S.3 Utility and Risk

A.1. (a) Denoting $ARA(w)$ as the absolute level of risk aversion at wealth w,

$$ARA(w) = -\frac{u''(w)}{u'(w)} = \frac{a\exp(-aw)}{\exp(-aw)} = a$$

(b) $\lim_{a\to 0} u'(w) = \lim_{a\to 0}\exp(-aw) = 1$

(c) $\lim_{a\to\infty}\mathbb{E}u(\tilde{x}) = \lim_{a\to\infty}\sum_{x\geq 0}\left[p_x\frac{1-\exp(-ax)}{a}\right] + \sum_{x<0}\left[p_x\frac{1-\exp(-ax)}{a}\right] = -\infty$ since lottery x includes negative payoffs.

A.2. (a) Denoting the risk premium of the lottery by π,

$$u(4-\pi) = (4-\pi)^2 = \frac{1}{2}2^2 + \frac{1}{2}6^2 = 20$$

$\therefore \pi = 4-\sqrt{20} = -0.472$. The risk premium is negative because the utility function is convex over the range of interest ($u'' = 2 > 0$), and therefore the decision maker is willing to pay to take on a zero mean risk ('risk loving').

(b) Denoting the new risk premium of the lottery by π',

$$v(4-\pi') = (4-\pi')^4 = \frac{1}{2}2^4 + \frac{1}{2}6^4 = 656$$

$\therefore \pi' = 4-\sqrt[4]{656} = -1.061$, i.e. the risk premium decreases (or equivalently the decision maker is willing to pay a higher price to take on the zero mean risk). $v(w) = f(u(w))$ where $f(u) = u^2$, a convex function ($f''(u) > 0$).

A.3. $A(w) := -\frac{u''(w)}{u'(w)} = -\frac{d}{dw}\ln u'(w)$. Integrating over $[z_0, z]$ gives $\int_{z_0}^{z} A(w)dw := -\ln(u'(z)) + \ln(u'(z_0))$, and so $u'(z) = u'(z_0)\exp\left\{-\int_{z_0}^{z} A(w)dw\right\}$, which can be rewritten in the desired form.

B.1. (a) $\mathbb{E}[\tilde{y}] = \int_0^1 \lambda y^{\lambda-1} y dy = [\frac{\lambda}{\lambda+1}y^{\lambda+1}]_0^1 = \frac{\lambda}{\lambda+1}$

$\mathbb{E}[\tilde{y}^2] = \int_0^1 \lambda y^{\lambda-1} y^2 dy = [\frac{\lambda}{\lambda+2}y^{\lambda+2}]_0^1 = \frac{\lambda}{\lambda+2}$

$\therefore Var[\tilde{y}] = \frac{\lambda}{\lambda+2} - \left[\frac{\lambda}{\lambda+1}\right]^2$

(b) $\mathbb{P}[\tilde{y} \geq 0.5] = \int_{0.5}^1 \lambda y^{\lambda-1} dy = [y^\lambda]_{0.5}^1 = 1 - 0.5^\lambda$

(c) $u'(y) = (1-r)y^{-r}$. $u''(y) = -r(1-r)y^{-r-1}$ which is strictly negative for $r \in (0,1)$ and so the consumer is strictly risk averse.

$\mathbb{E}u(\tilde{y}) = \int_0^1 \lambda y^{\lambda-1} y^{1-r} dy = [\frac{\lambda}{\lambda-r+1}y^{\lambda-r+1}]_0^1 = \frac{\lambda}{\lambda-r+1}$

$u(\mathbb{E}(\tilde{y})) = \left(\frac{\lambda}{\lambda+1}\right)^{1-r}$

Consider

$$\ln\left[\frac{u(\mathbb{E}(\tilde{y}))}{\mathbb{E}(u(\tilde{y}))}\right] = \ln\left[\left(\frac{\lambda}{\lambda+1}\right)^{1-r} \times \frac{\lambda-r+1}{\lambda}\right]$$
$$= (1-r)\ln\left[\frac{\lambda}{\lambda+1}\right] + \ln\left[\frac{\lambda-r+1}{\lambda}\right]$$
$$\ln\left[\frac{u(\mathbb{E}(\tilde{y}))}{\mathbb{E}(u(\tilde{y}))}\right]|_{r=0} = 0$$
$$\ln\left[\frac{u(\mathbb{E}(\tilde{y}))}{\mathbb{E}(u(\tilde{y}))}\right]|_{r=1} = 0$$
$$\frac{d^2}{dr^2}\ln\left[\frac{u(\mathbb{E}(\tilde{y}))}{\mathbb{E}(u(\tilde{y}))}\right] = -\frac{1}{(\lambda-r+1)^2} < 0$$

So $\ln\left[\frac{u(\mathbb{E}(\tilde{y}))}{\mathbb{E}(u(\tilde{y}))}\right]$ is strictly concave and takes the value 0 at $r=0$ and $r=1$. We must therefore have that $u(\mathbb{E}(\tilde{y})) > \mathbb{E}(u(\tilde{y}))$ for $0 < r < 1$.

B.2. Y_i are i.i.d. with mean μ and variance σ^2, so $\sum_{i=1}^n Y_i$ has mean $n\mu$ and variance $n\sigma^2$.

(a) Total premiums $= nA = n\mu + 10\sigma\sqrt{n}$. The probability that claims exceed premiums is

$$\mathbb{P}\left(\sum Y_i > nA\right) = \mathbb{P}\left(\sum Y_i - \mathbb{E}\sum Y_i > 10\sigma\sqrt{n}\right) \leq \frac{\text{Var}\left(\sum Y_i\right)}{100\sigma^2 n} = 1/100,$$

using Chebyshev's inequality.

(b) As above, the probability that claims exceed premiums is

$$\mathbb{P}\left(\sum Y_i - \mathbb{E}\sum Y_i > 3\sigma\sqrt{n}\right).$$

By the CLT for the i.i.d. variables Y_i, this probability tends to $\mathbb{P}(Z > 3)$ as $n \to \infty$, where Z has a standard normal distribution. From the table, the probability on the RHS is 0.00135, so for large n the probability that claims exceed premiums is less than 0.01 as required.

B.3. (a) Denoting the certainty equivalent and risk premium of x by CE_x and RP_x respectively,

$$u(10-RP_x) = u(10+CE_x) = (10+CE_x)^{1/2} = \mathbb{E}u(10+\tilde{x}) = \frac{1}{2}u(4)+\frac{1}{2}u(16) = 3$$

$\therefore CE_x = 9 - 10 = -1$ and $RP_x = +1$.

(b) The Arrow-Pratt approximation of the risk premium

$$\hat{RP}_x = \frac{1}{2}\mathbb{E}\tilde{x}^2 ARA(10) = \frac{1}{2} \times 36 \times \frac{\frac{1}{4}10^{-3/2}}{\frac{1}{2}10^{-1/2}} = \frac{36}{40} = 0.9$$

(c) $ARA(w) = -\frac{u'(w)}{u''(w)} = \frac{\frac{1}{4}w^{-3/2}}{\frac{1}{2}w^{-1/2}} = \frac{1}{2w}$ which is positive and decreasing with w.
$RRA(w) = -w\frac{u'(w)}{u''(w)} = \frac{1}{2}$ which is constant.

(d) Denoting the certainty equivalent and risk premium of x by CE_x and RP_x respectively,

$$\begin{aligned} v(10 - RP_x) = v(10 + CE_x) = (10 + CE_x)^{1/4} &= \mathbb{E}v(10 + \tilde{x}) \\ = \frac{1}{2}v(4) + \frac{1}{2}v(16) &= \frac{1}{\sqrt{2}} + 1 = 1.707 \text{ (4sf)} \end{aligned}$$

$\therefore CE_x = 1.707^4 - 10 = -1.507$ and $RP_x = +1.507$. The risk premium has increased substantially. Note that $w^{1/4}$ is a concave transformation of $w^{1/2}$ and so v is more risk averse than u in the sense of Arrow-Pratt.

(e) The Arrow-Pratt approximation of the risk premium

$$\hat{RP}_x = \frac{1}{2}\mathbb{E}\tilde{y}^2 ARA(10) = \frac{1}{2} \times 9 \times \frac{\frac{1}{4}10^{-3/2}}{\frac{1}{2}10^{-1/2}} = \frac{9}{40} = \frac{0.9}{4}$$

The risk premium is proportional to the squared magnitude of the risk, which has decreased by a factor of 4 since the magnitude of the risk has decreased by a factor of 2.

B.4. If the decision maker has preferences that satisfy constant relative risk aversion then preferences can be represented by a utility function $u(w) = \frac{w^{1-\gamma}}{1-\gamma}$ for some $\gamma \in \mathbb{R}$ (or $\ln(w)$ for the case of $\gamma = 1$).

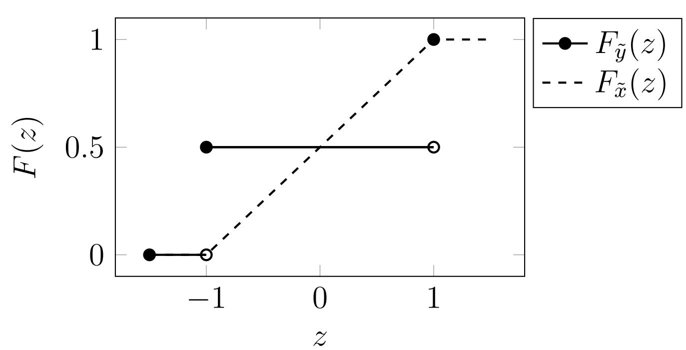
Utility of certain wealth w is $\frac{w^{1-\gamma}}{1-\gamma}$ and expected utility of lottery $w(1 + \tilde{x})$ is

$$\begin{aligned} \mathbb{E}\left[u(w(1 + \tilde{x}))\right] &= \mathbb{E}_{\tilde{x}}\left[\frac{[w(1 + \tilde{x})]^{1-\gamma}}{1 - \gamma}\right] \\ &= \frac{w^{1-\gamma}}{1 - \gamma} \times \mathbb{E}_{\tilde{x}}\left[(1 + \tilde{x})^{1-\gamma}\right] \end{aligned}$$

and so the lottery will be (strictly) preferred to certain wealth iff $\mathbb{E}\left[(1 + \tilde{x})^{1-\gamma}\right] \geq (>)1$. This condition does not depend on w.

Similarly, for logarithmic utility function $u(w) = \ln(w)$, $\mathbb{E}[u(w(1 + \tilde{x}))] = \ln(w) + \mathbb{E}[\ln(1 + \tilde{x})]$ which is (strictly) greater than $\ln(w)$ iff $\mathbb{E}[\ln(1 + \tilde{x})] \geq (>)0$. Again, this condition does not depend on w.

B.5. (a) See the following figure



(b) Both distributions have support on the interval $[-1,1]$. We have

$$\int_{-1}^{\theta} F_{\tilde{x}}(z)dz = \int_{-1}^{\theta} \frac{z+1}{2}dz = \left[\frac{z^2}{4}+\frac{z}{2}\right]_{-1}^{\theta} = \frac{\theta^2}{4}+\frac{\theta}{2}+\frac{1}{4}$$

$$\int_{-1}^{\theta} F_{\tilde{y}}(z)dz = \int_{-1}^{\theta} \frac{1}{2}dz = \left[\frac{z}{2}\right]_{-1}^{\theta} = \frac{\theta}{2}+\frac{1}{2}$$

$$\therefore \int_{-1}^{\theta} F_{\tilde{y}}(z) - F_{\tilde{x}}(z)dz = \frac{1-\theta^2}{4} \begin{cases} = 0 \text{ for } \theta = 1 \\ \geq 0 \text{ for } \theta \in [-1,1) \end{cases}$$

$\tilde{x}$ therefore second order stochastically dominates $\tilde{y}$, and so $\tilde{y}$ is said to be riskier than $\tilde{x}$.

(c) For each outcome x of $\tilde{x}$ add a zero mean noise $\tilde{\varepsilon}_x := \{1-x, \frac{1+x}{2}; -(1+x), \frac{1-x}{2}\}$. Conditional on any x, there are two potential outcomes, $x+1-x=+1$ and $x-(1+x)=-1$, and $\mathbb{E}[\tilde{\varepsilon}_x]=0$. By symmetry, and over all possible outcomes of x, these two outcomes of $\tilde{x}+\tilde{\varepsilon}$ have equal probability of $\frac{1}{2}$, where conditional on $\tilde{x}$, $\tilde{\varepsilon}$ has distribution $\tilde{\varepsilon}_x$, and so $\tilde{x}+\tilde{\varepsilon} \sim (-1,\frac{1}{2};+1,\frac{1}{2})$, which is the distribution of $\tilde{y}$.

B.6. Let $\tilde{A} = (80,\frac{1}{4};100,\frac{1}{4};120,\frac{1}{4};140,\frac{1}{4})$, $\tilde{B} = (90,\frac{1}{2};130,\frac{1}{2})$ and $\tilde{\varepsilon} = (+10,\frac{1}{2};-10,\frac{1}{2})$, and $\tilde{B}$ and $\tilde{\varepsilon}$ independent. Then $\tilde{B}+\tilde{\varepsilon} \sim \tilde{A}$. Conditional on each outcome of $\tilde{B}$, $\tilde{\varepsilon}$ has zero mean. Therefore by the Rothschild-Stiglitz Theorem, Project B SSD Project A.

B.7. (a) Since $g(x)$ is a probability density function with support $[0,1]$, we must have

$$\int_0^1 g(x)dx = 1 \Rightarrow c\int_0^1 x^2 - x + \frac{1}{4}dx = c\left[\frac{x^3}{3}-\frac{x^2}{2}+\frac{x}{4}\right]_0^1 = \frac{c}{12} = 1$$

and so $c=12$.

$$\int_0^1 x\, g(x)dx = 12\int_0^1 x^3 - x^2 + \frac{x}{4}\, dx = 12\left[\frac{x^4}{4}-\frac{x^3}{3}+\frac{x^2}{8}\right]_0^1 = 12\left(\frac{6}{24}-\frac{8}{24}+\frac{3}{24}\right) = \frac{1}{2}$$

This shows that the mean of L_b is equal to the mean of L_a.
Defining the interval $I := \left[\frac{1}{2}-\sqrt{1/12}, \frac{1}{2}+\sqrt{1/12}\right]$ and denoting the pdf of L_a as f, we therefore have $g(x) \geq f(x)$ for $x \in [0,1]\backslash I$ (the set $[0,1]$ excluding elements in I), and $g(x) \leq f(x)$ for $x \in I$. Distribution $g(x)$ can be obtained from distribution $f(x)$ by moving some probability mass from the interval I to outside of I without changing the mean, and so L_b is a mean-preserving spread of L_a.

(b) No. The cdf of lottery L_a is $F(x)=x$, $x \in [0,1]$ and the cdf of lottery L_b is

$$G(x) = \int_0^x g(z)dz = 12\left[\frac{x^3}{3}-\frac{x^2}{2}+\frac{x}{4}\right]$$

$G(1/4) = 0.4375 > 0.25 = F(1/4)$ and $G(3/4) = 0.5625 < 0.75 = F(3/4)$. Therefore neither $G(x) \geq F(x) \forall x \in [0,1]$ nor $G(x) \leq F(x) \forall x \in [0,1]$.

C.1. We are required to calculate the risk premium $\pi(w_0, u, \tilde{y})$ when $u(w) = -\exp\{-Aw\}$ and $\tilde{y} \sim N(0, \sigma^2)$. We have

$$\begin{aligned}
u(w_0 - \pi) &= -\exp\{-A(w_0 - \pi)\} \\
&= -\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\{-\frac{y^2}{2\sigma^2}\} \exp\{-A(w_0 + y)\} dy \\
&= -\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\{-\frac{y^2 + 2\sigma^2 A(w_0 + y)}{2\sigma^2}\} dy \\
&= -\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\{-\frac{(y + \sigma^2 A)^2}{2\sigma^2}\} \exp\{-Aw_0 + \frac{1}{2}\sigma^2 A^2\} dy \\
&= -\exp\{-Aw_0 + \frac{1}{2}\sigma^2 A^2\} \\
\therefore \pi &= \frac{1}{2}\sigma^2 A
\end{aligned}$$

Since $\tilde{y}$ has zero mean, $\mathbb{E}[y^2] = \sigma^2$, and since u satisfies CARA, $A(w_0) = A$, and so we have the required result.

C.2 Let $\tilde{y}$ denote the distribution of final wealth under the perfect diversification strategy. Now under any alternative feasible strategy $A = (\alpha_1, \ldots, \alpha_n)$ we have

$$\sum_{i=1}^{n} \alpha_i \tilde{x}_i = \tilde{y} + \sum_{i=1}^{n} \left(\alpha_i - \frac{1}{n}\right) \tilde{x}_i$$

From the Rothschild-Stiglitz theorem we are done if we can show that $\sum_{i=1}^{n} \left(\alpha_i - \frac{1}{n}\right) \tilde{x}_i$ has a zero mean conditional on $\tilde{y}$. Now, by joint independence of $\tilde{x}_i$ and symmetry of $\tilde{y}$, $\mathbb{E}[\tilde{x}_i|\tilde{y}]$ is independent of i. We may denote it k. Then, we have

$$\begin{aligned}
\mathbb{E}\left[\sum_{i=1}^{n} \left(\alpha_i - \frac{1}{n}\right) \tilde{x}_i | \tilde{y}\right] &= \sum_{i=1}^{n} \left(\alpha_i - \frac{1}{n}\right) \mathbb{E}[\tilde{x}_i|\tilde{y}] \\
&= k \sum_{i=1}^{n} \left(\alpha_i - \frac{1}{n}\right) = 0
\end{aligned}$$

This proves that the alternative strategy is second order stochastically dominated by the perfect diversification strategy.

C.3. (a) u is piecewise linear with gradient $u'(z) = 1$ for $z < z_0$ and $u'(z) = a < 1$ for $z > z_0$. We must show that for any $z, z' \in \mathbb{R}$, $\lambda \in [0, 1]$,

$$\lambda u(z) + (1 - \lambda)u(z') \leq u(\lambda z + (1 - \lambda)z').$$

If $z, z' \geq z_0$ or $z, z' \leq z_0$ then this weak inequality is satisfied with equality since u is linear between z and z'. If $z \leq z_0$ and $z' \geq z_0$ then from the definition of u,

$$\lambda u(z) + (1 - \lambda)u(z') = \lambda z + (1 - \lambda)az' + (1 - \lambda)(1 - a)z_0$$

There are two cases. For $\lambda z + (1-\lambda)z' \leq z_0$,

$$u(\lambda z + (1-\lambda)z') = \lambda z + (1-\lambda)z'$$
$$\therefore \lambda u(z) + (1-\lambda)u(z') - u(\lambda z + (1-\lambda)z') = (1-a)(1-\lambda)(z_0 - z') \leq 0$$

since $z' \geq z_0$ by assumption, and in the second case, $\lambda z + (1-\lambda)z' \geq z_0$,

$$u(\lambda z + (1-\lambda)z') = (1-a)z_0 + a(\lambda z + (1-\lambda)z')$$
$$\therefore \lambda u(z) + (1-\lambda)u(z') - u(\lambda z + (1-\lambda)z') = (1-a)\lambda(z - z_0) \leq 0$$

since $z \leq z_0$ by assumption.

(b) Denoting the cdf of $\tilde{x}$ as $F(x)$, where $\mathbb{E}[\tilde{x}] = 0$, the risk premium $\pi(z_0, u, k\tilde{x})$ is nonnegative since u is concave, and therefore satisfies

$$\begin{aligned}
u(z_0 - \pi) &= z_0 - \pi = \mathbb{E}[u(z_0 + k\tilde{x})] \\
&= \int_{-\infty}^{0} (z_0 + kx)dF(x) + \int_0^{\infty} ((1-a)z_0 + a(z_0 + kx))dF(x) \\
&= z_0 + k\left[\int_{-\infty}^{0} xdF(x) + a\int_0^{\infty} xdF(x)\right] \\
&= z_0 + k\left[\mathbb{E}[\tilde{x}] - (1-a)\int_0^{\infty} xdF(x)\right] \\
&= z_0 - k(1-a)\int_0^{\infty} xdF(x) \\
\therefore \pi(z_0, u, k\tilde{x}) &= k(1-a)\int_0^{\infty} xdF(x)
\end{aligned}$$

For $\tilde{x}$ non-degenerate, $\mathbb{E}[\tilde{x}] = 0$ implies $\int_0^{\infty} xdF(x)$ is positive and finite and so $\pi(z_0, u, k\tilde{x})$ is linear in k and $\lim_{k\to 0} \pi(z_0, u, k\tilde{x}) = 0$.

(c) Consider

$$\phi(z) := \begin{cases} z & \text{if } z \leq z_0 \\ z_0 + b(z - z_0) & \text{if } z > z_0 \end{cases}.$$

where $0 < b < 1$. We therefore have

$$\phi(u(z)) := \begin{cases} z & \text{if } z \leq z_) \\ z_0 + ab(z - z_0) & \text{if } z > z_0 \end{cases}.$$

ϕ is an increasing concave transformation from the first part of this exercise. Now consider a decrease in parameter a of function u to a'. The new utility function is the same as $\phi \circ u$ where ϕ is defined with parameter $b = a'/a \in [0, 1]$. A reduction in a to a' therefore increases the degree of risk aversion because it can result from transforming u with an increasing, concave function ϕ.

(d) We prove by finding a pure risk with zero risk at wealth w_0 and strictly positive risk premium at wealth $w_0' > w_0$. This is sufficient to show that u does not exhibit decreasing absolute risk aversion.

Let the pure risk to consider be the lottery $\tilde{x} := \left(+x, \frac{1}{2}; -x, \frac{1}{2}\right)$ and let $w_0, w_0 + x \leq z_0$. Then the risk premium is zero at wealth level w_0 since from the definition of the risk premium $w_0 - \pi(w_0, u, \tilde{x}) = 0.5(w_0 - x) + 0.5(w_0 + x) = w_0$. Now consider a level of wealth $w_0' > w_0$ such that $z_0 - x < w_0' < z_0$. The risk premium $\pi(w_0', u, \tilde{x})$ is defined by

$$0.5(w_0' - x) + 0.5((1-a)z_0 + a(w_0' + x)) = w_0' - \pi(w_0', u, \tilde{x})$$
$$\therefore \pi(w_0', u, \tilde{x}) = 0.5(1-a)(w_0' + x - z_0) > 0$$

since $w_0' + x > z_0$ by construction and $a \in (0, 1)$