S.1 Time value of money

A.1. There are four payments which accumulate under compound interest to

$$100(1.09) + 100(1.09)^{3/4} + 100(1.09)^{1/2} + 100(1.09)^{1/4} = 422.26$$

A.2. (a) $i = e^{\delta} - 1 \approx 7.788\%$

(b) $i = e^{\delta} - 1 = (1 - v)/v = d/(1 - d) \approx 9.890\%$

(c) $i = (1 + i^{(2)}/2)^2 - 1 = 8.16\%$

(d) $i = (1 + i^{(12)}/12)^{12} - 1 \approx 9.381\%$

A.3. Just calculate from the definition of the accumulation factor

$$\mathrm{Val}_2((0, 1000)) = 1,000 \exp\left\{\int_0^2 0.06(t + 1)dt\right\} = 1,000 \exp\{0.24\} = 1,271.25$$

A.4. (i) Present value of quarterly payments in arrears, of £0.25 each for 67 years:

$$a^{(4)}_{\overline{67}|} = \sum_{r=1}^{67\times 4} \frac{1}{4}(1 + i)^{-r/4} = \frac{1}{4}(1 + i)^{-1/4}\frac{1 - (1 + i)^{-67}}{1 - (1 + i)^{-1/4}} = 23.5391.$$

Notice also $a^{(4)}_{\overline{67}|} = \frac{i}{i^{(4)}} a_{\overline{67}|}$, See Garrett, section 4.2 for a fuller explanation.

(ii) Terminal accumulated value of monthly payments in arrears, of £1/12 each for 18 years:

$$s^{(12)}_{\overline{18}|} = (1 + i)^{18} a^{(12)}_{\overline{18}|} = (1 + i)^{18}\frac{1}{12}(1 + i)^{-1/12}\frac{1 - (1 + i)^{-18}}{1 - (1 + i)^{-1/12}} = 26.1122.$$

(iii) Present value of quarterly payments in advance, of £0.25 each for 16.5 years (198 months):

$$\ddot{a}^{(4)}_{\overline{16.5}|} = \sum_{r=0}^{16.5\times 4-1} \frac{1}{4}(1 + i)^{-r/4} = \frac{1}{4}\frac{1 - (1 + i)^{-16.5}}{1 - (1 + i)^{-1/4}} = 12.2078.$$

(iv) Accumulated value of monthly payments in advance, of £1/12 each for 15.25 years (61 quarters):

$$\ddot{s}^{(12)}_{\overline{15.25}|} = (1 + i)^{15.25}\ddot{a}^{(12)}_{\overline{15.25}|} = (1 + i)^{15.25}\frac{1}{12}\frac{1 - (1 + i)^{-15.25}}{1 - (1 + i)^{-1/12}} = 20.9079.$$

B.1. The fund is to provide three payments of amount X, say, at times 4, 12 and 16, if we define the death time as time 0 and the time unit as half-years. The sum of discounted present values of the three future payments must equal the present value of the fund which is £50,000:

$$(1.06)^{-4}X + (1.06)^{-12}X + (1.06)^{-16}X = 50,000 \Rightarrow X = 29,713.99$$

Each of the children obtain £29.713.99 when they turn 21.

This is not "fair" if there is inflation, as one would expect. Money will be worth less when the younger ones turn 21.

B.2. (i) One easily derives the following table from the definitions.

	δ	i	v	d
$\delta =$	δ	$\log(1+i)$	$-\log(v)$	$-\log(1-d)$
$i =$	$e^{\delta}-1$	i	$\frac{1-v}{v}$	$\frac{d}{1-d}$
$v =$	$e^{-\delta}$	$\frac{1}{1+i}$	v	$1-d$
$d =$	$1-e^{-\delta}$	$\frac{i}{1+i}$	$1-v$	d

(ii) $d = vi$ means that a discount of d at time 0 has the same value as interest of i at time 1.

(iii) $1+i = e^{\delta} \Rightarrow \delta = \log(1+i) = i - i^2/2 + o(i^2)$ by log expansion,
$d = vi = i/(1+i) = i(1-i+o(i)) = i - i^2 + o(i^2)$ by geometric expansion,
$d = 1 - v = 1 - e^{-\delta} = 1 - (1 - \delta + \delta^2/2 + o(\delta^2)) = \delta - \delta^2/2 + o(\delta^2)$ by exp expansion.

B.3. We calculate from the definition of the discount factor

$$\begin{aligned} v(t) &= \exp\left\{-\int_0^t \delta(y)dy\right\} \\ &= \exp\left\{-\int_0^t \left(p+s-\frac{rse^{sy}}{1+re^{sy}}\right) dy\right\} \\ &= \exp\{-(p+s)t\}\,\frac{1+re^{st}}{1+r} \\ &= \frac{1}{1+r}e^{-(p+s)t} + \frac{r}{1+r}e^{-pt} \end{aligned}$$

and we can read off $v_1 = e^{-(p+s)}$, $v_2 = e^{-p}$ and $\lambda = 1/(1+r)$.

This means that cash-flow valuations in this time-dependent interest model can be made by calculating weighted averages of fixed rate models.

B.4. (i) We transform the definition of $v(t)$ and differentiate

$$\int_0^t \delta(s)ds = -\log v(t) \quad \Rightarrow \quad \delta(t) = -\frac{v'(t)}{v(t)} = \frac{2t+2\alpha+1}{(t+\alpha)(t+\alpha+1)}$$

(ii) The effective rate of interest for the period from time n to time $n+1$ is given by

$$\begin{aligned} \exp\left\{\int_n^{n+1} \delta(t)dt\right\} - 1 &= \exp\left\{\int_0^{n+1} \delta(t)dt - \int_0^n \delta(t)dt\right\} - 1 \\ &= \frac{v(n)}{v(n+1)} - 1 = \frac{2}{n+\alpha} \end{aligned}$$

(iii) The required present value is given by

$$\sum_{r=1}^{n} v(r) = \sum_{r=1}^{n} \frac{\alpha(\alpha+1)}{(\alpha+r)(\alpha+r+1)}$$

The trick is to split the fraction

$$= \alpha(\alpha+1)\sum_{r=1}^{n}\left(\frac{1}{r+\alpha} - \frac{1}{r+\alpha+1}\right) = \frac{n\alpha}{n+\alpha+1}.$$

(iv) Let P be the level annual premium. The present value of 12 annual premium payments starting with year 0 is $P(1+a(11))$. The present value of the annuity paying £1,800 from year 12 to 21 is $1,800(a(21) - a(11))$. These have to be equal for the premium to be fair:

$$P(1 + a(11)) = 1,800(a(21) - a(11)) \quad \Rightarrow \quad P = 608.11$$

The annual premium is £608.11.

The value of the annuity at time 0 is $P(1 + a(11)) = 4,324.34$. For the value at time 12 we just divide by $v(12)$ to obtain $\pounds 13,621.66$.

B.5. The second investment gives a rate of interest i given by

$$4,000(1+i)^4 = 4,400 \quad \Rightarrow \quad i = (4400/4000)^{1/4} - 1 \approx 0.024 = 2.4\%$$

For the first investment, we cannot calculate the rate of interest explicitly, but we can see that if it was $i = 2.4\%$, we'd have

$$-4,000(1+i)^6 + 2,000(1+i)^4 + 2,400 = -14.76 < 0$$

so the actual rate must be lower to break even. Therefore, the second investment gives the higher rate of interest.

B.6. We need to solve $-X(1+y)^{-s} + Y(1+y)^{-t} = 0$. This gives $y = (Y/X)^{1/(t-s)} - 1$. As you would expect, the yield increases as Y increases or X decreases. If $Y > X$ the yield is positive, and then it decreases with the length of the term $t - s$; if $Y < X$ then the yield is negative and it increases towards zero as $t - s$ increases.

B.7. Let C be the advertised price. Then the equation of discounted values at time 0 is

$$0.95C = \frac{C}{15}(1.05)12a^{(12)}_{\overline{1.25}|i}$$

or

$$\frac{0.95}{1.05} \times 15 = \frac{1 - (1+i)^{-1.25}}{(1+i)^{1/12} - 1}$$

or

$$f(i) := \frac{1-(1+i)^{-1.25}}{(1+i)^{1/12}-1} - 13.5714 = 0.$$

The function $f(i)$ is decreasing in i and gives

$$f(10\%) = 0.5134, \quad f(20\%) = -0.2595$$

and by linear interpolation

$$i \approx \frac{20\% f(10\%) - 10\% f(20\%)}{f(10\%) - f(20\%)} \approx 16.6\%$$

We quote this as an approximate answer (or check $f(16.6\%) \approx -0.0137$, which is pretty good compared to $f(10\%)$ and $f(20\%)$, indeed it can be shown that 16.5% is correct to 1 d.p.)

C.1. (a) Let $\ddot{a}^*_{\overline{n}|i}$ denote the value of this annuity at an annual rate of interest i. We have

$$\ddot{a}^*_{\overline{n}|i} = \sum_{k=0}^{n-1}\left(\frac{1+r}{1+i}\right)^k = \sum_{k=0}^{n-1}(1+j)^{-k} = \ddot{a}_{\overline{n}|j}.$$

since

$$1+j = 1 + \frac{i-r}{1+r} = \frac{1+i}{1+r}.$$

(b) Let $a^*_{\overline{n}|i}$ denote the value of this annuity at an annual rate of interest i. We have

$$a^*_{\overline{n}|i} = \sum_{k=1}^{n}\frac{(1+r)^{k-1}}{(1+i)^k} = \frac{1}{1+r}\sum_{k=1}^{n}(1+j)^{-k} = \frac{1}{1+r}a_{\overline{n}|j}.$$

Hence, the present value of this annuity is not equal to $a_{\overline{n}|j}$ (unless $r = 0$).

(c) Let the first annuity payment be X. The equation of value is

$$10,000 = X(1.05)^{-1}a_{\overline{20}|j}$$

where

$$1+j = \frac{1.09}{1.05} \qquad \Rightarrow \qquad j = 0.03810.$$

Now $a_{\overline{20}|j} = 13.822455$ so that

$$10,000 = X(1.05)^{-1}a_{\overline{20}|j} \qquad \Rightarrow \qquad X = £759.63.$$

C.2. The income stream corresponds to interest at effective rate X/P paid on the investment P at the end of each time unit, with the capital P repaid at time n. So the yield is simply X/P. This can be verified formally from the yield equation.

C.3. The values of the increasing annuity at time 0 and 1 are given by

$$(Ia)_{\overline{n}|} = \sum_{k=1}^{n} kv^k \qquad \text{and} \qquad (1+i)(Ia)_{\overline{n}|} = \sum_{j=0}^{n-1} (j+1)v^j.$$

Subtracting the two, we obtain

$$i(Ia)_{\overline{n}|} = \ddot{a}_{\overline{n}|} - nv^n \quad \Rightarrow \quad (Ia)_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{i}.$$