## S.2 lifetime distributions and life products

A.1. Continuous cash flow at unit rate until death but for a minimum term of $n$.

A.2. (i) $\sum_{k\geq 1}(1+i)^{-k}(1-q)^k = \frac{1-q}{1+i}\frac{1}{1-(1-q)/(1+i)} = \frac{1-q}{i+q}.$

(ii) $\sum_{k\geq 1}(1+i)^{-k}(1+g)^k = \frac{1+g}{1+i}\frac{1}{1-(1+g)/(1+i)} = \frac{1+g}{i-g}.$

(iii) $\sum_{k\geq 1}(1+i)^{-k}(1+h)^k(1-q)^k = \frac{(1+h)(1-q)}{1+i}\frac{1}{1-(1+h)(1-q)/(1+i)} = \frac{(1+h)(1-q)}{i+q-h(1-q)}.$

A.3. By definition ${}_tp_x = \mathbb{P}(T_x > t)$. We defined the force of mortality as infinitesimal survival probability and can write this in terms of ${}_tp_x$ as

$$\mu_{x+t} = \lim_{\varepsilon\downarrow 0}\frac{{}_tp_x - {}_{t+\varepsilon}p_x}{\varepsilon \times {}_tp_x} = -\frac{\frac{\partial}{\partial t}{}_tp_x}{{}_tp_x} = -\frac{\partial}{\partial t}(\log({}_tp_x))$$

This differential equation is solved by simple integration

$$\log({}_tp_x) = -\int_0^t \mu_{x+s}ds + c \quad\Rightarrow\quad {}_tp_x = e^c\exp\left\{-\int_0^t \mu_{x+s}ds\right\}.$$

From ${}_{0-}p_x = \mathbb{P}(T_x \geq 0) = 1$, we get $c = 0$, hence the result.

A.4. $\mu_x^{(2)} = 2\mu_x^{(1)}$. Think of $\mu^{(1)}$ to be from a general life table, but of a population with a certain disease or other risks causing their mortality to be higher than in that lifetable. One way to express this increased mortality is by so-called proportional hazards, here proportionality factor 2.

(i) We have

$${}_tp_x^{(1)} = \exp\left\{-\int_x^{x+t}\mu_y^{(1)}dy\right\},$$

where now

$$\begin{aligned}\int_x^{x+t}\mu_y^{(1)}dy &= Bc^x\int_0^t c^z dz\\ &= Bc^x\int_0^t e^{z\log(c)}dz\\ &= \frac{Bc^x}{\log(c)}(c^t-1).\end{aligned}$$

This leads to the form $g^{c^x(c^t-1)}$ for $-B/\log(c) = \log(g)$ as required.

(ii) ${}_np_x^{(2)} = \exp\{-2\int_0^n \mu_{x+s}^{(1)}ds\} = ({}_np_x^{(1)})^2$ as required.

(iii) Under the Gompertz law

$$ {}_np_x^{(2)} = g^{2c^x(c^n-1)} = {}_np_{x+a}^{(1)} = g^{c^{x+a}(c^n-1)} $$

if and only if $c^a = 2$, i.e. $a = \log(2)/\log(c)$. This is saying that, under the Gompertz law, the model with mortality doubled is the same as a model with age shifted appropriately. The latter is another way of expressing the fact that a population with higher mortality may behave like an older population. For general mortality laws, these two models are different.

B.1. (i) This is similar, with the extra factor $\exp\{-\int_0^t A dy\} = s^t$ for $s = e^{-A}$.

(ii) We have three equations in three unknowns $s$, $c$ and $g$:

$$ {}_{10}p_{30} = s^{10}g^{c^{30}(c^{10}-1)} = \frac{\ell_{40}}{\ell_{30}} =: \alpha = 0.9912231 $$
$$ {}_{10}p_{40} = s^{10}g^{c^{40}(c^{10}-1)} = \frac{\ell_{50}}{\ell_{40}} =: \beta = 0.9740027 $$
$$ {}_{10}p_{50} = s^{10}g^{c^{50}(c^{10}-1)} = \frac{\ell_{60}}{\ell_{50}} =: \gamma = 0.919498. $$

First eliminate $s$ by taking ratios, taking log then gives

$$ c^{30}(c^{10}-1)^2\log(g) = \log(\beta/\alpha) = -0.0175256 $$

and

$$ c^{40}(c^{10}-1)^2\log(g) = \quad \log(\gamma/\beta) = -0.0575862 $$

and again taking ratios yields $c^{10} = \log(\gamma/\beta)/\log(\beta/\alpha) = 3.285833$ and so $c = 1.1265$.

Then the $\log(\beta/\alpha)$-equation gives $\log(g) = -9.4547 \times 10^{-5}$. So $B = -\log c \times \log g = 1.1247 \times 10^{-5}$.

Finally, the first equation gives $s^{10} = 0.998852$ and then $s = 0.9998852$, so that $A = -\log s = 1.149 \times 10^{-4}$.

B.2. (i) We split into death benefits during the period and survival to end of period, then use $q_{x+k} = 1 - p_{x+k}$:

$$ \begin{aligned} A_{x:\overline{n}|} &= \sum_{k=0}^{n-1} {}_kp_x q_{x+k} v^{k+1} + {}_np_x v^n \\ &= \sum_{k=0}^{n-1} {}_kp_x v^{k+1} - \sum_{k=0}^{n-2} {}_kp_x p_{x+k} v^{k+1} \\ &= v\ddot{a}_{x:\overline{n}|} - \sum_{j=1}^{n-1} {}_jp_x v^k \\ &= v\ddot{a}_{x:\overline{n}|} - a_{x:\overline{n-1}|}. \end{aligned} $$

The first term pays a benefit of $v$ at the *beginning* of every year $1, \ldots, n$ provided alive at the beginning. This is equivalent in value to paying a benefit of 1 at the *end* of every year $1, \ldots, n$, still provided alive at the *beginning*.

The second term takes away the benefit of 1 if alive at the *end* of the year, except in year $n$.

So the payments do not cancel in two cases: in the event that the life dies sometime during the year, or in the event the life survives the final year.

This means a payment at the end of the year of death, or $n$ if earlier; this is the cash flow of an endowment assurance, as required.

(ii) We calculate

$$\begin{aligned}
\ddot{a}_x - d(I\ddot{a})_x &= \sum_{k=0}^{\infty} {}_kp_x v^k - d\sum_{k=0}^{\infty} {}_kp_x v^k (k+1) \\
&= \sum_{k=0}^{\infty} {}_kp_x v^k (1-(1-v)(k+1)) \\
&= \sum_{k=0}^{\infty} {}_kp_x v^{k+1} (k+1) - \sum_{j=0}^{\infty} {}_{j+1}p_x v^{j+1}(j+1) \\
&= \sum_{k=0}^{\infty} ({}_kp_x - {}_{k+1}p_x) v^{k+1}(k+1) \\
&= \sum_{k=0}^{\infty} {}_kp_x q_{x+k} v^{k+1}(k+1) = (IA)_x.
\end{aligned}$$

$(IA)_x$ pays 1 for each beginning of a year we were alive, payment is at the end of the year of death. $\ddot{a}_x$ pays one for each beginning of a year we were alive, payment is at the beginning of each of these years. $-d(I\ddot{a})_x$ takes away the interest we earn on the early payments so that the accumulation at the time of death is $(IA)_x$.

B.3. (a) We have

$$\begin{aligned}
NPV_{[0,T_x]}(\delta) &= \int_0^{T_x} \rho(t)e^{-\delta t}dt \\
E\left(NPV_{[0,T_x]}(\delta)\right) &= E\left(\int_0^{\infty} 1_{\{T_x>t\}}\rho(t)e^{-\delta t}dt\right) \\
&= \int_0^{\infty} E(1_{\{T_x>t\}})\rho(t)e^{-\delta t}dt \\
&= \int_0^{\infty} P(T_x>t)\rho(t)e^{-\delta t}dt.
\end{aligned}$$

(b) Recall that

$$P(T_x > t) = \exp\left\{-\int_0^t \mu_{T_x}(s)ds\right\},$$

since

$$\frac{d}{dt}\log P(T_x > t)) = \frac{d}{dt}\log F_{T_x}(t) = -\frac{f_{T_x}(t)}{F_{T_x}(t)} = -\mu_{T_x}(t)$$

and $\log P(T_x > x) = \log 1 = 0$.
So with the given form of $\mu_{T_x}$ we have

$$P(T_x > t) = \exp\left\{-\int_0^t \mu_{T_x}(s)ds\right\} = \begin{cases} e^{-0.01t} & 0 \le t \le 5 \\ e^{-0.05-(t-5)0.02} & 5 < t \le 10 \end{cases}$$

and the answer is given by solving the integral in (a) to give

$$\begin{aligned} E\left(NPV_{[0,T_x]}(\delta)\right) &= 5000\int_0^5 e^{-0.01t}e^{-\delta t}dt + 10000\int_5^{10} e^{-0.05-(t-5)0.02}e^{-\delta t}dt \\ &= \ldots = 52851.69. \end{aligned}$$

(c) Here the cash flow is $C_2 = (T_x, 1_{\{T_x \le 10\}}40000)$ and we calculate the probability density function of $T_x$ from the survival function:

$$f_{T_x}(t) = -\frac{d}{dt}P(T_x > t) = \begin{cases} 0.01e^{-0.01t} & 0 \le t \le 5 \\ 0.02e^{-0.05-(t-5)0.02} & 5 < t \le 10 \end{cases}.$$

Then

$$\begin{aligned} E\left(NPV(\delta)\right) &= 40000E\left(e^{-\delta T_x}1_{\{T_x \le 10\}}\right) \\ &= 40000\int_0^{10} e^{-\delta t}f_{T_x}(t)dt = \ldots = 4228.14. \end{aligned}$$

B.4. The present value of benefits is

$$2,500_{20}|\ddot{a}_{40} = 2,500_{20}p_{40}(1.04)^{-20}\ddot{a}_{60}.$$

The present value of expenses is

$$.05P\ddot{a}_{40:\overline{20}|} + 5_{20}p_{40}(1.04)^{-20}\ddot{a}_{60},$$

where $P$ is the annual premium. The present value of premiums is

$$P\ddot{a}_{40:\overline{20}|}$$

The third quantity is paying for the sum of the first two, and we solve this equation for $P$ to get

$$P = \frac{2505_{20}p_{40}(1.04)^{-20}\ddot{a}_{60}}{.95\ddot{a}_{40:\overline{20}|}}.$$

B.5. (a) Using $d_x = \ell_x - \ell_{x+1} = 100,000$ for all $x$, we calculate

$$A_{50:\overline{10}|} = \frac{1}{\ell_{50}}\left(\sum_{k=0}^{9} d_{50+k}v^{k+1}\right) + \frac{\ell_{60}}{\ell_{50}}v^{10} = .645565$$

and

$$\ddot{a}_{50:\overline{10}|} = \frac{1}{\ell_{50}} \sum_{k=0}^{9} \ell_{50+k} v^k = 7.443129$$

or more elegantly using $A_{x:\overline{n}|} = 1 - d\ddot{a}_{x:\overline{n}|}$ where $d = i/(1+i)$.

(b) Now the net annual premium is

$$25,000 \frac{A_{50:\overline{10}|}}{\ddot{a}_{50:\overline{10}|}} = 2,168.33.$$

The policy value at age 53 is

$$R_{53} = 25,000 A_{53:\overline{7}|} - 2,168.33\, \ddot{a}_{53:\overline{7}|},$$

where, as above

$$A_{53:\overline{7}|} = .72795 \quad \text{and} \quad \ddot{a}_{50:\overline{7}|} = \frac{1 - A_{50:\overline{7}|}}{d} = 5.71306,$$

so that $R_{53} = 5810.96$. Similarly, at age 54, $R_{54} = 8016.91$.

(c) Let the office premium be $P$, then

$$P\ddot{a}_{50:\overline{10}|} = 25,000 A_{50:\overline{10}|} + 0.5P + 300 + 0.02 P a_{50:\overline{9}|} + 50 a_{50:\overline{9}|},$$

where

$$a_{50:\overline{9}|} = \ddot{a}_{50:\overline{10}|} - 1 = 6.443129,$$

so that, when we solve for $P$, we get $P = 2,459.74$.

B.6. (i) We have

$$\begin{aligned} \bar{A}_x &= \int_{y=0}^{\infty} \exp(-y\delta) \exp\left(-\int_0^y \mu_{x+s} ds\right) \mu_{x+y} dy, \\ \bar{a}_x &= \int_{y=0}^{\infty} \exp(-y\delta) \exp\left(-\int_0^y \mu_{x+s} ds\right) dy. \end{aligned}$$

Integrating the expression for $\bar{a}_x$ by parts one gets

$$\begin{aligned} \bar{a}_x &= \left[\exp\left(-\int_0^y \mu_{x+s} ds\right)\left(-\frac{\exp(-y\delta)}{\delta}\right)\right]_0^{\infty} \\ &\qquad - \int_{y=0}^{\infty} \left[-\frac{\exp(-y\delta)}{\delta}\right]\left[-\mu_{x+y} \exp\left(-\int_0^y \mu_{x+s} ds\right)\right] dy \\ &= \frac{1}{\delta} - \frac{\bar{A}_x}{\delta}, \end{aligned}$$

which gives $\bar{A}_x = 1 - \delta \bar{a}_x$ as required.

[Alternative proof: $\bar{a}_x = \mathbb{E}\left[\frac{1-e^{-\delta T_x}}{\delta}\right] = \frac{1-\mathbb{E}[e^{-\delta T_x}]}{\delta} = \frac{1-\bar{A}_x}{\delta}$]

(ii) The premium for the insurance is paid continuously at rate $\bar{P}_x = \bar{A}_x/\bar{a}_x$. The reserve after time $t$ has passed is equal to the expected present value of future benefits minus the expected present value of future premiums, so that

$$ {}_t\bar{V}_x = \bar{A}_{x+t} - \bar{P}_x \bar{a}_{x+t} = \bar{A}_{x+t} - \frac{\bar{A}_x}{\bar{a}_x}\bar{a}_{x+t} = 1 - \delta\bar{a}_{x+t} - (1-\delta\bar{a}_x)\frac{\bar{a}_{x+t}}{\bar{a}_x} = 1 - \frac{\bar{a}_{x_x+t}}{\bar{a}_x}. $$

C.1. (a) $\ddot{a}_x = \sum_{k=0}^{\infty} v^k {}_kp_x = 1 + vp_x \sum_{k=1}^{\infty} v^{k-1} {}_{k-1}p_{x+1} = 1 + vp_x \ddot{a}_{x+1}$

(b) $\ddot{a}_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^k {}_kp_x = \sum_{k=1}^{n} v^k {}_kp_x + 1 - v^n {}_np_x = a_{x:\overline{n}|} + 1 - A_{x:\overline{n}|}^{\;\;1}$.

(c) $v\ddot{a}_x - a_x = v\sum_{k=0}^{\infty} v^k {}_kp_x - \sum_{j=1}^{\infty} v^j {}_jp_x = \sum_{j=1}^{\infty} v^j ({}_{j-1}p_x - {}_jp_x) = \sum_{j=1}^{\infty} v^j {}_{j-1}p_x (1 - p_{x+j-1}) = A_x$

C.2. (a) (i) $C_1 = ((n, B))$ where $B = I(K \geq n)$.
(ii) $C_2 = ((k, B_k))_{k\geq 1}$ where $B_k = I(k = K + 1)$.
(iii) $C_3 = (k, B_k)_{k=1,\ldots,n}$, with $B_k$ as in (ii).
(iv) $C_4 = (k, B_k)_{k=1,\ldots,n}$, with $B_k = I(k = K + 1 \leq n) + I(k = n \leq K)$.

(b) (i) $A_1(\delta) = E(NPV_1(\delta)) = e^{-\delta n}P(K \geq n)$, $V_1 = Var(NPV_1(\delta)) = E(NPV^2) - E(NPV)^2 = A_1(2\delta) - (A_1(\delta))^2$
(ii) $A_2(\delta) = E(NPV_2(\delta)) = \sum_{k=1}^{\infty} e^{-\delta k}P(K + 1 = k) = E(e^{-\delta(K+1)})$,
$V_2 = Var(NPV_2(\delta)) = A_2(2\delta) - (A_2(\delta))^2$
(iii) $A_3(\delta) = E(NPV_3(\delta)) = \sum_{k=1}^{n} e^{-\delta k}P(K + 1 = k) = E(e^{-\delta(K+1)}1_{\{K\leq n-1\}})$,
$V_3 = Var(NPV_3(\delta)) = A_3(2\delta) - (A_3(\delta))^2$
(iv) $A_4(\delta) = E(NPV_4(\delta)) = \sum_{k=1}^{n-1} e^{-\delta k}P(K + 1 = k) + e^{-\delta n}P(K \geq n - 1) = E(e^{-\delta(min(K+1,n))})$, $V_4 = Var(NPV_4(\delta)) = A_4(2\delta) - (A_4(\delta))^2$

(c) $C_4 = C_1 + C_3$, $A_4 = A_1 + A_3$, $V_4 = V_1 + V_3 + 2Cov(NPV_1(\delta), NPV_3(\delta)) = V_1 + V_3 - 2A_1A_3$ since $NPV_1 \times NPV_3 = 0$ always.

(d) Only the pure endowment can still be expressed in the form asked for in part (a): $C_1 = ((n, B))$ where $B = 1(T \geq n)$. For (ii)-(iv), the time of the payment is now a continuous random variable.

The expressions of for net premiums are essentially the same as in the discrete case, but replacing the discrete probability mass function of $K + 1$ by the density function of $T$, and integrating rather than summing. For example for part (ii), $\tilde{A}_2(\delta) = \int_0^{\infty} e^{-\delta t} f(t)dt$.

In each case the variance again has the form $V = A(2\delta) - A(\delta)^2$.

C.3. First calculate the two factors

$$ \begin{aligned} {}_tV_{x:\overline{n}|}^{\;\;1} &= A_{x+t:\overline{n-t}|}^{\;\;\;\;\;\;1} - P_{x:\overline{n}|}^{\;\;1}\ddot{a}_{x+t:\overline{n-t}|} = A_{x+t:\overline{n-t}|}^{\;\;\;\;\;\;1} - \frac{A_{x:\overline{n}|}^{\;\;1}}{\ddot{a}_{x:\overline{n}|}}\ddot{a}_{x+t:\overline{n-t}|} \\ &= v^{n-t}{}_{n-t}p_{x+t} - \frac{\ddot{a}_{x+t:\overline{n-t}|}}{\ddot{a}_{x:\overline{n}|}} v^n {}_np_x \end{aligned} $$

and

$$P_{x:\overline{t}|}^{\ 1} = \frac{A_{x:\overline{t}|}^{\ 1}}{\ddot{a}_{x:\overline{t}|}} = \frac{1}{\ddot{a}_{x:\overline{t}|}} v^t {}_t p_x.$$

Now multiply these two and, for convenience, their denominators $\ddot{a}_{x:\overline{n}|}\ddot{a}_{x:\overline{t}|}$ to get

$$\begin{aligned} {}_tV_{x:\overline{n}|}^{\ 1} P_{x:\overline{t}|}^{\ 1} \ddot{a}_{x:\overline{n}|}\ddot{a}_{x:\overline{t}|} &= v^n {}_n p_x \ddot{a}_{x:\overline{n}|} - v^{n+t} {}_n p_x {}_t p_x \ddot{a}_{x+t:\overline{n-t}|} \\ &= v^n {}_n p_x \left(\ddot{a}_{x:\overline{n}|} - v^t {}_t p_x \ddot{a}_{x+t:\overline{n-t}|}\right) \\ &= v^n {}_n p_x \left(\ddot{a}_{x:\overline{n}|} - (\ddot{a}_{x:\overline{n}|} - \ddot{a}_{x:\overline{t}|})\right) \\ &= v^n {}_n p_x \ddot{a}_{x:\overline{t}|} = A_{x:\overline{n}|}^{\ 1} \ddot{a}_{x:\overline{t}|}, \end{aligned}$$

and we see that

$${}_tV_{x:\overline{n}|}^{\ 1} P_{x:\overline{t}|}^{\ 1} = \frac{A_{x:\overline{n}|}^{\ 1}}{\ddot{a}_{x:\overline{n}|}} = P_{x:\overline{n}|}^{\ 1}$$

does not depend on $t$.

By general reasoning, ${}_tV_{x:\overline{n}|}^{\ 1}$ is the amount that (on average) can be paid to the holders of $n$-year pure endowments after $t$ years. If they are paid out this reserve, they realise a $t$-year pure endowment with sum assured ${}_tV_{x:\overline{n}|}^{\ 1}$ which would have required an annual premium of ${}_tV_{x:\overline{n}|}^{\ 1} P_{x:\overline{t}|}^{\ 1}$ whereas they paid $P_{x:\overline{n}|}^{\ 1}$. Therefore, the two must be equal.