geq 0$ when loss is x, and is priced with a loading of $\lambda \geq 0$.

Example 80. Sempronius only has one available ship to transport his wealth home. However, he is offered insurance against any loss up to 5,000 ducats with a loading of 0.2. Assuming he has utility function $u(w) = \sqrt{w}$, would he buy the insurance?

- We can describe this as $w_0 = 12\,000$, facing a risky loss of $\tilde{x} : \left(8\,000, \frac{1}{10}; 0, \frac{9}{10}\right)$
- $I(x) = \min(x, 5\,000)$ and $\lambda = 0.2$
- So $P = (1+\lambda)\mathbb{E}[I(x)] = 1.2 \times \left(\frac{1}{10}\min(8\,000, 5\,000) + \frac{9}{10}\min(0, 5\,000)\right) = 600$
- Expected utility assuming no insurance is 104.91 (see section 9.2)
- Expected utility assuming purchasing insurance is

$$\begin{aligned}
\mathbb{E}\left[w_0 - P - \tilde{x} + I(\tilde{x})\right] &= \frac{1}{10}\sqrt{(12\,000 - 600 - 8\,000 + 5\,000)} + \frac{9}{10}\sqrt{(12\,000 - 600)} \\
&= \frac{1}{10}\sqrt{8\,400} + \frac{9}{10}\sqrt{11\,400} \\
&= 105.26
\end{aligned}$$

This shows that Sempronius has higher expected utility with the insurance and so would be willing to buy it.

13.2 Optimal Coinsurance (Mossin's Theorem)

Suppose that the decision-maker must choose an optimal level of *coinsurance* $\beta \in [0, 1]$:

- $I_\beta(x) = \beta x$ when the loss is x
- Premium $P(\beta) = (1+\lambda)\mathbb{E}[\beta\tilde{x}] = \beta P_0$ where $P_0 = (1+\lambda)\mathbb{E}[\tilde{x}]$
- Denote realised wealth $\tilde{y} = w_0 - \beta P_0 - \tilde{x} + \beta\tilde{x}$

Mossin's Theorem. Full insurance ($\beta^* = 1$) is optimal at an actuarially fair price ($\lambda = 0$), while partial coverage ($\beta^* < 1$) is optimal at an actuarially unfair price ($\lambda > 0$).

Proof. Let $H(\beta) = \mathbb{E}[u(\tilde{y})]$. Our decision-maker will seek to maximise expected utility, and therefore our optimisation problem is

$$\max_\beta [H(\beta)] \; s.t. \; \beta \in [0, 1]$$

$$\begin{aligned}
H(\beta) &= \mathbb{E}[u(\tilde{y})] = \mathbb{E}[u(w_0 - \beta P_0 - \tilde{x} + \beta\tilde{x})] \\
H'(\beta) &= \mathbb{E}[(\tilde{x} - P_0)u'(\tilde{y})] \\
H''(\beta) &= \mathbb{E}[(\tilde{x} - P_0)^2\, u''(\tilde{y})]
\end{aligned}$$

We can see that $H''(\beta) < 0$ and therefore there will be a unique maximum β^*. Note that it might be that $H'(\beta^*) = 0$ or it might be a corner solution so that $\beta^* = 0$ or $\beta^* = 1$.

First consider

$$\begin{aligned}
H'(1) &= \mathbb{E}[(\tilde{x} - P_0)u'(w_0 - P_0)] \\
&= u'(w_0 - P_0)(\mathbb{E}[\tilde{x}] - P_0) \\
&= u'(w_0 - P_0)(\mathbb{E}[\tilde{x}] - (1+\lambda)\mathbb{E}[\tilde{x}]) \\
&= -\lambda u'(w_0 - P_0)\mathbb{E}[\tilde{x}]
\end{aligned}$$

This shows that if $\lambda = 0$ then $H'(1) = 0$, and so we have our unique solution at $\beta^* = 1$. This should not surprise us: a risk averse decision-maker prefers to swap uncertain wealth for certain wealth with the same expected value.

We can also see that if λ were permitted to be *less* than zero (e.g. perhaps some insurance is subsidised or marketed as a 'loss leader') then $H'(1) > 0$. i.e. it is still increasing at $\beta = 1$, the permitted maximum, which would be our solution.

Next consider

$$\begin{aligned}
H'(0) &= \mathbb{E}[(\tilde{x} - P_0)u'(w_0 - \tilde{x})] \\
&= \mathbb{E}[(\tilde{x} - \mathbb{E}[\tilde{x}])u'(w_0 - \tilde{x})] - \mathbb{E}[(P_0 - \mathbb{E}[\tilde{x}])u'(w_0 - \tilde{x})] \\
&= cov(\tilde{x}, u'(w_0 - \tilde{x})) - \lambda\mathbb{E}[\tilde{x}]\mathbb{E}[u'(w_0 - \tilde{x})]
\end{aligned}$$

Now if $H'(0) \leq 0$ then that implies the corner solution $\beta^* = 0$ is optimal (higher values of β will reduce expected utility, since $H(\beta)$ is single peaked). This will occur when

$$\lambda \geq \lambda^* = \frac{\mathrm{cov}(\tilde{x}, u'(w_0 - \tilde{x}))}{\mathbb{E}[\tilde{x}]\mathbb{E}[u'(w_0 - \tilde{x})]}$$

Again, we should not be surprised that such a limit exists. As the insurance becomes more expensive, eventually our risk averse agent will decline to take insurance at all, because of the risk-reward trade-off.

For $0 < \lambda < \lambda^*$, $H'(1) < 0 < H'(0)$. This shows there must be a solution to $H'(\beta) = 0$ between zero and one, so partial insurance is optimal: $0 < \beta^* < 1$. □

It helps interpreting these results if we remember that risk aversion is a second order effect. As long as $\lambda > 0$ then the agent is trading-off risk against reward. And since, for very small risks, our risk averse agent tends towards risk neutral, she will always be willing to accept a little extra risk when reward is offered.

13.2.1 Impact of increasing risk aversion on the level of co-insurance

In this section we consider how we can compare the optimal level of coinsurance between two agents.

Proposition 81. Consider two utility functions u and v that are increasing and concave, and suppose that u is more risk averse than v (in the sense of Arrow-Pratt). Then, the optimal coinsurance rate β^* is higher for u than v: $\beta^*_u \geq \beta^*_v$

Proof. If $\lambda = 0$ then trivial: $\beta_u^* = \beta_v^* = 1$

If $\lambda > 0$ then suppose that $u'(w_0 - P_0) = v'(w_0 - P_0)$. (We can do this without loss of generality since expected utility is cardinal; we can take a linear transformation of one of the functions without affecting how it represents preferences.)

Because u is more concave than v it must be that:

$$u'(y) \geq v'(y) \quad \forall y < w_0 - P_0$$

$$u'(y) \leq v'(y) \quad \forall y > w_0 - P_0$$

And so

$$(x - P_0)v'(w_0 - (1 - \beta_u^*)x - \beta_u^* P_0) \leq (x - P_0)u'(w_0 - (1 - \beta_u^*)x - \beta_u^* P_0) \quad \forall x$$

Since for $x > P_0$ both sides are positive and the right hand side is larger, and for $x < P_0$ both sides are negative and the right hand is is less negative. Now we can take the expectation on each side, which gives the evaluation of the $H'(\beta)$ function at $\beta = \beta_u^*$

$$\begin{gathered} \mathbb{E}[(x - P_0)v'(w_0 - (1 - \beta_u^*)x - \beta_u^* P_0)] \leq \mathbb{E}[(x - P_0)u'(w_0 - (1 - \beta_u^*)x - \beta_u^* P_0)] \\ H_v'(\beta_u^*) \leq H_u'(\beta_u^*) = 0 \end{gathered}$$

We know that the final equality holds because β_u^* is the unique maximum for H_u. Furthermore, we know that H_v is concave, and so if sloping downwards at β_u^* it has already passed its own unique maximum, so it follows that $\beta_v^* \leq \beta_u^*$. □

13.2.2 Impact of increasing wealth on the level of coinsurance

In this section we consider how a change in the initial wealth will affect the level of optimal coinsurance.

Proposition 82. If u exhibits decreasing absolute risk aversion then an increase in initial wealth will decrease the optimal rate of coinsurance β^*.

Proof. Let β^* be optimal for $w = w_0$. Now consider

$$\begin{aligned} \frac{\partial H'(\beta)}{\partial w} &= \frac{\partial^2 \mathbb{E}[u(\tilde{y})]}{\partial \beta \, \partial w} \\ &= \frac{\partial \mathbb{E}[u'(\tilde{y})]}{\partial \beta} \text{ (reversing the order of differentiation)} \\ \Rightarrow -\frac{\partial H'(\beta)}{\partial w} &= \frac{\partial \mathbb{E}[-u'(\tilde{y})]}{\partial \beta} \quad (*) \end{aligned}$$

We saw in the proof of Proposition 77 that we can treat $v = -u'$ as a utility function, and where u is DARA, v is more concave than u. By Proposition 81 we know that the optimal level of coinsurance for $v = -u'$ is *higher* than the level for u, since here v is more risk averse in the sense of Arrow-Pratt. So the gradient of $H_v(\beta)$ evaluated at β^* must be greater than zero, which corresponds to the right hand side of $(*)$. So $\frac{\partial H'(\beta)}{\partial w}$ must be negative when evaluated at $\beta = \beta^*$. This means that an increase in w will take $H'(\beta)$ below zero, and so the new unique maximum must lie to the left, i.e. a lower level of coinsurance. □

13.3 Optimality of Deductibles

In this section we are interested in the ideal form of the insurance contract. The following proposition is due to Arrow (1971).

Optimality of Deductibles (Arrow's Theorem). Suppose a risk averse policy-holder selects an insurance contract $(P, I(\cdot))$ with $P = (1+\lambda)\mathbb{E}[I(\tilde{x})]$ and with $I(x)$ non-decreasing and $I(x) \geq 0$ for all x. Then the optimal contract contains a straight deductible D; that is $I^*(x) = \max(0, x - D)$ is optimal for some D.

Proof. We prove for a discrete loss distribution where p_x denotes $\mathbb{P}[\tilde{x} = x]$ The realised net wealth under the indemnity function $I^*(x)$ is

$$w^*(x) = w_0 - P - \min(x, D)$$

This has a minimum of $w_0 - P - D$ where $x \geq D$.
Consider an alternative insurance policy $(P, I(\cdot))$ with the same premium P. Since the loading is the same we must have $\mathbb{E}[I(x)] = \mathbb{E}[I^*(x)]$ and the net wealth is

$$w(x) = w_0 - P - x + I(x)$$

Now we consider the relationship between the two. We consider some x_i such that $I(x_i) = I^*(x_i) + \varepsilon_i$ where $\varepsilon_i > 0$. Now because $\mathbb{E}[I(x)] = \mathbb{E}[I^*(x)]$ then there must be some other loss level(s) x_j such that $I(x_j) = I^*(x_j) - \varepsilon_j$ for $\varepsilon_j > 0$ and $\sum p_i\varepsilon_i = \sum p_j\varepsilon_j$. Since each $I(x_j) \geq 0$ it must be that each $I^*(x_j) > 0$. This means that the optimal policy is paying out in these states, so these must be states where the claim exceeds the deductible, or equivalently, $x_j > D$.

This leads to the following changes in final wealth:

$$\begin{aligned} \text{at } x_i : w(x_i) &= w^*(x_i) + \varepsilon_i = w_0 - P - \min(x_i, D) + \varepsilon_i \\ \text{at each } x_j : w(x_j) &= w^*(x_j) - \varepsilon_j = w_0 - P - D - \varepsilon_j \end{aligned}$$

This

- *increases* net wealth in states with $w^*(x) \geq w_0 - P - D$ and

- *decreases* net wealth in states with $w^*(x) = w_0 - P - D$

Therefore $w(x)$ can be obtained from $w^*(x)$ by a mean preserving spread around net wealth of $w_0 - P - D$. Therefore $w^*(x)$ SSD $w(x)$. □

Remember that the theorem depends on the insurance loading being constant. It means that, given this budget constraint, the optimal insurance contract concentrates indemnification on the worst outcomes, where the marginal utility of additional wealth is highest ($u'(w)$ is highest when w is lowest if u is concave.) The implication is that we expect investors to insure their 'large risks', e.g. their most expensive assets like a house or car, and to ignore the small risks.

Lecture 14

Static Portfolio Choices

Reading: Eeckhoudt-Gollier-Schlesinger chapter 4

So far we have been thinking about insurance decisions: in this lecture we will see how an investor choosing a portfolio of assets faces equivalent problems with equivalent solutions. We will consider the following environment:

- The decision-maker is strictly non-satiated and risk averse with twice differentiable utility function u ($u'(x) > 0, u''(x) < 0$).
- Initial wealth of w_0 which can be invested in one risky asset (amount α and uncertain return $\tilde{x}$) and one risk-free asset (amount $w_0 - \alpha$ with certain return r) .
- The value of the realised portfolio is

$$\begin{aligned}(w_0 - \alpha)(1 + r) + \alpha(1 + \tilde{x}) &= w_0(1 + r) + \alpha(\tilde{x} - r) \\ &= w + \alpha\tilde{y}\end{aligned}$$

  Where $w = w_0(1 + r)$ is future wealth under the risk-free strategy and $\tilde{y} = \tilde{x} - r$, the excess return on the risky asset.

The problem of the investor is to choose α in order to maximise expected utility.

$$\alpha^* \in \arg\max_{\alpha} \mathbb{E}[u(w + \alpha\tilde{y})]$$

The problem is formally equivalent to the optimal level of coinsurance. To see this, we define:

$$w \equiv w' - P_0 \qquad \alpha \equiv (1 - \beta)P_0 \qquad \tilde{y} \equiv \frac{P_0 - \tilde{x}}{P_0}$$

Consequently

$$\begin{aligned}\mathbb{E}[u(w + \alpha\tilde{y})] &= \mathbb{E}\left[u\left((w' - P_0) + (1 - \beta)P_0\frac{P_0 - \tilde{x}}{P_0}\right)\right] \\ &= \mathbb{E}[u(w' - \beta P_0 - (1 - \beta)\tilde{x})]\end{aligned}$$

We can interpret $\alpha = 0$ as full insurance (zero risk), and so by increasing α (i.e. decreasing the coinsurance level β) the investor accepts some risk in exchange for a higher expected final wealth.

Note that here we have not yet imposed a constraint on the level of α or the mean of $\tilde{y}$. In general, we expect $\mathbb{E}[\tilde{y}] > 0$ as being equivalent to $\lambda > 0$ in the coinsurance model, and if we restrict α/w_0 to the range $[0,1]$ this is equivalent to the same restriction for β in the coinsurance model.

Example 83. Assume we have an investor with utility function $u(w) = \sqrt{w}$, with initial wealth of $1,000$, and access to:

- a risk-free asset with an annual return of 5%
- a risky asset with a random return $\tilde{x} : \left(-5\%, \frac{1}{2}; 15.5\%, \frac{1}{2}\right)$

Assuming our investor wants to maximise her utility at the end of one year, how much would she invest in each asset at the beginning of the year? (We do not permit any changes to the initial allocation mid-year.) We want to maximise expected utility with respect to α:

$$\begin{aligned}
\mathbb{E}\left[u(\tilde{w})\right] &= \mathbb{E}\left[(1.05(1\,000-\alpha)+(1+\tilde{x})\alpha)^{\frac{1}{2}}\right] \\
&= \mathbb{E}\left[(1\,050+(\tilde{x}-0.05)\alpha)^{\frac{1}{2}}\right] \\
&= \frac{1}{2}\left(1\,050-0.10\alpha\right)^{\frac{1}{2}} + \frac{1}{2}\left(1\,050+0.105\alpha\right)^{\frac{1}{2}} \\
\frac{\partial \mathbb{E}\left[u(\tilde{w})\right]}{\partial \alpha} &= -0.1\frac{1}{4}\left(1\,050-0.10\alpha\right)^{-\frac{1}{2}} + 0.105\frac{1}{4}\left(1\,050+0.105\alpha\right)^{-\frac{1}{2}} \\
\Rightarrow \left(1\,050-0.10\alpha\right)^{-\frac{1}{2}} &= 1.05\left(1\,050+0.105\alpha\right)^{-\frac{1}{2}} \quad \text{(setting line above to zero)} \\
\Rightarrow 1\,050+0.105\alpha &= \pm 1.05^2\left(1\,050-0.10\alpha\right) \\
\Rightarrow \alpha &= 500
\end{aligned}$$

Our investor will chose to invest 500 in the risky asset and the remaining 500 in the risk-free asset.[1]

Proposition 84. Consider the static portfolio choice problem, where $\tilde{y}$ is the return of the asset over the risk-free rate, and α^* is the optimal dollar investment in the risky asset. Then the optimal investment in the risky asset is positive iff the expected excess return is positive: $\alpha^* = 0$ if $\mathbb{E}[\tilde{y}] = 0$ and $\alpha^* > 0$ if $\mathbb{E}[\tilde{y}] > 0$. Moreover, when the expected excess return is positive,

- α^* is reduced when the risk aversion of the investor is increased in the sense of Arrow and Pratt;
- α^* is increasing in wealth if absolute risk aversion is decreasing.

Proof. This follows immediately from the corresponding proofs for the optimal level of coinsurance. □

[1] We can ignore the alternative solution because we were restricted to the positive domain of our utility function.

14.1 Static Portfolio choices under CRRA

We now consider an example of the static portfolio choice.

Proposition 85. Under constant relative risk aversion, the demand for the risky asset is proportional to wealth: $\alpha^*(w) = kw$.

Proof. Suppose that $u'(x) = x^{-\gamma}$ where γ is the coefficient of relative risk aversion. Under this specification, the first order condition may be written as

$$\begin{aligned} 0 &= \mathbb{E}[\tilde{y}\, u'(w + \alpha^* \tilde{y})] \\ &= \mathbb{E}[\tilde{y}\, (w + \alpha^* \tilde{y})^{-\gamma}] \\ \Rightarrow 0 &= \mathbb{E}\left[\tilde{y}\, (1 + \frac{\alpha^*}{w} \tilde{y})^{-\gamma}\right] \end{aligned}$$

So as w changes $\frac{\alpha^*}{w}$ must be constant, and so $\alpha^* = kw$ for some k. □

14.2 Static Portfolio Choice with more than one risky asset

We can also demonstrate that a static portfolio choice with more than one risky asset is equivalent to the diversification between risks we saw in the earlier part of the course - e.g. Sempronius and his two ships. We will look at the most simple case, where our investor has a choice between two risky assets (with returns $\tilde{x}_1$ and $\tilde{x}_2$) which we assume are independent and identically distributed. If she invests α in $\tilde{x}_1$ then her final wealth will be $\alpha(1 + \tilde{x}_1) + (w_0 - \alpha)(1 + \tilde{x}_2)$ and so we want to solve the following

$$\max_{\alpha} \mathbb{E}[u(w_0 + \alpha \tilde{x}_1 + (w_0 - \alpha)\tilde{x}_2)]$$

We expect the result as follows:

Proposition 86. A risk averse investor faced with an opportunity set of two independent and identically distributed risky assets will optimally invest half her wealth in each asset.

Proof. We will prove that $\alpha^* = \frac{1}{2} w_0$ SSD any other chosen value for α.

Observe that $w_0 + \alpha \tilde{x}_1 + (w_0 - \alpha)\tilde{x}_2$ can be rewritten as

$$w_0 \left(1 + \frac{\tilde{x}_1 + \tilde{x}_2}{2}\right) + \tilde{\varepsilon}$$

where

$$\tilde{\varepsilon} \equiv (\alpha - \tfrac{1}{2} w_0)(\tilde{x}_1 - \tilde{x}_2)$$

Now we need to check that the conditional mean of $\tilde{\varepsilon}$ is zero.

$$
\begin{aligned}
&\mathbb{E}\left[\tilde{\varepsilon}\,\middle|\, w_0\left(1+\frac{\tilde{x}_1+\tilde{x}_2}{2}\right)\right] \\
&= \mathbb{E}\left[(\alpha-\tfrac{1}{2}w_0)(\tilde{x}_1-\tilde{x}_2)\,\middle|\, 1+\frac{\tilde{x}_1+\tilde{x}_2}{2}\right] \\
&= (\alpha-\tfrac{1}{2}w_0)\mathbb{E}[\,\tilde{x}_1-\tilde{x}_2 \mid \tilde{x}_1+\tilde{x}_2\,] \\
&= (\alpha-\tfrac{1}{2}w_0)\left(\mathbb{E}[\,\tilde{x}_1 \mid \tilde{x}_1+\tilde{x}_2\,]-\mathbb{E}[\,\tilde{x}_2 \mid \tilde{x}_1+\tilde{x}_2\,]\right) \\
&= 0 \quad \text{since } \tilde{x}_1 \text{ and } \tilde{x}_2 \text{ are identically distributed.}
\end{aligned}
$$

So we have shown that any other distribution can be obtained by adding zero-mean noise, and therefore $\alpha^* = \frac{1}{2}w_0$ SSD any alternative split between the two assets. □

Lecture 15

Consumption and Saving

Reading: Eeckhoudt-Gollier-Schlesinger chapter 6

Until now we have assumed that our decision-maker lives for one period and derives utility from consuming 'wealth'. In this chapter we extend the choice over multiple time periods.

15.1 Consumption over a lifetime

Assume that a decision-maker lives for n periods and consumes c_t in each period $t = 1, ..., n$. Furthermore, assume that the decision-maker derives a lifetime utility $U(c_1, ..., c_n)$ from the consumption schedule $(c_1, ..., c_n)$.

We assume that the utility function is a linear function of individual *felicity functions*, of the following form

$$U(c) = \sum_{t=1}^{n} \beta_t u(c_t)$$

Where

- $u(c_t)$ is the felicity of consuming c_t at time t
- $\beta_t > 0$ is the discount factor applied to felicity occurring at time t

This form of the model means that preferences over any consumption pair (c_{t_1}, c_{t_2}) are independent of all other c_t, $t \neq t_1$, t_2, and that the shape of the felicity function remains constant over time.

Interpretation of the felicity function Our usual assumption will be the felicity function is concave, that is $u' > 0$, $u'' < 0$. Note that this will then ensure that U is concave, as a linear combination of concave functions.

The interpretation is that this gives us not just risk aversion, but a preference for smooth consumption over time. How 'smooth' will be determined by β_t.

15.2 Lifetime consumption under certainty

To explore the consumption smoothing we first look at the position in the absence of risk. Suppose that our decision-maker is endowed with a certain flow of income.

Let y_t be a certain flow of income at time t, and let i be the risk-free rate of interest. Note that we assume both borrowing and lending is available at i.

Let x_t be the accumulated saving just after time t (just after, so that it records the position after the consumption of c_t at t).

We assume that our decision-maker starts with savings of $z_0 \geq 0$, and we include the condition that $z_n \geq 0$. This prevents our decision-maker being in debt at the end of her lifetime; we assume lenders won't agree to lending if they won't be repaid.

This gives us a *dynamic budget constraint* of the form

$$z_t = (1+i)z_{t-1} + y_t - c_t, \qquad t = 1, ..., n$$

We can now rewrite this

$$\begin{aligned}
z_t &= (1+i)z_{t-1} + y_t - c_t & t = 1, ..., n \\
\Rightarrow z_n &= (1+i)^{n-t+1} z_{t-1} + \sum_{s=t}^{n} (1+i)^{n-s}(y_s - c_s), & t = 1, ..., n \\
\Rightarrow z_n &= (1+i)^n z_0 + \sum_{s=1}^{n} (1+i)^{n-s}(y_s - c_s) & \\
\Rightarrow v^n z_n &= z_0 + \sum_{s=1}^{n} v^s (y_s - c_s) \geq 0 & \text{since } z_n \geq 0 \\
\Rightarrow \sum_{t=1}^{n} v^t c_t &\leq z_0 + \sum_{t=1}^{n} v^t y_t = w_0 &
\end{aligned}$$

This states that the present value of future consumption must not be more than the present value of future income and initial wealth. So w_0 can be thought of as the present value of lifetime wealth, where $\sum_{t=1}^{n} v^t y_t$ is called 'human capital', since it may be modelled as an asset.

15.2.1 The maximisation problem under certainty

Our decision-maker's maximisation problem is therefore

$$\max_{c_1,...,c_n} \sum_{t=1}^{n} \beta_t u(c_t) \quad \textit{subject to} \quad \sum_{t=1}^{n} v^t c_t = w_0$$

We have replaced '$\leq$' with '$=$' in the budget constraint since $u' > 0$ and so optimally the decision-maker will always consume the maximum amount left in the final period.

We may write the Lagrangian as

$$L\left(c_1, ..., c_n, \lambda\right) = \sum_{t=1}^{n} \beta_t u(c_t) + \lambda \left(w_0 - \sum_{t=1}^{n} v^t c_t \right)$$

The objective function is a sum of concave functions, and the condition is linear, so this is enough for the following first order conditions to be necessary and sufficient for optimality:

$$\frac{\partial L}{\partial c_t} : \quad \beta_t u'(c_t) - \lambda v^t = 0 \quad \text{for all } t = 1, ..., n$$

$$\frac{\partial L}{\partial \lambda} : \quad \sum_{t=1}^{n} v^t c_t = w_0$$

This shows us that the shape of optimal consumption over a lifetime depends on the income only through λ. The timing of income doesn't affect optimal consumption except through the net present value. The split of total wealth w_0 between savings z_0 and human capital doesn't affect the optimal consumption strategy.

Now we can rearrange the first order condition to get the ratio of successive marginal utilities:

$$\frac{u'(c_t)}{u'(c_{t+1})} = (1+i)\frac{\beta_{t+1}}{\beta_t}$$

We can also use the properties $u' > 0$ and $u'' < 0$ to establish a relationship between successive consumption states. For example:

$$c_{t+1} = c_t \quad \Longleftrightarrow \quad \frac{\beta_t}{\beta_{t+1}} = 1 + i$$

or

$$c_{t+1} > c_t \quad \Longleftrightarrow \quad \frac{\beta_t}{\beta_{t+1}} < 1 + i$$

What this says is that the 'smoothness' of consumption over a lifetime depends on the relationship of the subjective discount rate compared to the rate of interest.

Example: exponential discounting Let's assume that the felicity function has a CRRA form, that is

$$u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma}$$

And let

$$\beta_t = \frac{1}{(1+\delta)^t}$$

Then our first order conditions from $\frac{\partial L}{\partial c_t}$ become

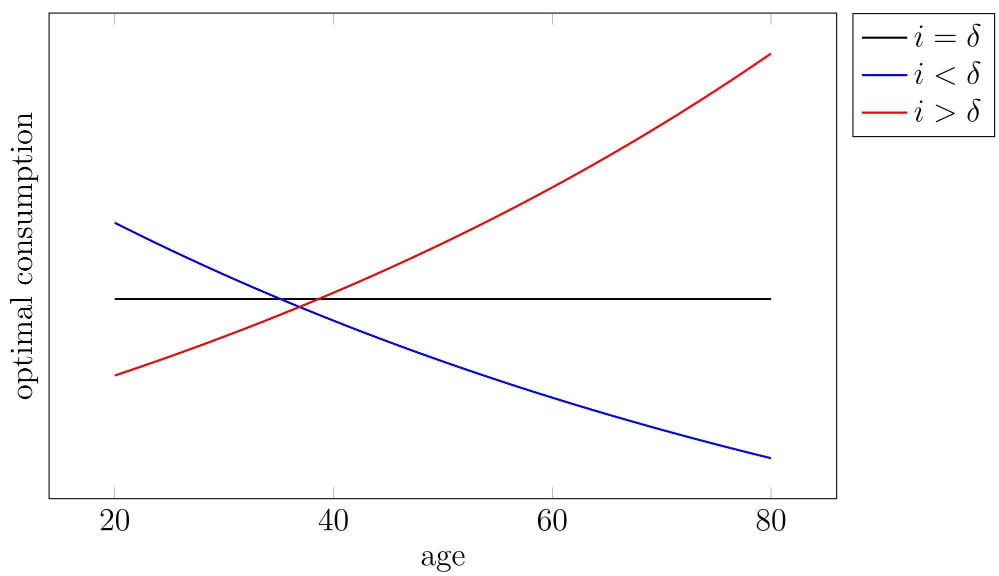
$$c_t^{-\gamma} = \lambda \left(\frac{1+\delta}{1+i}\right)^t$$

This can be rewritten as

$$c_t = c_0 a^t \quad \text{where } a = \left(\frac{1+i}{1+\delta}\right)^{1/\gamma}$$

Here c_0 is a constant chosen to satisfy the lifetime budget constraint. As above, we can see that

$$c_{t+1} > c_t \iff i > \delta$$



15.2.2 The Fischer Separation Theorem

We have assumed above that $y = (y_1, ..., y_n)$ is fixed. Now support that instead the decision-maker can choose between different income profiles - e.g. an individual choosing between different careers, or a firm choosing between competing projects.

Proposition 87. If financial markets are *frictionless*, then the decision-maker should optimally choose the income profile that maximises the NPV of income.

Proof: Since the decision-maker can use financial markets to rearrange the income schedule to achieve a desired consumption schedule, and this consumption schedule depends on y only through $\sum_{t=1}^{n} v^t y_t$. □

Here 'frictionless' means that there are no transaction costs and no 'wedge' between the cost of borrowing and the return on lending (both are at the risk-free rate).

15.3 Extending the model to uncertainty

There are three main ways where we can introduce risk into the model:

- Saving may be possible through risky assets.
- Labour income may be risky.
- Future lifetime itself may be risky

We can also introduce other elements of choice in relation to the income pattern (for example, over jobs, or retirement date). We can also introduce *market frictions*, for example, by charging a higher rate of interest for borrowing than for risk-free saving.

In the next sub-section we show an example of a simple two-period consumption problem with risk.

15.3.1 Precautionary saving under risk

We will introduce risk with an uncertain income $\tilde{y}_1$ in the second period, and certain income y_0 in the first period. This risk does not depend on the consumer's choice, so the only choice is how much to save, which we denote as s (the remainder of y_0 is consumed in period 0. This saving is invested, at the risk-free rate r, and consumed in period 1. Consumers select how much to save at date 0 in order to maximise their expected lifetime utility:

$$\max_s V(s) = u_0(y_0 - s) + \mathbb{E}\left[u_1\left((1+r)s + \tilde{y}_1\right)\right]$$

If we denote the optimal saving by s^*, then for the first order condition is written as

$$u_0'(y_0 - s^*) = (1+r)\mathbb{E}\left[u_1'\left((1+r)s^* + \tilde{y}_1\right)\ \right]$$

We want to compare this with optimal saving in the absence of risk, when the income in the second period is equal to $\mathbb{E}[\tilde{y}_1]$ with certainty. In this case the maximisation problem would be:

$$\max_s \hat{V}(s) = u_0(y_0 - s) + u_1\left((1+r)s + \mathbb{E}\left[\tilde{y}_1\right]\right)$$

We denote the solution as $\hat{s}$.

There will be precautionary saving if $s^* > \hat{s}$.

Now $\hat{V}(s)$ has a negative second derivative. So if the first derivative of $\hat{V}(s)$ evaluated at s^* is positive, then then $\hat{s}$ must be further to the right, that is greater, than s^*. Conversely, if it is negative, then $\hat{s}$ must be further to the left, that is less than, than s^*.

$$\begin{aligned}
\hat{V}'(s) &= -u_0'(y_0 - s) + (1+r)u_1'\left((1+r)s + \mathbb{E}\left[\tilde{y_1}\right]\right) \\
\Rightarrow \hat{V}'(s^*) &= -u_0'(y_0 - s^*) + (1+r)u_1'\left((1+r)s^* + \mathbb{E}\left[\tilde{y_1}\right]\right) \\
&= -(1+r)\mathbb{E}\left[u_1'\left((1+r)s^* + \tilde{y_1}\right)\right] + (1+r)u_1'\left((1+r)s^* + \mathbb{E}\left[\tilde{y_1}\right]\right) \\
&= (1+r)\left(u_1'\left((1+r)s^* + \mathbb{E}\left[\tilde{y_1}\right]\right) - \mathbb{E}\left[u_1'\left((1+r)s^* + \tilde{y_1}\right)\right]\right)
\end{aligned}$$

$$\text{So } \hat{V}'(s^*) \leq 0 \iff u_1'\left((1+r)s^* + \mathbb{E}\left[\tilde{y_1}\right]\right) \leq \mathbb{E}\left[u_1'\left((1+r)s^* + \tilde{y_1}\right)\right]$$

Now this holds, by Jensen's inequality, if and only if u_1' is a convex function. That is, if u_1''' is positive. This is referred to as *prudence*. Risk aversion on its own is not sufficient to cause precautionary saving for all possible distributions of the future risk.

15.3.2 Risk Aversion and Prudence

Proposition 88. A decision-maker who exhibits decreasing absolute risk aversion is prudent.

Proof: If the decision maker exhibits DARA then $A'(w) \leq 0 \quad \forall w$.

$$\begin{aligned}
A(w) &= \frac{-u''(w)}{u'(w)} \\
A'(w) &= \frac{-u'''(w)}{u'(w)} - \frac{-u''(w)u''(w)}{(u'(w))^2} \\
&= \frac{-u''(w)}{u'(w)}\left(\frac{-u'''(w)}{-u''(w)} - \frac{u''(w)}{u'(w)}\right) \\
&= A(w)\left(A(w) - \frac{-u'''(w)}{u''(w)}\right) \\
&= A(w)\left(A(w) - P(w)\right) \qquad \text{where } P(w) = \frac{-u'''(w)}{u''(w)}
\end{aligned}$$

Now since under risk aversion $A(w)$ is positive, $u'(w)$ is positive, and $u'(w)$ is negative, then:

$$A'(w) \leq 0 \iff P(w) \geq A(w) \Rightarrow u'''(w) > 0$$

$\square$

$P(w)$ is known as the *measure of absolute prudence*. If we believe that DARA is intuitive then this supports the idea that precautionary saving is too. Note that the implication does not hold in reverse. It is possible to be prudent without exhibiting decreasing absolute risk aversion. In fact, it is even possible to be prudent and risk-loving!

15.4 Time Consistency

In the model above with only two consumption dates, decisions are made at $t = 0$ which determine present and future consumption; at the second date, all remaining wealth is consumed, and there is no scope for changing one's mind. We now consider a model with three (or more) consumption dates.

Let us consider the 3-period problem, with no uncertainty. At $t = 0$ the agent will choose the optimum consumption schedule (c_0, c_1, c_2) so as to maximise expected (discounted) lifetime utility, $\sum_{t=0}^{2} \beta_t u(c_t)$ subject to a lifetime budget constraint, $\sum_{t=0}^{2} v^t c_t = w_0$.

Then, as above, we have first order conditions which can be rearranged to include:

$$\frac{u'(c_1)}{u'(c_2)} = (1+i)\frac{\beta_2}{\beta_1}$$

Now consider the agent's position at $t = 1$ (just before consumption). She has the chance to revisit the choice between consumption and saving, and will act to maximise expected discounted utility again. She has remaining wealth $w_1 = w_0 - (1+r)c_0$. Using the same discounted utility function, but now discounting back to $t = 1$ not $t = 0$, this changes our problem to:

$$\max_{c_1,c_2} \sum_{t=1}^{2} \beta_{t-1} u(c_t) \quad \textit{subject to} \quad \sum_{s=1}^{2} v^{t-1} c_t = w_1$$

This has the first order condition

$$\frac{u'(c_1)}{u'(c_2)} = (1+i)\frac{\beta_1}{\beta_0}$$

This shows that the agent will be *time consistent* if and only if $\frac{\beta_1}{\beta_0} = \frac{\beta_2}{\beta_1}$, or more generally iff $\frac{\beta_{t+1}}{\beta_t} = \frac{\beta_{t+2}}{\beta_{t+1}}$, for the case where $n \geq 3$.

We can see that this time-consistent β_t will be of the form $a\beta^t$, which is the exponential discounting example we showed above. The assumption is convenient. We might also argue that it provides for a reasonable model of rational behaviour.

Time-inconsistent discounting, on the other hand, will lead to some very interesting results, and may provide a model of some sorts of behaviour. For example, an agent who is addicted to consumption might find it optimal to consume now and start saving in the next period, but when the next period arrives, put saving off again. In such a model, removing some choices from an agent might boost overall welfare. But this also leads to interesting philosophical problems; if the perspective of the agent changes as to what is 'best' then how would we evaluate between competing perspectives?[1]

[1] If you want to get into this, I suggest Derek Parfit's *Reasons and Persons* (1984, Oxford University Press).

Lecture 16

The Utility of Annuitisation

In this chapter we extend the concept of utility to the whole of a future lifetime at retirement, and consider the implications of competing strategies.

16.1 Optimal consumption plan in the absence of life annuities

We will assume that our agent retires at $t = 0$ at age x with total net worth of w_0 and will live for a maximum of n periods. We can think of n as $\omega - x$.

We will maintain our fixed interest model - so assume that savings earn an annual interest at rate i. At this stage we have no randomness on the asset side; the unknown future lifetime is the sole source of randomness.

A consumption strategy is a choice of c_t for each time $t \in 1, ..., n$.

The decision maker discounts future consumption using annual rate δ and derives per-period felicity at time t of $\frac{c_t^{1-\rho}-1}{1-\rho}$ if she is alive at time t or zero otherwise.

Note that this means the felicity function takes CRRA form, and the agent gains no utility from bequests. By 'bequests' we mean money left over on death - e.g. to a dependant.

We want to optimise lifetime utility, so the programme is:

$$\max_{c_1,\dots,c_N} \sum_{t=1}^{n} \frac{{}_tp_x}{(1+\delta)^t} \frac{c_t^{1-\rho}-1}{1-\rho} \qquad \textit{subject to } w_0 = \sum_{t=1}^{n} \frac{c_t}{(1+i)^t}$$

We can see here that the constraint is that the total value of *possible* consumption is equal to the starting wealth. The agent is having to choose her consumption pattern now, but probability of survival only features in the utility function.

We can solve using the following Lagrangian

$$\max_{c_1,\dots,c_N} L(c_1, ..., c_N, \lambda) = \sum_{t=1}^{n} \frac{{}_tp_x}{(1+\delta)^t} \frac{c_t^{1-\rho}-1}{1-\rho} + \lambda \left(w_0 - \sum_{t=1}^{n} \frac{c_t}{(1+i)^t} \right)$$

First order conditions are

$$\frac{\partial L}{\partial c_t} = \frac{{}_tp_x}{(1+\delta)^t} c_t^{-\rho} - \frac{\lambda}{(1+i)^t} = 0 \quad \text{for } t = 1, ..., n$$

$$\frac{\partial L}{\partial \lambda} = w_0 - \sum_{t=1}^{n} \frac{c_t}{(1+i)^t} = 0$$

$$\frac{\partial L}{\partial c_t} = \frac{{}_tp_x}{(1+\delta)^t} c_t^{-\rho} - \frac{\lambda}{(1+i)^t} = 0$$

$$\Rightarrow c_t^{-\rho} = \frac{\lambda(1+\delta)^t}{{}_tp_x(1+i)^t}$$

$$\Rightarrow c_t = \left(\frac{{}_tp_x(1+i)^t}{\lambda(1+\delta)^t}\right)^{1/\rho}$$

$$\Rightarrow \frac{c_{t+1}}{c_t} = \left(\frac{{}_{t+1}p_x(1+i)}{{}_tp_x(1+\delta)}\right)^{1/\rho}$$

$$\Rightarrow c_{t+1} = c_t \left(p_{x+t} \frac{1+i}{1+\delta}\right)^{1/\rho}$$

If $i = \delta$ then $c_{t+1} < c_t$ because $p_{x+t} < 1$. When there was no uncertainty over future lifetime, we saw that when $i = \delta$ we would have the optimal choice showing flat consumption, that is $c_{t+1} = c_t$. Now we see that when there is mortality, and life annuities are not available, rational expected utility maximisers optimally consume less as they age.

16.2 Optimal retirement strategy with life annuities

At this stage we assume the presence of an actuarially fair life annuity market - that is we are ignoring costs and expenses and other frictions. The optimisation programme, in the presence of a complete market, changes the budged constraint so that the expected present value of consumption is equal to the starting wealth. So the programme becomes:

$$\max_{c_1,\ldots,c_N} \sum_{t=1}^{n} \frac{{}_tp_x}{(1+\delta)^t} \frac{c_t^{1-\rho}-1}{1-\rho} \qquad \textit{subject to } w_0 = \sum_{t=1}^{n} \frac{{}_tp_x\, c_t}{(1+i)^t}$$

And then the Lagrangian becomes

$$\max_{c_1,\ldots,c_N} L(c_1, ..., c_N, \lambda) = \sum_{t=1}^{n} \frac{{}_tp_x}{(1+\delta)^t} \frac{c_t^{1-\rho}-1}{1-\rho} + \lambda \left(w_0 - \sum_{t=1}^{n} \frac{{}_tp_x\, c_t}{(1+i)^t}\right)$$

The first order conditions are then:

$$\frac{\partial L}{\partial c_t} = \frac{{}_tp_x}{(1+\delta)^t} c_t^{-\rho} - \frac{\lambda\, {}_tp_x}{(1+i)^t} = 0 \quad \text{for } t = 1, ..., n$$

$$\frac{\partial L}{\partial \lambda} = w_0 - \sum_{t=1}^{n} \frac{{}_t p_x \, c_t}{(1+i)^t} = 0$$

We can see that ${}_t p_x$ cancels in the first set of equations and we have

$$c_t^{-1/\rho} = \lambda \left(\frac{1+\delta}{1+i} \right)^t \Rightarrow \frac{c_{t+1}}{c_t} = \left(\frac{1+i}{1+\delta} \right)^{1/\rho}$$

So where $i = \delta$ the consumption will be smooth across time. This is the pattern for a standard annuity.

Even where $i \neq \delta$ the slope of the optimal c_t will be higher. It achieves this by relaxing the budget constraint, allowing a higher present value of consumption.

Mortality credit One way of thinking about the benefit to pensioners from life annuities is the following. Suppose you are aged x and want to invest for t periods.

If you invest 1 in the risk-free asset you receive $(1+i)^t$.

If you invest 1 in an actuarially fair survival benefit you receive:

- $\frac{(1+i)^t}{{}_t p_x} \geq (1+i)^t$ if you survive until $t+1$.
- $0 \leq (1+i)^t$ otherwise.

You earn more than the risk free rate if you survive, but less if you don't.

The additional return on survival $\frac{1}{{}_t p_x} > 1$ is called a *mortality credit*, since this is the extra amount by which savings are accumulated if the life survives. We can think of an annuity has having a mortality credit of $\frac{1}{p_x + j}$ for each year j of the annuity.

Possible extensions of the model We can combine the idea of random future lifetime with the ideas presented in the earlier lectures:

- Expenses and profit margin of the life office act to move the annuity away from actuarially fair, and similarly, borrowing is likely to be more expensive than saving.
- A risky asset may be available as well as the risk-free rate (we assume with a higher expected return).
- Labour income may be risky – we would have a multi-period model of income throughout life.

This can get particularly interesting and complicated if we allow for choices around careers and even retirement age. Different careers might give a different pattern of earnings growth, different volatility, and different levels of correlation with the risky assets. Some career choices will even change the mortality characteristics.

Desynchronisation and the Market Economy

The Purpose of the Financial System

What is the primary purpose of the financial system? Why is such a large proportion of our economy dedicated to financial services? Why are the pay and bonuses for bankers so enormous? A decade on from the Financial Crisis, when much of the industry has been described as "socially useless", these questions have never been more relevant.

The aim of this course is to provide a prism through which to view the financial system, and to start to answer these questions. We can argue that the primary purpose of the financial system is to allow *desynchronisation* of the income and consumption streams of agents. There are two dimensions to this desynchronisation: time and risk. Income will be concentrated during working years and the 'good times': when you are healthy, when the jobs market is good, when the harvest is good. But we need and want much smoother consumption, to see us through our early years and retirement, and through the 'bad times': ill-health, the flooded house, the bad harvest. This is what borrowing, saving and insurance permit.

- Banks are other financial institutions are intermediaries between individuals who want to borrow and individuals who want to lend. The supply and demand will determine the interest rate.
- Insurance companies and pension funds can pool diversifiable risks, taking idiosyncratic risks away from individuals.
- Asset markets (such as stock markets) can allow individuals to take on undiversifiable risks, according to their own risk and reward preferences.

In all these examples there will be costs to this intermediation - ultimately consumers of financial services have to pay for the administration, pricing and managing of the risks - which includes the costs of financial regulation.

We should be able to judge the financial system as 'socially useful' to the extent it facilitates this desynchronisation - in terms of its availability, reliability and affordability.