# P.1 Time value of money

*Assignment sheets have 3 sections. Section A is meant to be straightforward familiarisation with definitions, Questions in Sections B and C are more varied in difficulty.* **Please hand in your answers to problems in Section B only.** *Solutions to all questions will be provided.*

A.1. If the effective annual rate of interest is 9%, calculate the total accumulated value at 1 January 2010 of payments of £100 made on 1 January 2009, 1 April 2009, 1 July 2009 and 1 October 2009.

A.2. Calculate the equivalent effective annual rate of interest of

(a) a force of interest of 7.5% p.a.

(b) a discount rate of 9% p.a.

(c) a nominal rate of interest of 8% p.a. convertible half-yearly

(d) a nominal rate of interest of 9% p.a. convertible monthly.

A.3. Calculate the accumulated value of £1,000 after 2 years if $\delta(t) = 0.06(t+1)$ for $0 \le t \le 2$.

A.4. Find, on the basis of an effective interest rate of 4% per unit time, the values of

$$a^{(4)}_{\overline{67}|}, \quad s^{(12)}_{\overline{18}|}, \quad \ddot{a}^{(4)}_{\overline{16.5}|}, \quad \ddot{s}^{(12)}_{\overline{15.25}|}$$

Describe the meaning of each of the symbols.

B.1. A man stipulates in his will that £50,000 from his estate is to be placed in a fund from which his three children are each to receive an equal amount when they reach age 21. When the man dies, the children are ages 19, 15 and 13. If this fund earns 6% interest per half-year (i.e. nominal 12% p.a. compounded each 6 months), what is the amount that they each receive? Is the distribution fair?

B.2. (i) Establish a table of relationships between (constant) $\delta$, $i$, $v$ and $d$.

(ii) Show that $d = vi$ and interpret this.

(iii) Show that $\delta \approx i - i^2/2$ and $d \approx i - i^2$ for small $i$, and that $d \approx \delta - \delta^2/2$ for small $\delta$.

B.3. *Stoodley's formula.* Suppose the force of interest is given by

$$\delta(t) = p + \frac{s}{1 + re^{st}}, \qquad t \in \mathbb{R}_+$$

where $p, r \in \mathbb{R}_+$ and $s \ge -p$.

Calculate the discount factor $v(t)$, $t \in \mathbb{R}_+$, and show that the model can be reparametrized such that $v(t) = \lambda v_1^t + (1-\lambda)v_2^t$. Interpret this!

B.4. In valuing future payments an investor uses the formula

$$v(t) = \frac{\alpha(\alpha+1)}{(\alpha+t)(\alpha+t+1)}, \qquad t \in \mathbb{R}_+$$

where $\alpha$ is a given positive constant, for the value at time 0 of 1 due at time $t$ (measured in years).

Show that the above formula implies that

(i) the force of interest per annum at time $t$ will be

$$\delta(t) = \frac{2t+2\alpha+1}{(\alpha+t)(\alpha+t+1)},$$

(ii) the effective rate of interest for the period $r$ to $r+1$ will be $i(r) = \dfrac{2}{r+\alpha}$,

(iii) the present value of a series of $n$ payments, each of amount 1 (the $r$th payment being due at time $r$) is

$$a(n) = \frac{n\alpha}{n+\alpha+1}.$$

(iv) Suppose now that $\alpha = 15$. Find the level annual premium, payable in advance for twelve years, which will provide an annuity of £1,800 per annum, payable annually for ten years, the first annuity payment being made one year after payment of the final premium. What is the value at time 12 of the series of annuity payments, what at time 0?

B.5. Under the terms of a savings scheme an investor who makes an initial investment of £4,000 may receive either

- £2,000 after 2 years and a further £2,400 after 6 years; or
- £4,400 at the end of 4 years.

Which of these options corresponds to a higher rate of interest on the investor's money?

B.6. Find the yield of an investment of $X$ at time $s$, followed by a single payment of $Y$ received at time $t > s$. How does the yield change as $X$, $Y$, $s$, $t$ vary?

B.7. Buy now, pay later!! A luxury sofa is sold with the following payment options: either pay immediately and receive a 5% discount, or pay by monthly instalments (in arrears) for 15 months. To calculate the monthly payments, add a 5% finance charge to the purchase price, then divide the total amount into 15 equal payments. What annual effective rate of interest is being charged for paying by instalments?

C.1. (a) An annuity-certain is payable annually in advance for $n$ years. The first payment of the annuity is 1. Thereafter the amount of each payment is $(1+r)$ times that of the preceding payment.

Show that, on the basis of an interest rate of $i$ per annum, the present value of the annuity is $\ddot{a}_{\overline{n}|j}$ where $j = (i-r)/(1+r)$.

(b) Suppose instead that the annuity is payable annually in arrear. Is its present value (at rate $i$) now equal to $a_{\overline{n}|j}$?

(c) In return for a single premium of £10,000 an investor will receive an annuity payable annually in arrear for 20 years. The annuity payments increase from year to year at the (compound) rate of 5% per annum.

Given that the initial amount of the annuity is determined on the basis of an interest rate of 9% per annum, find the amount of the first payment.

C.2. Find the yield of a bond costing $P$ at time 0, which generates coupon payments of $X$ at times $1, 2, \ldots, n$ and a redemption payment of $P$ at time $n$.

C.3. The discounted present value of an *increasing annuity* with payments $j$ at time $j = 1, \ldots, n$ is denoted by $(Ia)_{\overline{n}|}$. Express $(Ia)_{\overline{n}|}$ in terms of ordinary annuity symbols.