P.2 Lifetime distributions and life products

Remember, only section B questions need to be handed in for marking.

A.1. Describe the benefit which has the present value random variable function given by Z below; T denotes the future lifetime of a life aged x.

$$Z = \begin{cases} \bar{a}_{\overline{n}|} & T \leq n \\ \bar{a}_{\overline{T}|} & T > n \end{cases}$$

A.2. Consider the interest model of an effective annual rate of $i > 0$.

(i) Consider a lifetime random variable K with geometric distribution

$$\mathbb{P}(K = k) = (1-q)^{k-1}q, \qquad k \geq 1.$$

Show that the price of a whole-life annuity is $(1-q)/(i+q)$.

(ii) Consider a discounted dividend model, in which dividend payments D_k, $k \geq 1$, are such that $\mathbb{E}(D_k) = (1+g)^k$, $k \geq 1$. Show that the total expected present value of all dividend payments is $(1+g)/(i-g)$.

(iii) Consider a discounted dividend model for company shares, in which the company has a geometrically distributed insolvency time K. Dividends satisfy $\mathbb{E}(D_k|K \geq k) = (1+h)^k$. Show that the total expected present value of all dividend payments is $(1+h)(1-q)/(i+q-h(1-q))$.

A.3. Prove that ${}_tp_x = \exp\left\{-\int_0^t \mu_{x+s} ds\right\}$.

A.4. Gompertz's Law has $\mu_x^{(1)} = Bc^x$.

(i) Show that the corresponding survival function is given by ${}_tp_x^{(1)} = g^{c^x(c^t-1)}$ where $\log g = -B/\log c$.

The force of mortality for table 2 has twice the force of mortality for table 1.

(ii) Show that the probability of survival for n years under table 2 is the square of that under table 1.

(iii) Suppose that table 1 follows the Gompertz Law. Show that the probability of survival for n years for a life aged x under table 2 is the same as that under table 1 for a life aged $x+a$, for some $a > 0$. Find a. Comment on the result.

B.1. (i) Makeham's Law has $\mu_x^{(2)} = A + Bc^x$. Show that ${}_tp_x^{(2)} = s^t {}_tp_x^{(1)}$ where $s = e^{-A}$.

(ii) If it is assumed that A1967-70 table follows Makeham's Law, use ℓ_{30}, ℓ_{40}, ℓ_{50} and ℓ_{60} to find A, B and c. ($\ell_{30} = 33839$, $\ell_{40} = 33542$, and $\ell_{50} = 32670$, $\ell_{60} = 30040$).

B.2. (i) Show algebraically that $A_{x:\overline{n}|} = v\ddot{a}_{x:\overline{n}|} - a_{x:\overline{n-1}|}$. Also demonstrate the result verbally.

(ii) Show algebraically that $(IA)_x = \ddot{a}_x - d(I\ddot{a})_x$. Demonstrate the result verbally.

B.3. A cash-flow is payable continuously at a rate of $\rho(t)$ per annum at time t provided a life who is aged x at time 0 is still alive. T_x is a random variable which models the residual lifetime in years of a life aged x.

(a) Write down an expression, in terms of T_x, for the (random) present value at time 0 of this cash-flow, at a constant force of interest δ p.a., and show that the expected present value at time 0 of the cash-flow is equal to

$$\int_0^\infty e^{-\delta s}\rho(s)\mathbb{P}(T_x > s)ds.$$

(b) An annuity is payable continuously during the lifetime of a life now aged 30, but for at most 10 years. The rate of payment at all times t during the first 5 years is £5,000 p.a., and thereafter £10,000 p.a. The force of mortality to which this life is subject is assumed to be 0.01 p.a. at all ages between 30 and 35, and 0.02 p.a. between 35 and 40. Find the expected present value of this annuity at a force of interest of 0.05 p.a.

(c) If the mortality and interest assumptions are as in (b), find the expected present value of the benefits of a term assurance, issued to the life in (ii), which pays £40,000 immediately on death within 10 years.

B.4. A deferred annuity is purchased by 20 annual payments payable by a life aged 40 for a year annuity in advance of £2,500 a year, commencing in 20 years, for life. Find an expression for the premium on the basis of 4% pa interest with expenses of 5% of each premium and £5 at each annuity payment.

B.5. Suppose that $l_x = 100,000(100 - x)$, where $0 \le x \le 100$, and the interest rate is 5%.

(a) Calculate $A_{50:\overline{10}|}$ and $\ddot{a}_{50:\overline{10}|}$.

(b) Calculate the net annual premium for a 10 year endowment assurance for £25,000 to someone aged 50 and the policy values of years 3 and 4 using the values above.

(c) Suppose that expenses are as follows

Commission:	50% of First Premium
	2% of Subsequent Premiums
General Expenses:	£300 Initially
	£50 in each subsequent year

Calculate the office premium for the policy in (b).

B.6. (i) Show that $\overline{A}_x = 1 - \delta \overline{a}_x$.

(ii) Consider a whole life assurance with sum assured 1 payable at the point of death, with a constant premium paid continuously. Show that the reserve at time t satisfies

$$_t\overline{V}_x = 1 - \frac{\overline{a}_{x+t}}{\overline{a}_x}$$

C.1. (a) Show that

$$\ddot{a}_x = 1 + (1+i)^{-1} p_x \ddot{a}_{x+1}.$$

(b) Show that

$$\ddot{a}_{x:\overline{n}|} = a_{x:\overline{n}|} + 1 - A_{x:\overline{n}|}^{\;\;1}.$$

(c) Show that

$$A_x = (1+i)^{-1}\ddot{a}_x - a_x.$$

C.2. Given a curtate lifetime $K = [T]$, consider the following insurance products:

(i) pure endowment with term n;

(ii) whole life assurance;

(iii) term assurance with term n;

(iv) endowment assurance with term n (under this product, there is a payment of one unit at the end of the year of death or at end of term whichever is earlier).

Also consider the constant-δ interest model.

(a) Write each one as a random cash-flow of the form $C = \big((t_k, c_k B_k)\big)_{k=1,2,\ldots}$, where the B_k are Bernoulli random variables defined in terms of K.

(b) Find expressions for the net premiums and variances of these products.

(c) Relate the products, premiums and variances of (iv) and (i) and (iii).

(d) Comment on (a) and (b) for the corresponding products where payment is made at death rather than at the end of the year of death (and n is not necessarily an integer).

C.3. Show algebraically that the product of

- the reserve after t years for an annual premium n-year pure endowment issued to a life aged x

and

- the annual premium for pure endowment of like amount issued at the same age but maturing in t years

is constant for all values of t. (Ignore expenses.)

Try to find an argument for this by general reasoning also.