P.3 Utility and risk

Remember, only section B questions need to be handed in for marking.

A.1. Consider the family of exponential utility functions $u(w) = \frac{1-\exp(-aw)}{a}$

(a) Show that a is the coefficient of absolute risk aversion for all levels of wealth.

(b) Show that u becomes linear in w when a tends to zero.

(c) Consider lottery $\tilde{x}$ with positive and negative payoffs. Determine the value of $\mathbb{E}u(\tilde{x})$ when a tends to infinity.

A.2. Let $u = w^2$ for $w \geq 0$.

(a) Compute the exact risk premium if initial wealth is 4 and if a decision maker faces the lottery $(-2, \frac{1}{2}; +2, \frac{1}{2})$. Explain why the risk premium is negative.

(b) If the utility function becomes $v = w^4$, what happens to the risk premium? Show that v is a convex transformation of u.

A.3. Show that

$$u'(z) = k \exp\left[-\int^z A(w)dw\right]$$

where $A(w)$ is the Arrow-Pratt coefficient of absolute risk aversion.

B.1. Suppose that a consumer's income, $\tilde{y}$, is a random variable with density function $f(y) = \lambda y^{\lambda-1}$, $\lambda > 0$ if $0 \leq y \leq 1$ and $f(y) = 0$ otherwise.

(a) Calculate the mean and variance of income.

(b) What is the probability that income is greater than 0.5?

(c) If the utility of income is $u(y) = y^{1-r}$ where $0 < r < 1$, show that the consumer is risk averse, calculate his expected utility, and compare it with the utility of expected income.

B.2. An insurer issues n identical policies. Let Y_j be the claim amount from the jth policy, and suppose that the random variables Y_j, $j = 1, \ldots, n$ are i.i.d. with mean $\mu > 0$ and variance σ^2. The insurer charges a premium of A for each policy.

(a) Show that if $A = \mu + 10\sigma n^{-1/2}$, then the probability that total claims exceed total premiums is no more than 1%, for any value of n.

(b) Use the Central Limit Theorem to show that if instead $A = \mu + 3\sigma n^{-1/2}$, then this probability is still less than 1%, provided n is large enough.

B.3. Consider an individual with utility function $u(w) = w^{1/2}$. Her initial wealth is 10 and she faces the lottery $\tilde{x} : \left(-6, \frac{1}{2}; +6, \frac{1}{2}\right)$.

(a) Compute the exact values of this individual's certainty equivalent and risk premium for lottery $\tilde{x}$.

(b) Obtain the Arrow-Pratt approximation of the risk premium.

(c) Show that with this utility function absolute risk aversion is decreasing while the relative risk aversion is constant in (initial) wealth.

(d) If the utility becomes $v(w) = w^{1/4}$ again answer part (a). Are you surprised by the changes in certainty equivalent and risk premium? Relate this change to the notion of 'more risk averse'.

(e) If the risk becomes $\tilde{y} : \left(-3, \frac{1}{2}; +3, \frac{1}{2}\right)$, compute the new (approximate) risk premium. Why is the approximated risk premium four times smaller than the risk premium for $\tilde{x}$.

B.4. Suppose that a decision maker has a utility function that satisfies constant relative risk aversion. Show that the decision maker's preference over certain wealth of w and lottery $w(1 + \tilde{x})$ does not depend on $w \geq 0$, where discrete random variable $\tilde{x}$ takes values $\{x_i, x_i > -1\}_{i=1,\dots,n}$ with probabilities $\{p_i\}_{i=1,\dots,n}$.

B.5. Consider the following two random variables: $\tilde{x}$ has a (continuous) uniform density on the interval $[-1, +1]$, while $\tilde{y}$ is a discrete random variable defined by $(-1, \frac{1}{2}; +1, \frac{1}{2})$.

(a) Draw the cumulative distributions of $\tilde{x}$ and $\tilde{y}$.

(b) By applying the Rothschild-Stiglitz 'integral conditions' or otherwise determine which random variable is riskier.

(c) Find the distributions of the 'white noise' that must be added to the less risky lottery to obtain the riskier one.

B.6. A corporation must decide between two mutually exclusive projects. Both projects require an initial outlay of £100 million, and they generate cash flows that are independent of the growth of the economy. Project A has an equal probability of four gross payoffs: £80 million; £100 million; £120 million; or £140 million. Project B has a 50:50 chance of paying either £90 million or £130 million. Assuming that shareholders are all risk averse, show that they unanimously prefer Project B to Project A.

B.7. Consider two lotteries L_a and L_b. The outcomes of lottery L_a are uniformly distributed on the unit interval. The probability density function of L_b is given by $g(x) = c\left(x - \frac{1}{2}\right)^2$ for $x \in [0, 1]$, and $g(x) = 0$ otherwise.

(a) Show that lottery L_b is a mean-preserving spread of lottery L_a.

(b) Does one of the lotteries L_a and L_b first-order stochastically dominate the other?

C.1. If it held for all risks, the Arrow-Pratt approximation for a small zero mean risk $\tilde{y} = k\tilde{x}$, $\pi(w_0, u, \tilde{y}) \simeq 0.5\mathbb{E}\tilde{y}^2 A(w_0)$, would make the EUT model a particular case of the mean-variance (MV) model. The theory of finance usually limits the analysis to the mean and variance by restricting the set of utility functions and distributions that make the Arrow-Pratt approximation exact. Show that the Arrow-Pratt approximation is exact when the utility function is CARA and $\tilde{y}$ is normally distributed.

C.2. An investor has wealth to invest in a set of n independent and identically distributed lotteries, $\tilde{x}_1, \ldots, \tilde{x}_n$. Let $\alpha_i \in [0, 1]$ denote the share of wealth invested in lottery i, where $\sum_{i=1}^{n} \alpha_i = 1$. Show that the distribution of final wealth generated by the perfect diversification strategy $\alpha = (\frac{1}{n}, \ldots, \frac{1}{n})$ second order stochastically dominates the distribution of final wealth generated by any feasible strategy.

[*Hint: Consider adding* $\sum_{i=1}^{n}(\alpha_i - \frac{1}{n})\tilde{x}_i$ *to each potential outcome of the perfect diversification strategy.*]

C.3. Consider the utility function

$$u(z) = \begin{cases} z & \text{if } z \leq z_0 \\ z_0 + a(z - z_0) & \text{if } z > z_0 \end{cases}.$$

where $0 < a < 1$.

(a) Show that u is concave.

(b) Show that u exhibits first-order risk aversion at $z = z_0$, namely that the risk premium $\pi(z_0, u, k\tilde{x})$ tends to zero when k tends to zero as k, rather than as k^2.

(c) Show that a reduction in the constant a increases the degree of risk aversion.

(d) Does it satisfy DARA?