INTERVAL PARTITION EVOLUTIONS WITH EMIGRATION RELATED TO THE ALDOUS DIFFUSION

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Abstract. We construct a stationary Markov process corresponding to the evolution of masses and distances of subtrees along the spine from the root to a branch point in a conjectured stationary, continuum random tree-valued diffusion that was proposed by David Aldous. As a corollary this Markov process induces a recurrent extension, with Dirichlet stationary distribution, of a Wright–Fisher diffusion for which zero is an exit boundary of the coordinate processes. This extends previous work of Pal who argued a Wright–Fisher limit for the three-mass process under the conjectured Aldous diffusion until the disappearance of the branch point. In particular, the construction here yields the first stationary, Markovian projection of the conjectured diffusion. Our construction follows from that of a pair of interval partition-valued diffusions that were previously introduced by the current authors as continuum analogues of down-up chains on ordered Chinese restaurants with parameters $\left(\frac{1}{2}, \frac{1}{2}\right)$ and $\left(\frac{1}{2}, 0\right)$. These two diffusions are given by an underlying Crump–Mode–Jagers branching process, respectively with or without immigration. In particular, we adapt the previous construction to build a continuum analogue of a down-up ordered Chinese restaurant process with the unusual parameters $\left(\frac{1}{2}, -\frac{1}{2}\right)$, for which the underlying branching process has emigration.

1. Introduction

The Aldous chain is a Markov chain on the space of rooted binary trees with n labeled leaves. Each transition of the Aldous chain, called a down-up move, has two steps. In the down-move a uniform random leaf is deleted and its parent branch point is contracted away. In the up-move a uniform random edge is selected, a branch point is inserted into the middle of the edge, and the leaf is reattached at that point. See Figure 1. David Aldous [5] studied the analogue of this chain on unrooted trees.

The unique stationary distribution of the Aldous chain on rooted n-leaf labeled binary trees is the uniform distribution. Consider an n-leaf binary tree as a metric space where each edge has a length of $1/\sqrt{n}$. Then the scaling limit of the sequence of uniform n-leaf binary trees, as n tends to infinity, is the Brownian Continuum Random Tree (CRT) [1]. This fundamental limiting random metric space can alternatively be described as being encoded by a Brownian excursion. Aldous [3, 5] conjectured a "diffusion on continuum trees" that can be thought of as a continuum analogue of the Aldous Markov chain.

In order to understand this difficult and abstract conjectured diffusion, it is natural to search for simpler "finite-dimensional projections" that are also Markovian and easier to analyze. Such a projection was suggested by Aldous for the Markov chain and later analyzed by Pal [29]. Specifically, suppose $(T_n(j), j \geq 0)$ is the Aldous chain on trees with n leaves. Any branch point naturally partitions the tree $T_n(0)$ into three components. As the Aldous chain runs, leaves move among components until the branch point disappears, i.e. a component becomes empty. Until that time, let $m_i(j)$, $i \in \{1, 2, 3\}$, be the proportions of leaves in these components, with m_3 referring to the root component. Then

$$\left(\left(m_1^{(n)}(\lfloor n^2 u \rfloor), m_2^{(n)}(\lfloor n^2 u \rfloor), m_3^{(n)}(\lfloor n^2 u \rfloor) \right), \, u \geq 0 \right) \xrightarrow[n \to \infty]{d} ((X_1(u), X_2(u), X_3(u)), \, u \geq 0), \quad (1)$$
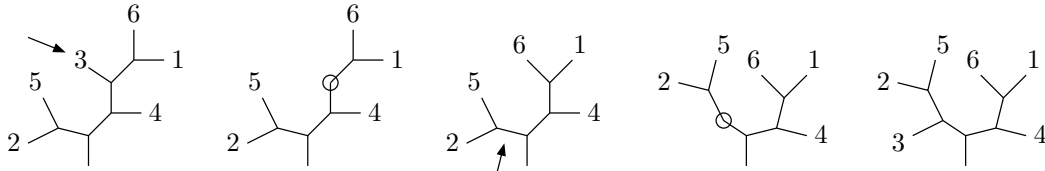
FIGURE 1. From left to right, one Aldous down-up move.

where the right hand side is a generalized Wright–Fisher diffusion with mutation rate parameters $(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$, stopped when one of the first two coordinates vanishes. Since zero is an exit boundary for the coordinates of a Wright–Fisher diffusion that have negative mutation rates, the limiting process does not shed light on how to continue beyond the disappearance of a branch point.

In our previous work on the discrete Aldous chain [14] we have provided a natural mechanism for selecting a new branch point when the old one disappears, in such a way that the projected mass evolutions remain Markovian. The primary purpose of this paper is to construct a diffusion analogue of this strategy for the case of one branch point. To this end, we construct a process on a space of interval partitions as in [11], which can be projected down onto a three-mass process, and which has the added benefit of describing certain lengths in the conjectured CRT-valued diffusion. For the discrete chain, this idea was described in [14, Appendix A].

An *interval partition* (IP) in the sense of [4, 32] is a set β of disjoint, open subintervals of some interval $[0, M]$, that cover $[0, M]$ up to a Lebesgue null set. We refer to $M \geq 0$ as the *mass* of β and generally use notation $\|\beta\|$ for M. We refer to the subintervals comprising the interval partition as its *blocks*. We denote by $\mathcal{I}_H$ the set of all interval partitions and by $\mathcal{I}$ the subset of "Brownian-like" interval partitions β *with diversity* [31], i.e. for which the limit

$$\mathscr{D}_\beta(t) = \sqrt{\pi} \lim_{h \downarrow 0} \sqrt{h} \#\{(a, b) \in \beta \colon |b - a| > h, b \leq t\} \tag{2}$$

exists for all $t \in [0, \|\beta\|]$. In the context of a rooted Brownian CRT $(\mathcal{T}, d, \rho, \mu)$ with root ρ, a "2-tree" with an associated interval partition can be extracted as follows. We independently sample two leaves $\Sigma_1, \Sigma_2 \sim \mu$. Consider the geodesic paths $[[\rho, \Sigma_1]], [[\rho, \Sigma_2]] \subset \mathcal{T}$, their intersection $[[\rho, b_{1,2}]] = [[\rho, \Sigma_1]] \cap [[\rho, \Sigma_2]]$, which defines a branch point $b_{1,2} \in \mathcal{T}$, and the masses $\mu(\mathcal{C})$ of all connected components $\mathcal{C}$ of $\mathcal{T} \setminus [[\rho, b_{1,2}]]$. Record (X_1, X_2, β), where X_1 and X_2 are the μ-masses of the connected components containing Σ_1 and Σ_2, respectively, and β is the *spinal interval partition* of total mass $1 - X_1 - X_2$ that captures in its interval lengths the μ-masses of the remaining components, in the order of decreasing distance from ρ. It is well-known [2, 33] that $(X_1, X_2, 1 - X_1 - X_2)$ has law $\mathtt{Dir}\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$, and $\beta/(1 - X_1 - X_2)$ is independent with law $\mathtt{PDIP}\left(\frac{1}{2}, \frac{1}{2}\right)$. Here $\mathtt{PDIP}\left(\frac{1}{2}, \frac{1}{2}\right)$, which stands for Poisson–Dirichlet Interval Partition, is law of the random interval partition of the unit interval obtained from the excursion intervals of a standard Brownian bridge [30, 35, 18]. Furthermore, the total diversity $\mathscr{D}_\beta(\|\beta\|)$, from (2), is also equal to $d(\rho, b_{1,2})$, the length of the spine from ρ to $b_{1,2}$ in $\mathcal{T}$. Our aim is to construct a Markov process on such 2-tree structures, triplets of an interval partition and two top masses, with total mass one, that is stationary with respect to the law of (X_1, X_2, β) described above.

In [11], the present authors introduced a related IP-valued process called type-1 evolution, or $\left(\frac{1}{2}, 0\right)$-IP evolution. We recall its definition in Section 2, but for now we recall three properties.

(i) It is a path-continuous Hunt process on a space $(\mathcal{I}, d_\mathcal{I})$ with continuously evolving diversities [11, Theorem 1.4]. The metric $d_\mathcal{I}$ is defined in Definition 14.
(ii) The total mass of the interval partition evolves as a $\mathtt{BESQ}(0)$, the squared-Bessel diffusion of dimension 0 [11, Theorem 1.5]. In particular, the type-1 evolution is eventually absorbed (we say it dies) at the empty interval partition state, $\emptyset$.
(iii) At Lebesgue almost every time prior to its death, the evolving interval partition has a leftmost block [11, Proposition 4.30, Lemma 5.1].

In this paper we find it convenient to represent a type-1 evolution by a pair, $((m^y, \beta^y), y \geq 0)$, rather than just an evolving interval partition, with m^y denoting the mass of the leftmost block

and β^y denoting the remaining interval partition, shifted down so that its left end lines up with zero. We take the convention that $m^y = 0$ at the exceptional times y at which there is no leftmost block and after the death of the process.

A type-2 evolution, or $(\frac{1}{2}, -\frac{1}{2})$-IP evolution, is a process that has two leftmost blocks at almost every time. We can represent the two leftmost blocks by just their masses and consider such a process on either of the following state spaces:

$$\mathcal{J}^\circ := \left\{ (a, b, \beta) \in [0, \infty)^2 \times \mathcal{I}, \ a + b > 0 \right\} \cup \{(0, 0, \emptyset)\},$$
$$\mathcal{I}^\circ := \{\beta \in \mathcal{I} \colon \exists a > 0 \text{ s.t. } (0, a) \in \beta\} \cup \{\emptyset\} = \left\{ (0, a) \star (0, b) \star \beta \colon (a, b, \beta) \in \mathcal{J}^\circ \right\}. \tag{3}$$

Here $\star$ means a natural concatenation of blocks. Let $d_{\mathcal{J}}$ denote the metric on $\mathcal{J}^\circ$ given by

$$d_{\mathcal{J}} \left((a_1, b_1, \beta_1), (a_2, b_2, \beta_2) \right) = |a_1 - a_2| + |b_1 - b_2| + d_{\mathcal{I}}(\beta_1, \beta_2).$$

Let $\mathtt{BESQ}_a(-1)$ denote the squared Bessel diffusion of dimension -1 starting from $a \geq 0$. This process is killed upon hitting zero. If $\mathbf{f} \sim \mathtt{BESQ}_a(-1)$, for some $a \geq 0$, let $\zeta(\mathbf{f})$ denote the *lifetime* of the process $\mathbf{f}$.

Definition 1. Let $(a, b, \beta) \in \mathcal{J}^\circ$. A type-2 evolution starting from (a, b, β) is a $\mathcal{J}^\circ$-valued process of the form $((m_1^y, m_2^y, \alpha^y), y \geq 0)$, with $(m_1^0, m_2^0, \alpha^0) = (a, b, \beta)$. Its IP-valued variant is a process on state space $\mathcal{I}^\circ$ that starts from $(0, a) \star (0, b) \star \beta$. The distributions of these processes are specified by the following construction.

Let $\left(\mathbf{m}^{(0)}, \gamma^{(0)} \right)$ be a type-1 evolution starting with the initial condition (b, β) and independent of $\mathbf{f}^{(0)} \sim \mathtt{BESQ}_a(-1)$. Let $Y_1 = \zeta(\mathbf{f}^{(0)})$. For $0 \leq y \leq Y_1$, define the type-2 evolution as

$$(m_1^y, m_2^y, \alpha^y) := \left(\mathbf{f}^{(0)}(y), \mathbf{m}^{(0)}(y), \gamma^{(0)}(y) \right), \quad 0 \leq y \leq Y_1,$$

while its IP-valued variant is the process

$$(0, \mathbf{f}^{(0)}(y)) \star (0, \mathbf{m}^{(0)}(y)) \star \gamma^{(0)}(y), \quad 0 \leq y \leq Y_1.$$

Now proceed inductively. Suppose, for some $n \geq 1$, these processes have been constructed until time Y_n with $m_1^{Y_n} + m_2^{Y_n} > 0$. Conditionally given this history, consider a type-1 evolution $(\mathbf{m}^{(n)}, \gamma^{(n)})$ with initial condition $(0, \alpha^{Y_n}) = (0, \gamma^{Y_{n-1}}(Y_n - Y_{n-1}))$ that is independent of $\mathbf{f}^{(n)}$, a $\mathtt{BESQ}(-1)$ diffusion with initial value $\mathbf{m}^{(n-1)}(Y_n - Y_{n-1})$. The latter equals $m_2^{Y_n}$ if n is odd or $m_1^{Y_n}$ if n is even. Set $Y_{n+1} = Y_n + \zeta(\mathbf{f}^{(n)})$. For $y \in (0, Y_{n+1} - Y_n]$, define

$$\left(m_1^{Y_n + y}, m_2^{Y_n + y}, \alpha^{Y_n + y} \right) := \begin{cases} (\mathbf{m}^{(n)}(y), \mathbf{f}^{(n)}(y), \gamma^{(n)}(y)), & \text{if } n \text{ is odd}, \\ (\mathbf{f}^{(n)}(y), \mathbf{m}^{(n)}(y), \gamma^{(n)}(y)), & \text{if } n \text{ is even}. \end{cases}$$

The IP-valued variant of the process does not switch between the top two masses and is always defined as

$$(0, \mathbf{f}^{(n)}(y - Y_n)) \star (0, \mathbf{m}^{(n)}(y - Y_n)) \star \gamma^{(n)}(y - Y_n), \quad Y_n < y \leq Y_{n+1}.$$

If, for some $n \geq 1$, $m_1^{Y_n} + m_2^{Y_n} = 0$, set $(m_1^y, m_2^y, \alpha^y) := (0, 0, \emptyset)$ for all $y > Y_n$ and $Y_{n+1} := \infty$.

The difference between the two variants of type-2 evolutions is that in one the top two masses are labeled by 1 and 2 which jump as a mass hits zero, while in the other the top two masses are unlabeled and simply drop out of the interval partition as empty blocks when they hit zero. The former allows a stationary construction, while the latter is necessary for continuity.

Theorem 2. *Type-2 evolutions are Borel right Markov processes on $(\mathcal{J}^\circ, d_{\mathcal{J}})$. IP-valued type-2 evolutions are path-continuous Hunt processes on $(\mathcal{I}^\circ, d_{\mathcal{I}})$.*

Theorem 3. *For a type-2 evolution $((m_1^y, m_2^y, \alpha^y), y \geq 0)$, the total mass process $(m_1^y + m_2^y + \|\alpha^y\|, y \geq 0)$ is a $\mathtt{BESQ}(-1)$ process.*

Since a $\mathtt{BESQ}(-1)$ process eventually gets killed at zero, a type-2 evolution is not stationary. However, we obtain a stationary variant by modifying the process in two ways: de-Poissonization and resampling. De-Poissonization means that we normalize so that the total mass remains constant at one, and then we apply a time-change. De-Poissonization was used in [11] to obtain
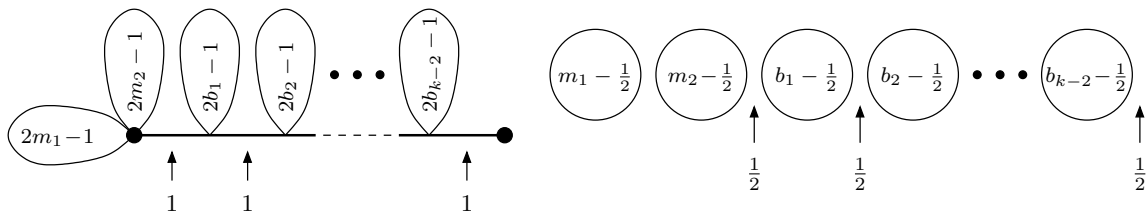
Figure 2. Up-move weights for 2-tree projection of Aldous chain correspond to seating rule for $\mathtt{oCRP}\left(\frac{1}{2}, -\frac{1}{2}\right)$.

a stationary variant of type-1 evolution and has previously been applied in related settings in [28, 29, 39]. Resampling is a new idea in this context. We will see that the type-2 evolution eventually *degenerates*, entering a state of only having a single block: either $m_1^y = \|\alpha^y\| = 0 < m_2^y$ or $m_2^y = \|\alpha^y\| = 0 < m_1^y$. At that time we will have the process jump into an independent state sampled from the law described above as a 2-tree projection of the Brownian CRT; see Definition 43. The state spaces of the resampling de-Poissonized processes are

$$\mathcal{J}_1^* := \{(a, b, \beta) \in \mathcal{J}^\circ : a + b + \|\beta\| = 1; \ a, b, \|\beta\| < 1\},$$
$$\mathcal{I}_1^* := \left\{\beta \in \mathcal{I}^\circ : \|\beta\| = 1, \ \beta \neq \{(0, 1)\}\right\} = \left\{(0, a) \star (0, b) \star \beta : (a, b, \beta) \in \mathcal{J}_1^*\right\}. \tag{4}$$

Theorem 4. *The resampling, de-Poissonized type-2 evolution (which we also call a 2-tree evolution) is a Borel right Markov process on $(\mathcal{J}_1^*, d_\mathcal{J})$. The IP-valued variant is a Borel right Markov process on $(\mathcal{I}_1^*, d_\mathcal{I})$ and is path-continuous except on a discrete set of resampling times.*

Consider $(A_1, A_2, A_3) \sim \mathtt{Dir}\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ and an independent interval partition $\overline{\beta} \sim \mathtt{PDIP}\left(\frac{1}{2}, \frac{1}{2}\right)$. The law of $(A_1, A_2, A_3\overline{\beta})$ is the unique stationary distribution for the 2-tree evolution on $\mathcal{J}_1^*$.

Consider the map $\pi_2^\bullet$ on $\mathcal{J}_1^*$ given by $(a, b, \beta) \mapsto (a, b, \|\beta\|)$. The range of this map is the set $\Delta := \{(p_1, p_2, p_3) \in [0, 1)^3, \sum_{i=1}^3 p_i = 1\}$. Also consider the stochastic kernel Λ from Δ to $\mathcal{J}_1^*$ that maps (p_1, p_2, p_3) to the law of $(p_1, p_2, p_3\overline{\beta})$, where $\overline{\beta} \sim \mathtt{PDIP}\left(\frac{1}{2}, \frac{1}{2}\right)$. Given $(p_1, p_2, p_3) \in \Delta$, run a resampling, de-Poissonized type-2 evolution $(T^u, u \geq 0)$ with initial condition $\Lambda(p_1, p_2, p_3)$. The *induced 3-mass process* is then $(X_1(u), X_2(u), X_3(u)) := \pi_2^\bullet(T^u), u \geq 0$.

Theorem 5. *The induced 3-mass process is a recurrent Markovian extension of the Wright–Fisher$(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$ diffusion, described in (1), in the following sense. Let U be the first time u when either $X_1(u) = 0$ or $X_2(u) = 0$. Then the process killed at U is the killed Wright–Fisher diffusion. The 3-mass process is intertwined with the resampling, de-Poissonized type-2 evolution, and it converges to its unique stationary law $\mathtt{Dir}\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$.*

Notice that the 3-mass process jumps back into the interior of the simplex immediately after either of the first two coordinates vanish. This extension of the generalized Wright–Fisher diffusion is natural from the perspective of the Aldous chain and is the continuum analogue of the construction in [14].

1.1. From the Aldous chain to $\left(\frac{1}{2}, -\frac{1}{2}\right)$-Chinese restaurants.

In this and the next subsection, we informally discuss a discrete counterpart to the 2-tree evolutions, giving a preliminary overview of the construction of type-1 evolution and its connection to 2-trees.

Consider the following decomposition of a rooted binary tree, in analogy with the decomposition of the BCRT described below equation (2). Select a branch point. We decompose the tree into two *top subtrees* above the branch point and a sequence of *spinal subtrees* branching off of the path, called the *spine*, from the branch point to the root. We represent partial information about the tree via the masses (leaf counts) in these subtrees: a pair of top masses (m_1, m_2), followed by a finite sequence of spinal masses $(b_1, \ldots, b_{k-2})$, ordered by decreasing distance from the root. We call this representation a *discrete 2-tree*.

The Aldous down-up moves act on the discrete 2-tree as follows. In the down-move, we make a size-biased pick among the masses and reduce that mass by one. If the mass is reduced to zero, it is removed from the list. For up-moves, we choose a mass m with probability proportional

to $2m - 1$, or choose any edge along the spine with probability proportional to 1; see Figure 2. If a mass is chosen, it is incremented by 1; if a spinal edge is chosen, a '1' is inserted into the sequence of spinal masses at that point, representing the appearance of a new spinal subtree. We adopt the rule that if, after a down-move, one of the two top subtree masses is reduced to zero, then the first mass along the spine replaces it as a new top mass. A generalization of this projected Aldous chain is studied in [15, Appendix A].

The up-move weights of Figure 2 are very close to the seating rule for an ordered Chinese restaurant process (oCRP) [33, 11]. The oCRP(α, θ) begins with a single customer sitting alone at a table. New customers enter one by one. Upon entering, the $n + 1^{\text{st}}$ customer chooses to join a table that already has m customers with probability $(m - \alpha)/(n + \theta)$; sits alone at a new table inserted at the far left end of the restaurant with probability $\theta/(n + \theta)$; or sits alone at a new table, inserted to the right of any particular table already present, with weight $\alpha/(n + \theta)$, so that the total probability to sit alone is $(k\alpha + \theta)/(n + \theta)$, where k is the number of tables already present. If we ignore the left-to-right order of these tables, then this is the well-known (unordered) CRP(α, θ) due to Dubins and Pitman [32, §3.2]. The distribution of an oCRP after n customers have arrived is the discrete analogue to the PDIP.

If we take $(\alpha, \theta) = \left(\frac{1}{2}, 0\right)$, then this seating rule differs from the up-move probabilities in Figure 2 only in that, in the oCRP, a new table can be introduced between the two leftmost tables, whereas in the 2-tree no new mass can be inserted in between the two leftmost masses, representing the two top subtrees, which are not separated by an edge but only by a branch point. We refer to the probabilities in Figure 2 as the seating rule for the oCRP$\left(\frac{1}{2}, -\frac{1}{2}\right)$, as, under this rule, if there are a total of k masses (2 top masses and $k - 2$ spinal masses) then the probability for insertion of a new mass '1' is $(k - 1)/(2n - 1) = \left(k\frac{1}{2} - \frac{1}{2}\right)/\left(n - \frac{1}{2}\right)$.

This is outside of the usual parameter range considered for the CRP. Indeed, if we start a CRP$\left(\frac{1}{2}, -\frac{1}{2}\right)$ with a single customer, as described above, then all subsequent customers will be forced to join the first at a single table, as the probability to sit alone will be zero. However, if we start with two customers sitting separately, then the oCRP$\left(\frac{1}{2}, -\frac{1}{2}\right)$ seating rule produces a non-trivial configuration distributed as the 2-tree projection of a uniform random rooted binary tree with labeled leaves. We remark that Poisson–Dirichlet distributions with "forbidden" parameters have been considered before, e.g. in the context of σ-finite dislocation measures of fragmentation processes and related discrete splitting probabilities [27, 6, 25, 20].

In the setting of the Chinese restaurant analogy Aldous's down-up moves become *re-seating*: a uniform random customer leaves their seat; their table is removed if empty; and they choose a new seat according to the seating rule, as if entering for the first time.

1.2. **Discrete scaffolding, spindles, and skewer.** We simplify matters by *Poissonizing* the Aldous chain. In the Poissonized Aldous chain, each leaf is removed in a down-move after an independent exponential time with rate 1. That leaf is not immediately re-inserted into the tree. Rather, up-moves occur at each edge after an exponential time with rate $\frac{1}{2}$ (since there are roughly twice as many edges as leaves). This allows the total number of leaves in the tree to fluctuate, but it results in a process in which disjoint subtrees evolve independently. Scaling limits of some statistics of this Poissonized chain have been rigorously connected to the Aldous chain via de-Poissonization in [29].

When we project to the discrete 2-tree as before, the sequence of subtree masses evolves as a *Poissonized down-up* oCRP$\left(\frac{1}{2}, -\frac{1}{2}\right)$. In this process, each table population m decreases by 1 with rate m, or increases by 1 with rate $m - \frac{1}{2}$. To the right of any table except for the leftmost (i.e. not between the two leftmost), a new table of population 1 appears with rate $\frac{1}{2}$.

Due to Poissonization, the table populations evolve independently of each other. Each one is a birth-and-death chain, having deaths with rate m and births with rate $m - \frac{1}{2}$ when the population is m, until absorption at population 0. Let μ denote the distribution of the lifetime of this birth-and-death chain, started from population 1.

This Poissonized down-up oCRP admits a surprising representation, which was introduced in [11] to describe continuum analogues of the Poissonized down-up oCRP$\left(\frac{1}{2}, 0\right)$ and oCRP$\left(\frac{1}{2}, \frac{1}{2}\right)$. We discuss here the $\left(\frac{1}{2}, 0\right)$ case and then the new extension to $\left(\frac{1}{2}, -\frac{1}{2}\right)$.
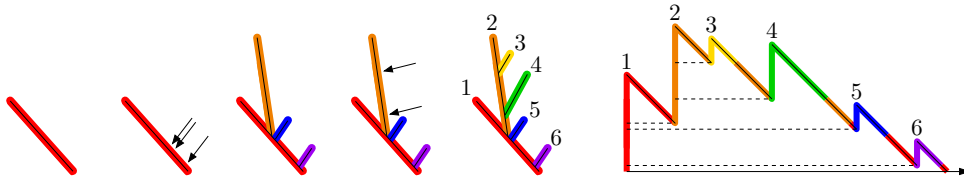
Figure 3. Iterative construction of splitting tree representation of tables in Poissonized down-up $\mathtt{oCRP}\left(\frac{1}{2}, 0\right)$, started with one customer, and JCCP.

We think of the tables that appear and vanish in this evolving $\mathtt{oCRP}$ as members of a family: when a new table is born, the table immediately to its left at that time is its parent. The number of tables is then evolving over time as a Crump–Mode–Jagers (CMJ) branching process [21]. The genealogy among these tables, and their lifetimes, can be represented in a splitting tree [17]. For our purposes, this can be formalized as a rooted plane tree with edge lengths.

Figure 3 depicts the construction of a splitting tree representation of the Poissonized down-up $\mathtt{oCRP}\left(\frac{1}{2}, 0\right)$, started with a single customer.

(1) Draw a line of random length, sampled from μ (the lifetime distribution of a table started with population 1); this represents the first table.
(2) Now, mark that line with Poisson points along its length, with rate $\frac{1}{2}$.
(3) At each marked point, attach a new "child" line, branching off to the right from its parent, with length independently sampled from μ. Each such line represents a table born, at some point, immediately to the right of the first table.
(4) Repeat steps (2), (3), and (4) on each of the newly drawn lines, if any.

It is not immediately obvious, but this procedure almost surely terminates for this choice of μ.

This tree can be represented by a *jumping chronological contour process* (JCCP) [16, 17], shown in Figure 3. Imagine a flea traveling around the splitting tree. It begins to the left of the root, and immediate jumps up to the top of the leftmost branch, representing the first table. It then slides down the right hand side of that branch at unit speed until its path is blocked by a branch sticking out to the right. When that happens, it jumps to the top of the new branch, and carries on in the same manner, until it finally reaches the root. The JCCP records the distance from the flea to the root, as a function of time.

The tables that arise in the evolving $\mathtt{oCRP}$ are in bijective correspondence with the jumps of the JCCP, with the levels of the bottom and top of each jump equaling the birth and death times of the corresponding table. The genealogy among tables can be recovered by looking to the bottom of each jump (a child), and drawing a horizontal line to the left from that point, seeing where it crosses another jump (its parent).

JCCP representations of splitting trees like ours are Lévy processes of positive jumps and negative drift [23]. Our particular JCCP has drift -1 and Lévy measure $\frac{1}{2}\mu$. *Levels* in the JCCP correspond to *times* in the evolving $\mathtt{oCRP}$. On the other hand, times in the JCCP have no simple meaning in the $\mathtt{oCRP}$, and serve mainly to record the left-to-right order of tables.

What is missing from this JCCP picture is the evolving table populations. Recall that each table population evolves as a birth-and-death chain with lifetime distribution μ. This is also the law of jump heights in our JCCP. We incorporate both the genealogy among tables and the evolving table populations into a single formal object by marking each jump with such a birth-and-death chain, with lifetime equal to the height of the jump.

We depict this object by representing each birth-and-death chain as a laterally symmetric "spindle" shape, beginning at the bottom of the jump and evolving towards its top, with width at each level describing the value of the chain at the corresponding time. In the context of this construction, we refer to the JCCP as *scaffolding* and the markings as *spindles*. See Figure 4.

Then, to recover the Poissonized down-up $\mathtt{oCRP}\left(\frac{1}{2}, 0\right)$ from the scaffolding and spindles representation, we apply a *skewer map*: for any $y \geq 0$, we draw a horizontal line through the picture at that level, and look at the cross-sections of spindles pierced by the line. The widths of these cross-sections represent populations of tables, and their left-to-right order corresponds to that
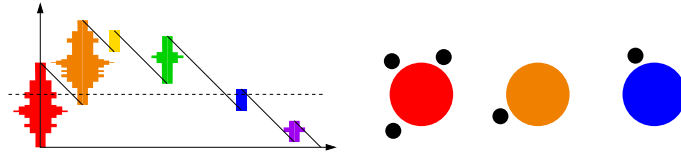
FIGURE 4. Scaffolding with spindles and `oCRP` arrangement described by the skewer.
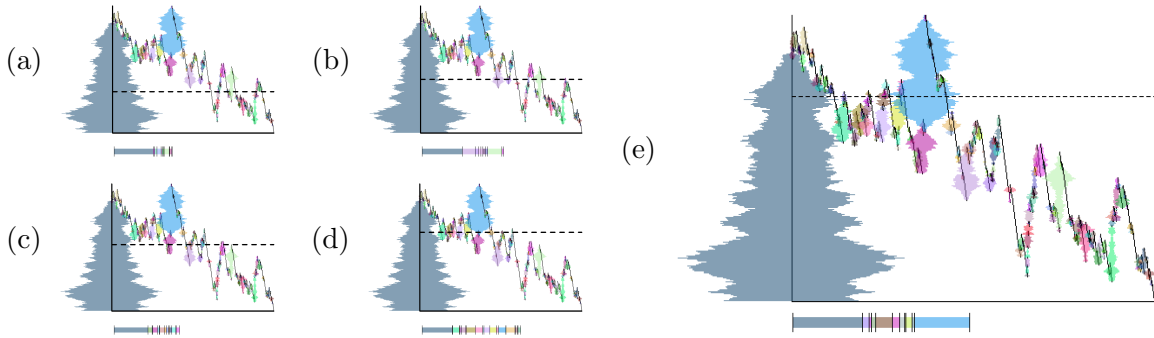

FIGURE 5. Type-1 interval partition evolution, constructed from scaffolding (slanted black lines), spindles (laterally symmetric colored shapes), and the skewer map. Simulation from [10].

in the `oCRP`. If we slide this horizontal line up continuously, then the cross-sections gradually change in width, with some dying out as the horizontal line passes the top of a jump, and new ones appearing as it reaches the bottom of a jump.

In scaling limits, the scaffolding converges to a $\mathtt{Stable}\!\left(\frac{3}{2}\right)$ Lévy process, and the law of the birth-and-death chain spindles converges to a σ-finite excursion measure associated with squared Bessel processes with parameter -1, abbreviated as $\mathtt{BESQ}(-1)$, studied in [34]. This motivated the construction, in [11], of *type-1 evolutions* by applying the skewer map to $\mathtt{Stable}\!\left(\frac{3}{2}\right)$ scaffolding marked by $\mathtt{BESQ}(-1)$ excursion spindles. A simulation of Poissonized down-up `oCRP` approximating the type-1 evolution is shown in Figure 5.

1.3. Poissonized discrete 2-tree evolution.

The difference between the Poissonized down-up $\mathtt{oCRP}\!\left(\frac{1}{2}, 0\right)$ and the Poissonized down-up $\mathtt{oCRP}\!\left(\frac{1}{2}, -\frac{1}{2}\right)$, corresponding to the 2-tree evolution, is that in the latter process no new tables can be born in between the two leftmost tables. This corresponds to no new subtrees appearing between the two top subtrees in the 2-tree evolution. At different times, different tables may become the leftmost. Such a table may have had children prior to becoming leftmost, but subsequently, it ceases to do so.

To construct such a process via scaffolding and spindles, we begin with a scaffolding-and-spindles construction of the Poissonized down-up $\mathtt{oCRP}\!\left(\frac{1}{2}, 0\right)$ with two initial tables. We then find all instances in which a child was born to a parent spindle at a level at which the parent was the overall leftmost spindle, and we delete all such children and their offspring; see Figure 6. We refer to these leftmost spindles as *clock spindles* and the transformation of deleting their descendants as *deletion clocking* or *emigration*. We think of these spindles, which correspond to the $\mathbf{f}^{(n)}$s in Definition 1, as timers. When they reach zero, at Y_{n+1}, we pass to the $n+1^{\text{st}}$ stage in the construction. We formalize deletion clocking in the continuum analogue in Definition 17.

2. Type-1 evolutions: preliminaries and representation as pairs

In this section, we recall from [11] the "scaffolding, spindles, and skewer" constructions of type-1 and type-0 interval partition evolutions. We also recall the main results of [11] and record some further consequences. Here, for brevity, we will construct type-0 and type-1 evolutions on a probability space; in [11], all of this work is carried out in terms of probability distributions and filtrations on a canonical space of counting measures.
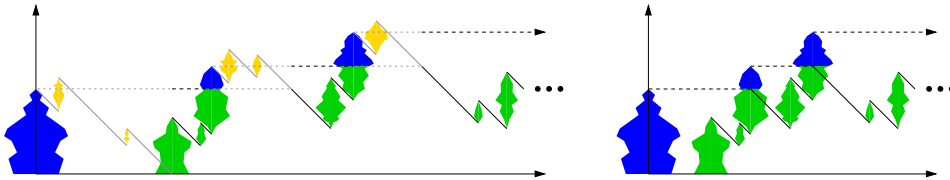
FIGURE 6. Deletion-clocking, to transform scaffolding-and-spindles representation of Poissonized down-up $\mathtt{oCRP}\left(\frac{1}{2}, 0\right)$ to representation of $\mathtt{oCRP}\left(\frac{1}{2}, -\frac{1}{2}\right)$.

2.1. **Preliminaries on type-1 and type-0 evolutions.** Recall the definition of the set $\mathcal{I}$ of interval partitions with diversity from the introduction. This space can be metrized by the Hausdorff metric between complements of interval partitions, but we prefer a stronger metric $d_{\mathcal{I}}$ that accounts for diversity. We formally define this metric later, in Definition 14. We define two probability distributions on this space, as in [11]: $\mathtt{PDIP}\left(\frac{1}{2}, 0\right)$, which is the reversal of the interval partition formed by excursion intervals of Brownian motion on time $[0, 1]$, including the incomplete final excursion; and $\mathtt{PDIP}\left(\frac{1}{2}, \frac{1}{2}\right)$, which is the partition formed by excursion intervals of Brownian bridge. The names are in recognition of the facts that the ranked excursion lengths are Poisson–Dirichlet distributed with respective parameters, $\mathtt{PD}(\frac{1}{2}, \frac{1}{2})$ and $\mathtt{PD}(\frac{1}{2}, 0)$, see e.g. [32].

We denote by $\mathcal{E}$ the space of càdlàg excursions away from zero. In the context of the following construction, we refer to continuous excursions $f \in \mathcal{E}$ as *spindles* and to excursions with a càdlàg jump at 0 and/or at their time of absorption $\zeta(f)$ as *cut-off spindles*.

Recall $\mathtt{BESQ}(-1)$ has an exit boundary at zero [19]. Despite this, methods of [34] allow the construction of a σ-finite excursion measure ν_{BES} associated with $\mathtt{BESQ}(-1)$. We choose the normalization constant so that

$$\nu_{\mathrm{BES}}(\zeta \in dy) = \frac{3}{2\pi\sqrt{2}} y^{-5/2} dy. \tag{5}$$

Let $\mathbf{N}$ be a Poisson random measure on $[0, \infty) \times \mathcal{E}$ with intensity measure $\mathrm{Leb} \times \nu_{\mathrm{BES}}$. For excursions f arising in this point process, we take their lifetimes $\zeta(f)$ to be jump heights for a Lévy process constructed from jumps and compensation:

$$\mathbf{X}(t) := \xi_{\mathbf{N}}(t) := \lim_{z \downarrow 0} \left(\int_{[0,t] \times \{g \in \mathcal{E} \,:\, \zeta(g) > z\}} \zeta(f) d\mathbf{N}(s, f) - \frac{3tz^{-1/2}}{\pi\sqrt{2}} \right), \quad t \ge 0. \tag{6}$$

We abbreviate $\xi(\mathbf{N}) := (\xi_{\mathbf{N}}(t), \, t \ge 0)$. This is a spectrally positive $\mathtt{Stable}\left(\frac{3}{2}\right)$ Lévy process, called *scaffolding*. The *aggregate mass process* and *skewer* of $\mathbf{N}$ at level $y \in \mathbb{R}$ are

$$M_{\mathbf{N}}^y(t) := \int_{(-\infty, t] \times \mathcal{E}} \max\left\{ f\big((y - \xi_{\mathbf{N}}(u-)) -\big), \, f\big(y - \xi_{\mathbf{N}}(u-)\big) \right\} d\mathbf{N}(u, f), \quad t \ge 0,$$

$$\mathrm{SKEWER}(y, \mathbf{N}) := \left\{ \big(M_{\mathbf{N}}^y(t-), M_{\mathbf{N}}^y(t)\big) : t \in \mathbb{R}, \, M_{\mathbf{N}}^y(t-) < M_{\mathbf{N}}^y(t) \right\}. \tag{7}$$

We abbreviate $\overline{\mathrm{SKEWER}}(\mathbf{N}) := \overline{\mathrm{SKEWER}}(\mathbf{N}, \xi(\mathbf{N})) := \big(\mathrm{SKEWER}(y, \mathbf{N}), \, y \ge 0\big)$. A simulation of this construction is depicted in Figure 5. Note that (7) is set up to allow counting measures N on $\mathbb{R} \times \mathcal{E}$, and not just on $[0, \infty) \times \mathcal{E}$, in anticipation of our construction of type-0 point measures.

We denote by $\mathcal{N}$ a space of point measures on $[0, \infty) \times \mathcal{E}$, supported on bounded time intervals $[0, T] \times \mathcal{E}$, in which $\overline{\mathrm{SKEWER}}(N)$ is well-defined and $d_{\mathcal{I}}$-continuous for each $N \in \mathcal{N}$; this was denoted by $\mathcal{N}_{\mathrm{fin}}^{\mathrm{sp},*}$ in [11, Definitions 3.16, 4.16]. We also define $\overleftarrow{\mathcal{N}}$ analogously, but for measures supported on $(-\infty, 0) \times \mathcal{E}$, not necessarily on a bounded time interval. Around (9), we will introduce one such random element of $\overleftarrow{\mathcal{N}}$, a random measure supported on unbounded time, for which the skewer at each level remains bounded and evolves continuously.

Now, consider $\mathbf{f} \sim \mathtt{BESQ}_x(-1)$ a $\mathtt{BESQ}(-1)$ started from $x > 0$. A *clade of initial mass x* is then a random counting measure $\mathbf{n} \in \mathcal{N}$, distributed as

$$\mathrm{CLADE}(\mathbf{f}, \mathbf{N}) := \delta(0, \mathbf{f}) + \mathbf{N}\big|_{(0, T_{-\zeta(\mathbf{f})}(\mathbf{N})] \times \mathcal{E}}, \quad \text{where} \quad T_{-y}(\mathbf{N}) := \inf\{t \ge 0 : \xi_{\mathbf{N}}(t) = -y\}. \tag{8}$$
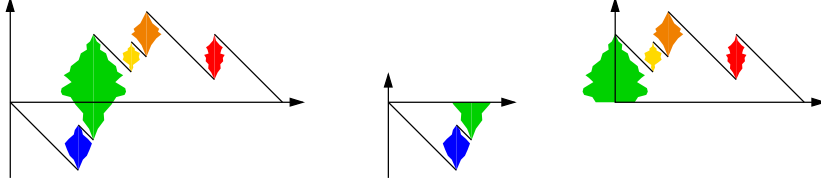
FIGURE 7. Decomposition of (discrete approximation of) a typical excursion of $\mathtt{Stable}\left(\frac{3}{2}\right)$ scaffolding with spindles into an initial component below the level and a subsequent component above. The latter is a *clade*.

In the following clade construction and elsewhere, the notion of "concatenation," denoted by $\star$, is in the sense of excursion theory: concatenating a sequence of excursions means running one after the other. This easily generalizes to totally ordered collections with summable excursion lengths. Concatenation of excursions induces a notion of concatenation of point measures of jumps and hence a notion for point measures of (jumps marked by) spindles. See [11] for details.

For any "initial" interval partition β, we denote by $\mathbf{P}_\beta^1$ the law of a random *type-1 point measure* $\mathbf{N}_\beta \in \mathcal{N}$ obtained by concatenating independent clades with initial masses equal to the lengths of the intervals in β.

Now, consider the point measure $\overleftarrow{\mathbf{N}}$ on $(-\infty, 0) \times \mathcal{E}$ formed by concatenating a sequence of independent copies of $\mathbf{N}|_{(0, T_{-1}(\mathbf{N})] \times \mathcal{E}}$, with each copy being concatenated to the left of the previous copies. We slightly modify (6) in this setting:

$$\overleftarrow{\mathbf{X}}(t) := \xi_{\overleftarrow{\mathbf{N}}}(t) := \lim_{z \downarrow 0} \left(-\int_{(t,0) \times \{g \in \mathcal{E}: \zeta(g) > z\}} \zeta(f) d\overleftarrow{\mathbf{N}}(s, f) + \frac{3|t|z^{-1/2}}{\pi\sqrt{2}} \right), \quad t \leq 0. \qquad (9)$$

As before, $\xi(\overleftarrow{\mathbf{N}}) := (\xi_{\overleftarrow{\mathbf{N}}}(t), \, t \leq 0)$. Informally, this is a spectrally positive $\mathtt{Stable}\left(\frac{3}{2}\right)$ first-passage descent from ∞ down to 0, arranged to arrive at 0 at time zero. This construction was discussed in [11, Remark 5.15].

If $\mathbf{N}_\beta$ is as above and independent of $\overleftarrow{\mathbf{N}}$, then $\overleftarrow{\mathbf{N}} + \mathbf{N}_\beta$ is a *type-0 point measure* with initial state β. In the sequel, we find it convenient to represent this as a *type-0 data pair* $(\overleftarrow{\mathbf{N}}, \mathbf{N}_\beta) \in \mathcal{N} \times \mathcal{N}$. We denote the law of this pair by $\mathbf{P}_\beta^0$. We take the convention that $\xi(\overleftarrow{\mathbf{N}} + \mathbf{N}_\beta)$ equals $\xi(\overleftarrow{\mathbf{N}})$ on $(-\infty, 0)$ and equals $\xi(\mathbf{N}_\beta)$ on $[0, \infty)$.

We showed in [11, Theorem 1.4] that the associated *type-1* and *type-0 evolutions*, respectively $\overline{\mathrm{SKEWER}}(\mathbf{N}_\beta)$ and $\overline{\mathrm{SKEWER}}(\overleftarrow{\mathbf{N}} + \mathbf{N}_\beta)$, are path-continuous strong Markov processes on $(\mathcal{I}, d_\mathcal{I})$.

We define the *shifted restrictions* of a point measure, denoted by $N|_{[a,b] \times \mathcal{E}}^{\leftarrow}$ and $N|_{[a,b] \times \mathcal{E}}^{\rightarrow}$ to be point measures obtained by first restricting support to the indicated region, and then shifting the resulting point measure to be supported on $[0, b-a] \times \mathcal{E}$ or $[a-b, 0] \times \mathcal{E}$, respectively. We denote by $\overleftarrow{T}_y(\overleftarrow{\mathbf{N}}) = \inf\{t \leq 0 : \xi_{\overleftarrow{\mathbf{N}}}(t) = y\}$, $y \geq 0$, the pre-0 downward first passage times of $\xi(\overleftarrow{\mathbf{N}})$. Just as we used $\mathbf{N}$ on $(0, T_{-\zeta(\mathbf{f})}(\mathbf{N})]$ to define $\mathrm{CLADE}(\mathbf{f}, \mathbf{N})$, we can use $\overleftarrow{\mathbf{N}}$ on the time interval $[\overleftarrow{T}_{\zeta(\mathbf{f})}(\overleftarrow{\mathbf{N}}), 0)$ to extend a spindle $\mathbf{f}$ to a clade $\mathrm{CLADE}(\mathbf{f}, \overleftarrow{\mathbf{N}}) := \delta(0, \mathbf{f}) + \overleftarrow{\mathbf{N}}|_{[\overleftarrow{T}_{\zeta(\mathbf{f})}(\overleftarrow{\mathbf{N}}), 0) \times \mathcal{E}}^{\leftarrow}$.

Lemma 6. *Let* $\Psi := (\mathbf{f}, \overleftarrow{\mathbf{N}}, \mathbf{N}_\gamma) \sim \mathbf{P}_{x,\gamma}^1 := \mathtt{BESQ}_x(-1) \otimes \mathbf{P}_\gamma^0$ *for some* $x \in [0, \infty)$ *and* $\gamma \in \mathcal{I}$. *Then the measure* $\mathbf{N}_* = \mathrm{CLADE}(\mathbf{f}, \overleftarrow{\mathbf{N}}) \star \mathbf{N}_\beta \sim \mathbf{P}_{(0,x) \star \gamma}^1$ *is a type-1 point measure.*

In light of this lemma, we refer to Ψ as a *type-1 data triple*. This construction may seem superfluous: why include $\mathbf{N}_\gamma$ as a member of a triple of objects, just to set up another point measure $\mathbf{N}_*$ of the same type? However, this sets up a parallel with type-0 data pairs leading to the definition of type-2 data quadruples. This parallel will be useful in forthcoming work on the Aldous diffusion [12, 13] involving all three processes.

To exhibit the Markovian nature of the skewer processes and underlying clade constructions, for $y \geq 0$ we decompose $\mathbf{N}_*$ into a point process $\mathbf{N}_*^y$ of spindles or cut-off spindles above level y and a point process $\mathbf{N}_*^{\leq y}$ of spindles or cut-off spindles below level y, as in Figure 7.
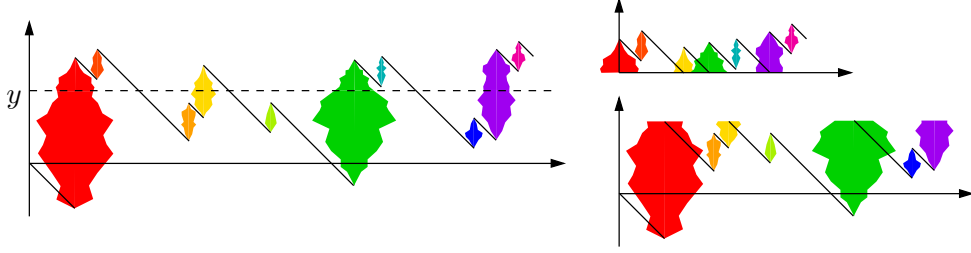
FIGURE 8. Illustration [11, Figure 3.3] of spindles cut at level y. Left: N. Right: N^y and $N^{\leq y}$.

More formally, for a type-0 data pair $\Psi = (\overleftarrow{\mathbf{N}}, \mathbf{N}_\beta)$, define $\overleftarrow{\mathbf{N}}^y_\Psi := \overleftarrow{\mathbf{N}}|^{\rightarrow}_{(-\infty, T_y(\overleftarrow{\mathbf{N}})) \times \mathcal{E}}$. This captures only spindles above level y. Beyond time $T_y(\overleftarrow{\mathbf{N}})$, each spindle f that crosses level y corresponds to the unique jump across y in an excursion of $\xi(\overleftarrow{\mathbf{N}} + \mathbf{N}_\beta)$ about level y. The cut-off spindle $\hat{f}^y$ together with the spindles following this jump in the excursion forms a clade; see Figure 7. We denote the concatenation of these subsequent clades by $\mathbf{N}^y_\Psi$. We also concatenate the point measures of the remaining spindles and cut-off spindles $\check{f}$ – the initial components, below level y, of each excursion of scaffolding, as in Figure 7 – into a point measure $\mathbf{N}^{\leq y}_\Psi$, and denote by $(\mathcal{F}^y, y \geq 0)$ the filtration generated by $(\mathbf{N}^{\leq y}_\Psi, y \geq 0)$.

By [11, Proposition 5.17 and its proof], type-0 data pairs have a Markov-like property.

Lemma 7. *Let* $\Psi = (\overleftarrow{\mathbf{N}}, \mathbf{N}_\beta) \sim \mathbf{P}^0_\beta$ *for some* $\beta \in \mathcal{I}$*. For all* $y \geq 0$*, conditionally given* $\mathcal{F}^y$*,*

$$(\overleftarrow{\mathbf{N}}^y_\Psi, \mathbf{N}^y_\Psi), \text{ defined above, has conditional distribution } \mathbf{P}^0_{\alpha^y}$$

where $(\alpha^z, z \geq 0) = \overline{\text{SKEWER}}(\overleftarrow{\mathbf{N}} + \mathbf{N}_\beta)$ *is the associated type-0 evolution.*

Recall from the type-1 setting of Lemma 6 that $\mathbf{N}_*$ is a function of a type-1 data triple Ψ. For $y \geq 0$, define

$$m^y(\mathbf{N}_*) := M^y_{\mathbf{N}_*}(\inf\{t \geq 0 \colon M^y_{\mathbf{N}_*}(t) > 0\})$$
$$\text{and} \quad \mathbf{f}^y_\Psi := \left(m^{y+z}(\mathbf{N}_*), 0 \leq z \leq \inf\{w \geq 0 \colon m^{y+w-}(\mathbf{N}_*) = 0\}\right), \tag{10}$$

respectively the mass of the leftmost block at level y and the leftmost spindle, evolving up from that level. These both vanish for $y \geq \sup \xi(\mathbf{N}_*)$ and for a Lebesgue null set of levels that are in the range of the running supremum process of $\xi(\mathbf{N}_*)$. For any other level, $\mathbf{f}^y_\Psi$ is associated with a jump of $\xi(\mathbf{N}_*)$ across level y, and along with the following spindles until $\xi(\mathbf{N}_*)$ first hits level y, it forms a clade $\delta(0, \mathbf{f}^y_\Psi) \star \overleftarrow{\mathbf{n}}^y$ for some point measure $\overleftarrow{\mathbf{n}}^y$. Beyond $\overleftarrow{\mathbf{n}}^y$, we collect clades above level y as for type 0 and concatenate these to form a point measure $\mathbf{N}^y_\Psi$. Let $\overleftarrow{\mathbf{N}}^y_\Psi = \overleftarrow{\mathbf{N}}|^{\rightarrow}_{(-\infty, T_{\zeta(\mathbf{f}^y_\Psi)}(\overleftarrow{\mathbf{N}})) \times \mathcal{E}} \star (\overleftarrow{\mathbf{n}}^y)^{\rightarrow}$.

We can alternatively represent this decomposition about level y via $\mathbf{N}^y_* := \text{CLADE}(\mathbf{f}^y_\Psi, \overleftarrow{\mathbf{N}}^y_\Psi) \star \mathbf{N}^y_\Psi$. Point processes $\mathbf{N}^y_*$ and $\mathbf{N}^{\leq y}_*$ can also be directly defined as above, from spindles in the excursions of $\xi(\mathbf{N}_*)$ about y. For type 1, augment the type-0 filtration so that $\mathbf{f}$ is also adapted. More specifically, we define $(\mathcal{F}^y, y \geq 0)$ as the natural filtration of $(\mathbf{f}(y), \mathbf{N}^{\leq y}_\Psi)$. Then $\mathbf{N}^{\leq y}_*$ is adapted. By [11, Proposition 5.6 (and Lemma 3.41)], we have the following.

Lemma 8. *In the setting of Lemma 6, for all* $y \geq 0$*, conditionally given* $\mathcal{F}^y$*,*

$$(\mathbf{f}^y_\Psi, \overleftarrow{\mathbf{N}}^y_\Psi, \mathbf{N}^y_\Psi) \qquad \text{has conditional distribution } \text{BESQ}_{m^y}(-1) \otimes \mathbf{P}^0_{\alpha^y},$$

where $m^y := m^y(\mathbf{N}_*)$ *and* $(\alpha^z, z \geq 0)$ *is such that* $\left((0, m^z) \star \alpha^z, z \geq 0\right) = \overline{\text{SKEWER}}(\mathbf{N}_*)$ *is the associated type-1 evolution. This includes the degenerate case* $m^y = 0$ *and* $\alpha^y = \emptyset$*.*

In the sequel, we will abuse terminology and also refer to $((m^y, \alpha^y), y \geq 0)$ as a type-1 evolution. Note that for type 1, the part $(\alpha^y, y \geq 0)$ is only a type-0 evolution up to the

random level $y = \zeta(\mathbf{f})$. Above this level, the point measure $\overleftarrow{\mathbf{N}}|_{(-\infty, T_{\zeta(\mathbf{f})}(\overleftarrow{\mathbf{N}})) \times \mathcal{E}}$ is redundant for the type-1 evolution, while it provides further blocks for the type-0 evolution. We recall some more facts about type-0 and type-1 evolutions from [11]. The following is an immediate consequence of the definitions.

Proposition 9. *Let $\mathbf{N}_\beta$ and $\mathbf{N}'_{\beta'}$ be two independent type-1 point measures. Then $\mathbf{N}_\beta \star \mathbf{N}'_{\beta'}$ is also a type-1 point measure. In particular, $\overline{\mathrm{SKEWER}}(\mathbf{N}_\beta \star \mathbf{N}'_{\beta'})$ is a type-1 evolution starting from $\beta \star \beta'$. Similarly, $(\overleftarrow{\mathbf{N}}, \mathbf{N}_\beta \star \mathbf{N}'_{\beta'})$ is a type-0 data pair.*

For type-0 and type-1 evolutions, the associated *total mass evolutions*, $(\|\mathrm{SKEWER}(y, \overleftarrow{\mathbf{N}} + \mathbf{N}_\beta)\|, \; y \geq 0)$ and $(\|\mathrm{SKEWER}(y, \mathbf{N}_*)\|, \; y \geq 0)$ respectively, are as follows.

Proposition 10 (Theorem 1.5 of [11]). *For any $\beta \in \mathcal{I}$, the total mass evolution under $\mathbf{P}_\beta^0$ is $\mathrm{BESQ}_{\|\beta\|}(1)$, while the total mass evolution under $\mathbf{P}_\beta^1$ or $\mathbf{P}_{x,\beta}^1$ is $\mathrm{BESQ}_{\|\beta\|}(0)$ or $\mathrm{BESQ}_{x+\|\beta\|}(0)$. In particular, the type-1 evolution a.s. is absorbed at $\emptyset$ in finite time.*

The transition kernels of type-1 and type-0 evolutions are given in [11, Propositions 4.30, 5.4 and 5.16]. Most relevant for us is the following.

Proposition 11 (Pseudo-stationarity for type-0 and type-1 evolutions; Theorem 6.1 of [11]). *Consider $\bar{\beta} \sim \mathrm{PDIP}(\frac{1}{2}, \frac{1}{2})$ and an independent random variable $M \in [0, \infty)$. Let $B \sim \mathrm{BESQ}_M(1)$ and α a type-0 evolution starting from $\beta = M\bar{\beta}$. Then α^y has the same distribution $B(y)\bar{\beta}$.*
The same result holds for type-1 evolutions α, if we take $\bar{\beta} \sim \mathrm{PDIP}(\frac{1}{2}, 0)$ and $B \sim \mathrm{BESQ}_M(0)$.

We refer to the respective distributions of $M\bar{\beta}$ as type-0 and type-1 *pseudo-stationary distributions*. We can also integrate this result over y to extend this to independent random times, and in the type-1 case, we can rephrase this as a result conditionally given that the process survives to level y, since this conditioning only involves the total mass; see [11, Theorem 6.9]. Specifically, we have the following.

Proposition 12 (Proposition 6.2 of [11]). *Consider a type-1 evolution $(\alpha^y, y \geq 0)$ starting from $M\bar{\beta}$, where $\bar{\beta} \sim \mathrm{PDIP}(\frac{1}{2}, 0)$ is independent of $M \sim \mathrm{Exponential}(\gamma)$ for some rate parameter $\gamma \in (0, \infty)$. Then the conditional distribution of α^y given $\alpha^y \neq \emptyset$ is the same as the (unconditional) distribution of $(2y\gamma + 1)\alpha^0$.*
If $\bar{\beta} \sim \mathrm{PDIP}(\frac{1}{2}, \frac{1}{2})$, $M \sim \mathrm{Gamma}(\frac{1}{2}, \gamma)$ for type 0, then $\alpha^y \sim (2y\gamma + 1)\alpha^0$ for all $y \geq 0$.

Corollary 13. *For a type-1 evolution $(\alpha^y, y \geq 0)$ starting from the pseudo-stationary distribution of random mass M, given $\alpha^y \neq \emptyset$, the mass $\|\alpha^y\|$ is conditionally independent of $\alpha^y/\|\alpha^y\|$. The former is conditionally distributed as $B(y)$, where $(B(z), z \geq 0)$ is a $\mathrm{BESQ}_M(0)$ conditioned to survive to time y, and the latter has conditional law $\mathrm{PDIP}(\frac{1}{2}, 0)$.*

Note that conditioning the total mass, $\mathrm{BESQ}(0)$, more strongly to never become extinct gives rise to a $\mathrm{BESQ}(4)$ process; see e.g. [34, p. 451].

2.2. Type-1 evolutions as pairs of leftmost blocks and remaining interval partition.
Recall Sharpe's definition [38] (see also [24, Definition A.18]) of Borel right Markov processes:
1. Lusin state space (homeomorphic to a Borel subset of a compact metric space),
2. right-continuous sample paths,
3. Borel measurable semi-group and strong Markov property.

It is additionally a Hunt process if it is quasi-left-continuous, i.e.
4. left-continuous along all increasing sequences of stopping times.

In preparation for a discussion of continuity, we recall the formal definition of $d_\mathcal{I}$ from [11].

Definition 14. We adopt the notation $[n] := \{1, 2, \ldots, n\}$. For $\beta, \gamma \in \mathcal{I}$, a *correspondence* from β to γ is a finite sequence of ordered pairs of intervals $(U_1, V_1), \ldots, (U_n, V_n) \in \beta \times \gamma$, $n \geq 0$, where the sequences $(U_j)_{j \in [n]}$ and $(V_j)_{j \in [n]}$ are each strictly increasing in the left-to-right ordering of the interval partitions.

The *distortion* of a correspondence $(U_j, V_j)_{j \in [n]}$ from β to γ, denoted by $\mathrm{dis}(\beta, \gamma, (U_j, V_j)_{j \in [n]})$, is defined to be the maximum of the following four quantities:

(i) $\sup_{j \in [n]} |\mathscr{D}_\beta(U_j) - \mathscr{D}_\gamma(V_j)|$,

(ii) $|\mathscr{D}_\beta(\infty) - \mathscr{D}_\gamma(\infty)|$,

(iii) $\sum_{j \in [n]} |\mathrm{Leb}(U_j) - \mathrm{Leb}(V_j)| + \|\beta\| - \sum_{j \in [n]} \mathrm{Leb}(U_j)$,

(iv) $\sum_{j \in [n]} |\mathrm{Leb}(U_j) - \mathrm{Leb}(V_j)| + \|\gamma\| - \sum_{j \in [n]} \mathrm{Leb}(V_j)$.

Note that the second of these quantities depends only on the partitions β and γ and not on the correspondence.

For $\beta, \gamma \in \mathcal{I}$ we define

$$d_\mathcal{I}(\beta, \gamma) := \inf_{n \geq 0, \, (U_j, V_j)_{j \in [n]}} \mathrm{dis}\big(\beta, \gamma, (U_j, V_j)_{j \in [n]}\big), \tag{11}$$

where the infimum is over all correspondences from β to γ.

Proposition 15 (Theorem 1.4 of [11])**.** *Type-1 and type-0 evolutions are path-continuous Hunt processes in $(\mathcal{I}, d_\mathcal{I})$ and are continuous in the initial condition.*

As noted above, a type-1 evolution has a leftmost block $m^y := m^y(\mathbf{N}_*) > 0$ at Leb-a.e. level a.s.. Let γ^y satisfy $(0, m^y) \star \gamma^y = \mathrm{SKEWER}(y, \mathbf{N}_*)$. Consider the continuous bijection $\varphi(m, \gamma) = (0, m) \star \gamma$ from $\mathcal{J}^\bullet := \{(m, \gamma) \in [0, \infty) \times \mathcal{I} : m > 0 \text{ or } \gamma \in \mathcal{I} \setminus \mathcal{I}^\circ\} \cup \{(0, \emptyset)\}$ to $\mathcal{I}$, which has a (discontinuous) measurable inverse. Then Proposition 15 has the following corollary.

Corollary 16. *Representations $((m^y, \gamma^y), y \geq 0) := (\varphi^{-1}(\beta^y), y \geq 0)$ of type-1 evolutions $(\beta^y, y \geq 0)$ are $\mathcal{J}^\bullet$-valued Borel right Markov processes, but not Hunt.*

Proof. 1. The space $\mathcal{J}^\bullet$, equipped with the metric $d_\bullet((m_1, \gamma_1), (m_2, \gamma_2)) = |m_1 - m_2| + d_\mathcal{I}(\gamma_1, \gamma_2)$, is as a Borel subset of a product of Lusin spaces and is therefore Lusin (see [11, Theorem 2.7] for the Lusin property of $(\mathcal{I}, d_\mathcal{I})$).

2. Consider $\mathbf{N}_* \sim \mathbf{P}^1_{(0,m) \times \gamma}$. It is a consequence of the clade construction and properties of $\mathtt{Stable}\big(\frac{3}{2}\big)$ processes that $y \mapsto m^y(\mathbf{N}^*)$ is càdlàg and the only jumps are up from zero, one at the starting level of each excursion of $\xi(\mathbf{N}_*)$ below the supremum. It is a.s. the case that no two such excursions share an endpoint. See [8] for details on fluctuation theory. It is not difficult to show that (m^y, γ^y) is also càdlàg since for $m_n \to m_0$ and $(m_n, \gamma_n) \in \mathcal{J}_\bullet$ for all $n \geq 0$, we have

$$d_\mathcal{I}(\gamma_n, \gamma_0) \to 0 \quad \text{if and only if} \quad d_\mathcal{I}((0, m_n) \star \gamma_n, (0, m_0) \star \gamma_0) \to 0.$$

3. Since φ and φ^{-1} are measurable bijections, the measurability of the semi-group and the strong Markov property follow from Proposition 15.

4. Consider two independent type-1 evolutions $(\beta^y, y \geq 0)$ and $(\gamma^y, y \geq 0)$. By Proposition 9, the concatenation $\beta^y \star \gamma^y$ defines a type-1 evolution. Consider $\eta_n = \inf\{y \geq 0 : \|\beta^y\| < 1/n\}$. Then η_n increases to $\eta = \inf\{y \geq 0 : \beta^y = \emptyset\}$. Then the top block at level η_n converges to 0, but the leftmost block of γ^η is non-zero with positive probability. $\qquad\square$

3. Type-2 evolutions

We will derive properties like càdlàg sample paths, strong Markov property and $\mathtt{BESQ}(-1)$ total mass directly from Definition 1, in Sections 3.2 and 3.3. However, it will extend our toolkit to rephrase this definition in the context of the scaffolding and spindles construction of type-1 evolutions. Indeed, the rephrasing also simplifies establishing some basic symmetry and non-accumulation properties of type-2 evolutions, which will be our starting point.

3.1. **Alternative definition of type-2 evolutions: deletion clocking.** Definition 1 constructs a type-2 evolution from sequences of $\mathtt{BESQ}(-1)$ processes and type-1 evolutions. In fact, we can construct a process with the same distribution using only a single $\mathbf{f}_1 \sim \mathtt{BESQ}_a(-1)$ and a single type-1 data triple $\Psi_0 = (\mathbf{f}_2, \overleftarrow{\mathbf{N}}, \mathbf{N}_\beta) \sim \mathbf{P}^1_{b,\beta}$. This is a continuum analogue of the construction described in Section 1.3.
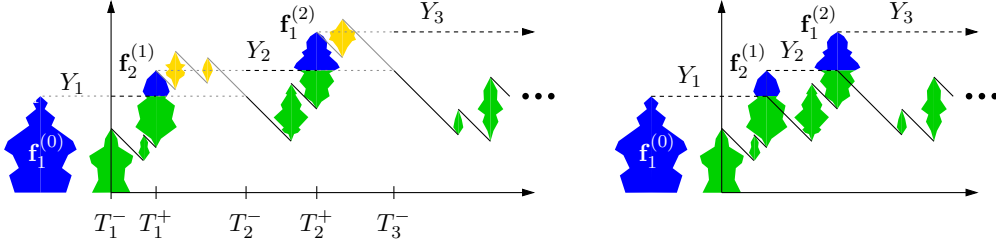
FIGURE 9. The effect of deletion clocking is to ignore intervals of spindles. Here, the clock spindles are dark blue, the ignored spindles are yellow, and other spindles are green.

Definition 17. For $(a, b, \beta) \in \mathcal{J}^\circ$, consider a *type-2 data quadruple with initial state* (a, b, β)

$$\Psi := (\mathbf{f}_1, \mathbf{f}_2, \overleftarrow{\mathbf{N}}, \mathbf{N}_\beta) \sim \text{BESQ}_a(-1) \otimes \text{BESQ}_b(-1) \otimes \mathbf{P}_\beta^0 =: \mathbf{P}_{a,b,\beta}^2.$$

Let $\mathbf{N}_* := \text{CLADE}(\mathbf{f}_2, \overleftarrow{\mathbf{N}}) \star \mathbf{N}_\beta$ and $\mathbf{X}_* := \xi(\mathbf{N}_*)$. We define $((m_1^y, m_2^y, \beta^y), y \geq 0)$ in four steps.

Step 1. We define *clock levels* (Y_n) and *clock change times* $(T_n^\pm)$ for $\mathbf{X}_*$ recursively. These quantities appear labeled in Figure 9. Set $Y_0 = 0$, $T_0^+ = T_1^- = 0$, $Y_1 := \zeta(\mathbf{f}_1)$, and for $n \geq 1$,

$$T_n^+ := \inf\{t \geq T_n^- : \mathbf{X}_*(t) > Y_n\}, \quad Y_{n+1} := \mathbf{X}_*(T_n^+), \quad T_{n+1}^- := \inf\{t > T_n^+ : \mathbf{X}_*(t) \leq Y_n\} \quad (12)$$

with the conventions $\inf \emptyset = \infty$ and $\mathbf{X}_*(\infty) = \infty$. Though we omit it from our notation, we view each of the preceding quantities as a function of Ψ.

Step 2. We define clock spindles. Let $\mathbf{f}^{(0)} = \mathbf{f}_1$. For $j \geq 1$, let $\mathbf{f}^{(n)}$ denote the cut-off top part $\hat{f}^{Y_n}$ of the spindle f that occurs at time T_n^+ in $\mathbf{N}_*$. Each $\mathbf{f}^{(n)}$ will be the clock spindle during the interval $[Y_n, Y_{n+1})$.

Step 3. We define type-1 data. Let $\Psi_0 := (\mathbf{f}_{\Psi_0}, \overleftarrow{\mathbf{N}}_{\Psi_0}, \mathbf{N}_{\Psi_0}) := (\mathbf{f}_2, \overleftarrow{\mathbf{N}}, \mathbf{N}_\beta)$. For $n \geq 1$, let

$$\Psi_n := \left(\mathbf{f}_{\Psi_n}, \overleftarrow{\mathbf{N}}_{\Psi_n}, \mathbf{N}_{\Psi_n}\right) := \left(0, \overleftarrow{\mathbf{N}}\big|_{(-\infty, T_{Y_n}(\overleftarrow{\mathbf{N}})] \times \mathcal{E}}^{\rightarrow}, \left(\mathbf{N}_*\big|_{(T_{n+1}^-, \infty) \times \mathcal{E}}^{\leftarrow}\right)^0\right) \quad \text{for } n \geq 1. \quad (13)$$

The superscript 0 on the rightmost term above is in the sense of the cutoff processes $\mathbf{N}^y$ described around Lemma 7, in which spindles below a given level are removed or cut off. Each Ψ_n is a type-1 data triple for the non-clock top mass and spinal masses during the interval $[Y_n, Y_{n+1})$.

Step 4. We define the evolution. For $n \geq 0$ even,

$$m_1^y := \mathbf{f}^{(n)}(y - Y_n), \quad (0, m_2^y) \star \alpha^y := \text{SKEWER}\left(y - Y_n, \mathbf{N}_*\big|_{(T_{n+1}^-, \infty) \times \mathcal{E}}^{\leftarrow}\right) \quad \text{for } y \in [Y_n, Y_{n+1}), \quad (14)$$

where $m_2^y = 0$ if and only if the skewer in the last expression has no leftmost block. For $n \geq 1$ odd, the definition is the same, but with m_1^y and m_2^y swapping roles.

The effect of this construction is to skip over intervals of spindles from $\mathbf{N}_*$, ensuring that they never contribute blocks to the skewer: for each $n \geq 1$, the process $\mathbf{N}_*\big|_{(T_n^+, T_{n+1}^-] \times \mathcal{E}}$ is redundant. We therefore refer to this construction as *deletion clocking*. This is illustrated in Figure 9. The time of the succession of clock spindles $\mathbf{f}^{(n)}$, which is the level of the scaffolding, is the time of the type-2 evolution. The deletions next to each clock spindle are naturally interpreted as *emigration* as each family of spindles in an excursion above the minimum of the $\text{Stable}(\frac{3}{2})$ process $\xi(\mathbf{N}_*)\big|_{(T_n^+, T_{n+1}^-]}$ is removed from the evolution and such excursions form a homogeneous Poisson point process up to the level where the last clock spindle dies.

Proposition 18. *The process constructed in Definition 17 is a type-2 evolution.*

Proof. Consider a data quadruple $\Psi = (\mathbf{f}_1, \mathbf{f}_2, \overleftarrow{\mathbf{N}}, \mathbf{N}_\beta) \sim \mathbf{P}_{a,b,\beta}^2$ and the filtration $(\mathcal{F}^y, y \geq 0)$ generated by $(\mathbf{f}_1(y), \mathbf{f}_2(y), \overleftarrow{\mathbf{N}}^{\leq y}, \mathbf{N}_\beta^{\leq y})$. We will use the notation of Definition 17 to inductively set up all random variables as needed for Definition 1, and we will show that Definitions 1 and 17, in this setup, yield pathwise the same process $((m_1^y, m_2^y, \alpha^y), y \geq 0)$. For the purpose of this proof we will mark all random variables appearing in Definition 1 by an underscore.

Now, $\underline{\mathbf{f}}^{(0)} := \mathbf{f}_1$ and $(0, \underline{\mathbf{m}}^{(0)}) \star \underline{\gamma}^{(0)} := \overline{\text{SKEWER}}(\text{CLADE}(\mathbf{f}_2, \overleftarrow{\mathbf{N}}) \star \mathbf{N}_\beta)$ have the appropriate joint distribution and achieve $((\underline{m}_1^y, \underline{m}_2^y, \underline{\alpha}^y), 0 \leq y \leq \underline{Y}_1) = ((m_1^y, m_2^y, \alpha^y), 0 \leq y \leq Y_1)$. Suppose we have defined up to $(\underline{\mathbf{f}}^{(n-1)}, \underline{\mathbf{m}}^{(n-1)}, \underline{\gamma}^{(n-1)})$ and identified $((\underline{m}_1^y, \underline{m}_2^y, \underline{\alpha}^y), 0 \leq y \leq \underline{Y}_n) = ((m_1^y, m_2^y, \alpha^y), 0 \leq y \leq Y_n)$ for some $n \geq 1$. Then given $\mathcal{F}^{Y_{n-1}}$, we apply Lemma 8, which is the Markov-like property of the type-1 data triple Ψ_{n-1} at the level $\zeta(\mathbf{f}^{(n-1)}) = Y_n - Y_{n-1}$, to find a post-$\zeta(\mathbf{f}^{(n-1)})$ data triple $\Psi_{n-1}^{\zeta(\mathbf{f}^{(n-1)})}$. The first component of this triple is $\mathbf{f}^{(n)}$ and the last component is $\mathbf{N}_{\Psi_n}$. Noting that $\mathbf{f}^{(n)}$ and $\mathbf{N}_{\Psi_n}$ are conditionally independent given the pre-$\zeta(\mathbf{f}_{n-1})$ data, indeed given $\mathcal{F}^{Y_n}$, we proceed as follows. Suppose n is even. First, $\underline{\mathbf{f}}^{(n)} := \mathbf{f}^{(n)} \sim \text{BESQ}_{m_1^{Y_n}}(-1)$, is as appropriate for Definition 1, since $\underline{Y}_n = Y_n$. Second, $\Psi_n \sim \mathbf{P}_{\alpha^{Y_n}}^1 = \mathbf{P}_{\underline{\alpha}^{Y_n}}^1$, which gives rise to a type-1 evolution $(0, \underline{\mathbf{m}}^{(n)}) \star \underline{\gamma}^{(n)} := \overline{\text{SKEWER}}(\mathbf{N}_{\Psi_n}) \sim \mathbb{P}_{\underline{\alpha}^{Y_n}}^1$, as required, since we have $\underline{m}_2^{Y_n} = \underline{\mathbf{f}}^{(n-1)}(\zeta(\underline{\mathbf{f}}^{(n-1)})) = 0$. This also implies that for all $y \in [0, Y_{n+1} - Y_n)$

$$(\underline{m}_1^{Y_n+y}, (0, \underline{m}_2^{Y_n+y}) \star \underline{\alpha}^{Y_n+y}) = (\underline{\mathbf{f}}^{(n)}(y), (0, \underline{\mathbf{m}}^{(n)}(y)) \star \underline{\gamma}^{(n)}(y)) = (\mathbf{f}^{(n)}(y), \text{SKEWER}(y, \mathbf{N}_{\Psi_n}))$$
$$= (m_1^{Y_n+y}, (0, m_2^{Y_n+y}) \star \alpha^{Y_n+y}),$$

as required. The same argument applies for n odd, with the roles of 1 and 2 interchanged. $\quad\square$

Lemma 19. *If we modify Definition 17 so that we let $\mathbf{N}_* = \text{CLADE}(\mathbf{f}_1, \overleftarrow{\mathbf{N}}) \star \mathbf{N}_\beta$ and $Y_1 = \zeta(\mathbf{f}_2)$ and accordingly swap the parity in Step 4., we obtain a type-2 evolution that is pathwise the same as in Definition 17, with identical sets $\{\zeta(\mathbf{f}_1), \zeta(\mathbf{f}_2)\} \cup \{Y_n, n \geq 0\}$. In particular, the point measure $\overleftarrow{\mathbf{N}}|_{(-\infty, T_{\min\{\zeta(\mathbf{f}_1), \zeta(\mathbf{f}_2)\}}(\overleftarrow{\mathbf{N}})) \times \mathcal{E}}$ is redundant for the type-2 evolution.*

Proof. For the purposes of this proof, we add underscores and write $\underline{Y}_j$, $\underline{\mathbf{f}}_i^{(j)}$, $\underline{\Psi}_j$, $i = 1$ or $i = 2$, $j \geq 0$, and $((\underline{m}_1^y, \underline{m}_2^y, \underline{\alpha}^y), y \geq 0)$ in the modification of Definition 17. We remark that the underscores here are unrelated to those in the previous proof. The main aim of this proof is to show the pathwise equality $((\underline{m}_1^y, \underline{m}_2^y, \underline{\alpha}^y), y \geq 0) = ((m_1^y, m_2^y, \alpha^y), y \geq 0)$. We only discuss the case where $a > 0$ and $b > 0$. The cases where $a = 0$ or $b = 0$ can then be checked similarly.

On the event $\{\zeta(\mathbf{f}_1) < \zeta(\mathbf{f}_2)\}$, we have $\underline{Y}_0 = 0 = Y_0 < Y_1 = \zeta(\mathbf{f}_1) < Y_2 = \zeta(\mathbf{f}_2) = \underline{Y}_1$, and we see inductively that $\underline{Y}_j = Y_{j+1}$, $\underline{\mathbf{f}}_i^{(j)} = \mathbf{f}_i^{(j+1)}$ and $\underline{\Psi}_j = \Psi_{j+1}$ for all $j \geq 1$, $i = 1, 2$. It is now easy to see that the pathwise equality holds on this event. Similarly, on $\{\zeta(\mathbf{f}_1) > \zeta(\mathbf{f}_2)\}$, we have $\underline{Y}_1 = \zeta(\mathbf{f}_2)$ and $\underline{Y}_{j+1} = Y_j$ for all $j \geq 1$, and the same argument applies.

In particular, the sets $\{Y_n, n \geq 0\}$ and $\{\underline{Y}_n, n \geq 0\}$ differ precisely by the omission of either $\zeta(\mathbf{f}_2)$ from the former or of $\zeta(\mathbf{f}_1)$ from the latter. The last statement of the lemma follows using the original definition on $\{\zeta(\mathbf{f}_1) < \zeta(\mathbf{f}_2)\}$ and the modified definition on $\{\zeta(\mathbf{f}_1) > \zeta(\mathbf{f}_2)\}$. $\quad\square$

It is not a priori clear in Definition 1, nor equivalently in Definition 17, that clock changes cannot accumulate at a finite level $Y_\infty = \sup_{n \geq 0} Y_n < \infty$. This would leave the type-2 evolution undefined for $y \geq Y_\infty$, so we address this point before establishing any further properties.

Lemma 20. *For all $(a, b, \beta) \in \mathcal{J}^\circ$, the type-2 evolution as constructed in Definition 17 is such that there is a.s. some finite $n \geq 0$ for which $Y_n < Y_{n+1} = \infty$ and as y increases to Y_n, the evolution (m_1^y, m_2^y, α^y) approaches $(0, 0, \emptyset)$.*

Proof. First, we prove the claimed convergence to $(0, 0, \emptyset)$. The events $\{T_1^+ = \infty\}$, $\{Y_2 = \infty\}$, and $\{\zeta(\overline{\text{SKEWER}}(\mathbf{N}_*)) < Y_1\}$ are equal up to null sets. On these events, m_1^y converges to 0 as y increases to Y_1, and (m_2^y, α^y) are already absorbed at $(0, \emptyset)$ prior to that level. We proceed inductively. On the event $\{T_n^+ < \infty\}$, this time T_n^+ is when the type-1 scaffolding $\mathbf{X}_*$ exceeds level Y_n. Since this scaffolding eventually dies at level 0, we get $T_{n+1}^- < \infty$ a.s.. Now, on the event $\{T_{n+1}^+ = \infty\}$, we apply the same argument as before to $\mathbf{N}_*|_{(T_{n+1}^-, \infty) \times \mathcal{E}}^{\overleftarrow{}}$ in place of $\mathbf{N}_*$, to conclude that (m_1^y, m_2^y, α^y) approaches $(0, 0, \emptyset)$ as y increases to Y_{n+1}.

It remains to show that $Y_n < Y_{n+1} = \infty$ for some $n \geq 1$. We claim that it suffices to prove the following.

(∗)   Consider any two spindles of heights $\zeta(f_1) = c_1$ and $\zeta(f_2) = c_2$ with $c_1 < c_2$. Apply the construction of Definition 17 to $(\mathbf{f}_1, \mathbf{f}_2, \overleftarrow{\mathbf{N}}, \mathbf{N}_\emptyset) = (f_1, f_2, \overleftarrow{\mathbf{N}}, 0)$ for $(\overleftarrow{\mathbf{N}}, 0) \sim \mathbf{P}_\emptyset^0$. Then there is some $n \geq 1$ for which $Y_n < Y_{n+1} = \infty$.

Indeed, once this is shown, $a + b > 0$ in the general case implies $Y_2 > 0$, and only finitely many clades of $\mathbf{N}_\beta$ survive to level Y_2. We apply (∗) to these clades one by one, with c_1 as the final clock level of the preceding clades and c_2 as the next level after c_1 at which the top mass of the next clade vanishes, to see that each clade contributes a finite number of clock change levels.

To prove (∗), we note that this can be read as a statement about the $\mathtt{Stable}\left(\frac{3}{2}\right)$ Lévy process $X = c_2 + \xi\left(\overleftarrow{\mathbf{N}}\big|^{\leftarrow}_{(T_{c_2}(\overleftarrow{\mathbf{N}}),0)\times\mathcal{E}}\right)$. Specifically, note that $\zeta\big(\mathbf{f}_1^{(j)}\big)$ or $\zeta\big(\mathbf{f}_2^{(j)}\big)$ is the overshoot $Y_{j+1} - Y_j$ of X when first crossing level Y_j.

Now we extend X to a $\mathtt{Stable}\left(\frac{3}{2}\right)$ process with infinite lifetime so that $T_j^+ < \infty$ for all $j \geq 1$, and we show that $Y_j \to \infty$. To this end, let $\Delta_n = Y_{n+1} - Y_n$ and $R_n = \Delta_{n+1}/\Delta_n$ for $n \geq 1$. By the strong Markov property of $\mathtt{Stable}\left(\frac{3}{2}\right)$, the conditional distribution of Δ_{n+1} given $\Delta_1, \ldots, \Delta_n$ equals the law of the overshoot of a $\mathtt{Stable}\left(\frac{3}{2}\right)$ process when first crossing Δ_n, which is the same as the overshoot of its $\mathtt{Stable}\left(\frac{1}{2}\right)$ ladder height subordinator [8]. By stable scaling, for each n, R_n is independent of Δ_n and is distributed like the overshoot of a $\mathtt{Stable}\left(\frac{1}{2}\right)$ subordinator across 1. So the sequence $(R_n, n \geq 1)$ is i.i.d. and

$$\Delta_{n+1} = \Delta_1 \cdot \prod_{i=1}^{n} R_i \qquad \text{for } n \geq 1.$$

Thus, $(\log(\Delta_n), n \geq 1)$ is a random walk. It suffices to show that the increments $\log(R_n)$, $n \geq 1$, of this walk have non-negative expected value.

We can get at the law of R_n by taking advantage of the $\mathtt{Stable}\left(\frac{1}{2}\right)$ inverse local time subordinator associated with one-dimensional Brownian motion, $(B(t), t \geq 0)$. In this setting, R_n is distributed like $T - 1$, where T is the time of the first return of B to zero, after time 1. By a calculation based on the reflection principle, we find $\mathbb{P}(T < t) = \frac{2}{\pi}\arctan(t-1)$. Thus,

$$\mathbb{E}\left(\log(R_n)\right) = \int_1^\infty \log(t-1)\mathbb{P}(T \in dt) = \frac{2}{\pi}\int_1^\infty \log(t-1)\frac{1}{(t-1)^2 + 1}dt = 0. \qquad \square$$

3.2. Type-2 evolutions as Borel right Markov processes.
In this section we will prove Theorem 2, i.e. that type-2 evolutions are Borel right Markov processes, and that the IP-valued variant is a path-continuous Hunt process. We listed the properties 1.-4. that this comprises before Proposition 15.

Proof of Theorem 2. 1. By Lemma 20, type-2 evolutions take values in $\mathcal{I}^\circ$ or $\mathcal{J}^\circ$ of (3), which are Lusin as Borel subsets of products of Lusin spaces (see [11, Theorem 2.7]).

2. We first prove the path-continuity of the $\mathcal{I}^\circ$-valued type-2 evolution. For $n \geq 0$, in between Y_n and Y_{n+1}, this process is formed by concatenating a $\mathtt{BESQ}(-1)$ block to the left of an $\mathcal{I}$-valued type-1 evolution, $(0, \mathbf{f}^{(n)}) \star (0, \mathbf{m}^{(n)}) \star \gamma^{(n)}$. The $\mathtt{BESQ}(-1)$ process is continuous and, as noted in Proposition 15, so is the type-1 evolution. By [11, Lemma 2.11], an interval partition process formed by concatenation of two continuous interval partition processes is again continuous. To see continuity at Y_n, first suppose that $n \geq 2$ is even. We note that as y approaches Y_n from below, the $\mathcal{I}^\circ$-valued process approaches $(0,0) \star (0, m_1^{Y_n}) \star \alpha^{Y_n}$, while for y approaching Y_n from above, it approaches $(0, m_1^{Y_n}) \star (0,0) \star \alpha^{Y_n}$, by the continuity of $\mathtt{BESQ}(-1)$ and $\mathcal{I}$-valued type-1 evolution. The argument for odd $n \geq 1$ is the same, with $m_2^{Y_n}$ in the place of $m_1^{Y_n}$.

The càdlàg property of $\mathcal{J}^\circ$-valued type-2 evolution follows similarly from the corresponding property of $\mathcal{J}^\bullet$-valued type-1 evolution proved in Corollary 16. Specifically, continuity at Y_n still holds by the same argument, using the path-continuity at independent random times of $\mathcal{J}^\bullet$-valued type-1 evolutions, which follows from the path-continuity at fixed levels, which in turn holds as no excursion of a $\mathtt{Stable}\left(\frac{3}{2}\right)$ process below the supremum starts from a fixed level.

3. The type of construction undertaken in Definition 1, in which a right Markov process with finite lifetime is reborn at the end of the lifetime according to a probability kernel, has

been studied by Meyer [26]. Type-1 evolutions and $\mathtt{BESQ}(-1)$ processes are Borel right Markov processes (see Corollary 16), and thus so too is the process $\left((m_1^y, m_2^y, \alpha^y, 1), 0 \leq y \leq Y_1\right)$ starting from any $(a, b, \beta) \in \mathcal{J}^\circ$ with $a > 0$ and killed at Y_1. By swapping the parity as in the statement of Lemma 19, we can similarly define $\left((m_1^y, m_2^y, \alpha^y, 2), 0 \leq y \leq Y_1\right)$ starting from $(a, b, \beta) \in \mathcal{J}^\circ$ with $b > 0$ and killed at Y_1, where the fourth component $I(y) = 1$ or $I(y) = 2$ records which of the two top blocks is evolving according to $\mathtt{BESQ}(-1)$ and which is forming a type-1 evolution with α^y. We define the deterministic kernel $\overline{N}((0, x, \beta, 1); \cdot) = \delta_{(0,x,\beta,2)}$, $\overline{N}((x, 0, \beta, 2); \cdot) = \delta_{(x,0,\beta,1)}$. As noted in [38, Definition 8.1], Borel right Markov processes are right Markov processes satisfying the hypothèses droites, in Meyer's sense. Therefore, we can apply [26, Théorème 1 and Remarque on p.474] to conclude that if we alternate killed processes with $I(y) = 1$ and $I(y) = 2$, using transitions according to $\overline{N}$ to determine initial states from the previous killing state,

$$\text{the process } \left((m_1^y, m_2^y, \alpha^y, I(y)), y \geq 0\right) \text{ is a right Markov process,} \tag{15}$$

satisfying the strong Markov property. It is not hard to show that the semigroup of this process is Borel, see e.g. the last point in the proof of [7, Théorème (3.18)]. In Proposition 24 we strengthen this to continuity.

Lemma 19 verifies Dynkin's criterion to show that the $\mathcal{J}^\circ$-valued type-2 evolution is a right Markov process as well. To see that the $\mathcal{I}^\circ$-valued type-2 evolution is a right Markov process, just note that every state $(0, a) \star (0, b) \star \beta \in \mathcal{I}^\circ$ corresponds to two states $(a, b, \beta, 1)$ and $(b, a, \beta, 2)$, but that both are based on $\mathtt{BESQ}_a(-1)$ and type-1 evolution from (b, β) and hence construct the same process, apart from maintaining opposite last components $I(y)$. Hence, Dynkin's criterion applies again.

4. The Hunt property of $\mathcal{I}^\circ$-valued type-2 evolutions holds since sample paths are continuous. $\qquad\square$

In Section 4.5 we prove a Hölder continuity result for type-2 evolutions started from certain initial distributions, with bounds on all moments of the Hölder constants. It is possible to mimic [11, Proof of Proposition 5.11] and appeal to the construction of Definition 17 to prove Hölder continuity with index $\theta \in (0, \frac{1}{4})$ at all times after time zero, from any initial state, but in this setting we could not also give the desired bounds, so we omit such arguments here.

In order to establish continuity of the semigroup of type-2 evolution in the initial condition we require some intermediate results.

Lemma 21. *Suppose that $((b_n, \beta_n), n \geq 1)$ is a sequence in $(\mathcal{J}^\bullet, d_\bullet)$ that converges to (b, β) and that $(x_n, n \geq 1)$ is a sequence of levels converging to $x > 0$. Let $((m_n^y, \gamma_n^y), y \geq 0)$ and $((m^y, \gamma^y), y \geq 0)$ be type-1 evolutions started from (b_n, β_n) and (b, β) respectively. If $f \colon \mathcal{J}^\bullet \to \mathbb{R}$ is bounded and continuous, then*

$$\mathbb{E}\left[f(m_n^{x_n}, \gamma_n^{x_n})\right] \to \mathbb{E}\left[f(m^x, \gamma^x)\right].$$

Proof. If $g \colon \mathcal{I} \to \mathbb{R}$ is bounded and continuous, then the fact that

$$\mathbb{E}\left[g\left((0, m_n^{x_n}) \star \gamma_n^{x_n}\right)\right] \to \mathbb{E}\left[g\left((0, m^x) \star \gamma^x\right)\right].$$

is established in the proof of [11, Proposition 5.20]. The slightly stronger version that separates out convergence of the top mass follows from the coupling used in that proof. Specifically, that proof reduces the argument to finitely many clades, each of which is composed of an initial spindle and an independent $\mathtt{Stable}\left(\frac{3}{2}\right)$ Lévy process. Furthermore, the ladder height process of a $\mathtt{Stable}\left(\frac{3}{2}\right)$ Lévy process, in which the leftmost spindle at each level can be found, is a $\mathtt{Stable}\left(\frac{1}{2}\right)$ subordinator. The probability that x is in its range is zero, so that the evolution of the leftmost mass is continuous around level x with probability one. $\qquad\square$

It will be convenient to augment the type-2 evolution $(\Gamma^y, y \geq 0)$ by the counting process $J(y) = \inf\{j \geq 0 \colon Y_{j+1} > y\}$ counting its clock changes. This process $((\Gamma^y, J(y)), y \geq 0)$ can be constructed as a strong Markov process as in (15) and similarly relates to $(\Gamma^y, y \geq 0)$ by

Dynkin's criterion. Let p be the parity map sending even numbers to 2 and odd numbers to 1. The state space for the evolution $((\Gamma^y, J(y)), y \geq 0)$ is the set

$$\mathcal{J}^+ = \{((m_1, m_2, \beta), j) \in \mathcal{J}^\circ \times \mathbb{N}_0 \colon m_{p(j+1)} > 0\}.$$

In the following lemma, we write $\mathbb{E}_{\gamma,j}$ to denote the expectation for the augmented process starting from $(\gamma, j) \in \mathcal{J}^+$.

Lemma 22. *Suppose that $(\Gamma^y, y \geq 0)$ is a type-2 evolution with clock change levels Y_j. Then*

(i) for all $f \colon \mathcal{J}^+ \to \mathbb{R}$ bounded and continuous

$$\mathbb{E}\left[f(\Gamma^{Y_j+u}, J(Y_j + u))\big|\mathcal{F}^{Y_j}\right] = \mathbb{E}_{\Gamma^{Y_j},j}\left[f(\Gamma^u, J(u))\right], \qquad \mathbb{P}\text{-}a.s..$$

(ii) for all $h \colon \mathcal{J}^\circ \to \mathbb{R}$ bounded and continuous and for $\mathbb{P}$-a.e. ω

$$\mathbb{E}\left[h(\Gamma^y)\mathbf{1}\{Y_j \leq y < Y_{j+1}\}\big|\mathcal{F}^{Y_j}\right](\omega)$$
$$= \mathbf{1}\{Y_j(\omega) \leq y\}\mathbb{E}_{\Gamma^{Y_j(\omega)}(\omega),j}\left[h(\Gamma^{y \vee Y_j(\omega) - Y_j(\omega)})\mathbf{1}\{y \vee Y_j(\omega) - Y_j(\omega) < Y_1\}\right].$$

Proof. The first claim is immediate from the construction of type-2 evolutions and the second follows from the proof of [9, Theorem 2.3.3] applied to the augmented Markov process $((\Gamma^y, J(y)), y \geq 0)$. The book [9] assumes that the Markov process takes place on a locally compact state space, but that is not needed in the proof of Theorem 2.3.3. The right-continuous dependence of the semigroup on time needed in the proof follows from the right-continuity of sample paths. $\qquad\square$

Next we establish weak continuity at clock levels.

Lemma 23. *Suppose that $(a_n, b_n, \beta_n) \to (a, b, \beta)$ in $(\mathcal{J}^\circ, d_\mathcal{J})$ with $a > 0$. Let $(\Gamma_n^y, y \geq 0)$ and $(\Gamma^y, y \geq 0)$ be type-2 evolutions started from (a_n, b_n, β_n) and (a, b, β) respectively with respective clock levels Y_k^n and Y_k. Then*

$$\mathbb{E}\left[f\left(\Gamma_n^{Y_j^n}, Y_j^n\right)\right] \to \mathbb{E}\left[f\left(\Gamma^{Y_j}, Y_j\right)\right].$$

Proof. We first establish the claim for $j = 1$. Let $((\Gamma_n^y, J_n(y)), y \geq 0)$ and $((\Gamma^y, J(y)), y \geq 0)$ be the augmented type-2 evolutions started from $(a_n, b_n, \beta_n, 0)$ and $(a, b, \beta, 0)$. Let $\mathbf{f}^{(0)}$ be a $\mathtt{BESQ}(-1)$ started from a, let $(\mathbf{m}^{(0)}, \gamma^{(0)})$ be an independent type-1 evolution started from (b, β), and let $(\mathbf{m}_n^{(0)}, \gamma_n^{(0)})$ be a type-1 evolution, independent from $\mathbf{f}^{(0)}$, and started from (b_n, β_n). From the construction of type-2 evolutions, we see that

$$(\Gamma^y, 0 \leq y \leq Y_1) \overset{d}{=} \left(\left(\mathbf{f}^{(0)}(y), \mathbf{m}^{(0)}(y), \gamma^{(0)}(y)\right), 0 \leq y \leq Y_1\right)$$

and

$$(\Gamma_n^y, 0 \leq y \leq Y_1^n) \overset{d}{=} \left(\left(\frac{a_n}{a}\mathbf{f}^{(0)}\left(\frac{a}{a_n}y\right), \mathbf{m}_n^{(0)}(y), \gamma_n^{(0)}(y)\right), 0 \leq y \leq Y_1^n\right). \qquad (16)$$

Note that, from this construction, $Y_1^n = (a_n/a)Y_1$. Furthermore, from [19, Equation (13)] we see that Y_1 is distributed like $a/(2G)$ where $G \sim \mathtt{Gamma}\left(\frac{3}{2}, 1\right)$. In particular, Y_1 has a continuous density q on $(0, \infty)$. Disintegrating based on the value of Y_1^n, we see that

$$\mathbb{E}\left[f(\mathbf{f}_n^{(0)}(Y_1^n), \mathbf{m}_n^{(0)}(Y_1^n), \gamma_n^{(0)}(Y_1^n), Y_1^n)\right] = \int_0^\infty \mathbb{E}\left[f\left(0, \mathbf{m}_n^{(0)}(x), \gamma_n^{(0)}(x), x\right)\right] \frac{a}{a_n} q\left(\frac{ax}{a_n}\right) dx.$$

It follows from Lemma 21 and a version of the dominated convergence theorem (e.g. [22, Theorem 1.21]) that

$$\mathbb{E}_{(a_n,b_n,\beta_n),0}\left[f\left(\Gamma^{Y_1}, Y_1\right)\right] \to \mathbb{E}_{(a,b,\beta),0}\left[f\left(\Gamma^{Y_1}, Y_1\right)\right]. \qquad (17)$$

This completes the proof for $j = 1$, for all $a > 0$, $b \geq 0$ and $\beta \in \mathcal{I}$. The same proof applied to augmented type-2 evolutions started from $(a_n, b_n, \beta_n, 1)$ and $(a, b, \beta, 1)$ shows

$$\mathbb{E}_{(a_n,b_n,\beta_n),1}\left[f\left(\Gamma^{Y_1}, Y_1\right)\right] \to \mathbb{E}_{(a,b,\beta),1}\left[f\left(\Gamma^{Y_1}, Y_1\right)\right], \qquad (18)$$

for all $a \geq 0$, $b > 0$ and $\beta \in \mathcal{I}$. The inductive step $j \to j + 1$ follows from the strong Markov property of the augmented type-2 evolutions at clock levels Y_j^n and Y_j, applying (18) for odd j and (17) for even j. $\qquad\square$

Proposition 24. *Fix $y \geq 0$ and define $F_y \colon \mathcal{J}^\circ \to \mathcal{P}(\mathcal{J}^\circ)$, by letting $F_y(a, b, \beta)$ be the law at level y of a type-2 evolution starting from the initial state $(a, b, \beta) \in \mathcal{J}^\circ$. Then $(a, b, \beta) \mapsto F_y(a, b, \beta)$ is weakly continuous. Similarly define $G_y(\gamma)$ as the law at level y for the IP-valued variant starting from $\gamma \in \mathcal{I}^\circ$. Then $\gamma \mapsto G_y(\gamma)$ is weakly continuous on $\mathcal{I}^\circ$.*

Proof. We first prove the $\mathcal{J}^\circ$-valued case. Suppose that $(a_n, b_n, \beta_n) \to (a, b, \beta)$ in $(\mathcal{J}^\circ, d_\mathcal{J})$, i.e. $a_n \to a$, $b_n \to b$ and $d_\mathcal{I}(\beta_n, \beta) \to 0$. We may assume without loss of generality that $a > 0$. Once the proof is complete for this subcase, we can apply Lemma 19 to deduce the subcase $a = 0$, $b > 0$; the subcase $a = b = 0$, $\beta = \emptyset$ is trivial. Let $(\Gamma_n^y, y \geq 0)$ and $(\Gamma^y, y \geq 0)$ be $\mathcal{J}^\circ$-valued type-2 evolutions started from (a_n, b_n, β_n) and (a, b, β), respectively, with respective clock levels $(Y_j^n)_{j \geq 0}$ and $(Y_j)_{j \geq 0}$. Observe that for all bounded continuous $f \colon \mathcal{J}^\circ \to \mathbb{R}$

$$\mathbb{E}\left[f\left(\Gamma_n^y\right)\right] = \sum_{j=0}^\infty \mathbb{E}\left[f\left(\Gamma_n^y\right)\mathbf{1}\{Y_j^n \leq y < Y_{j+1}^n\}\right]. \tag{19}$$

By Lemma 23 and the Skorohod representation theorem, we may now assume $Y_j^n(\omega) \to Y_j(\omega)$, $d_\mathcal{J}\left(\Gamma_n^{Y_j^n(\omega)}(\omega), \Gamma^{Y_j(\omega)}(\omega)\right) \to 0$, and since $\mathbb{P}(Y_j = y) = 0$, also $\mathbf{1}\{Y_j^n(\omega) \leq y\} \to \mathbf{1}\{Y_j(\omega) \leq y\}$ for $\mathbb{P}$-a.e. ω. Recall that Y_1 and the initial clock spindle under $\mathbb{P}_{\gamma,j}$ are associated with the block labeled 1 when j is even and with the block labeled 2 when j is odd. For $\gamma = \Gamma^{Y_j(\omega)}(\omega)$ or $\Gamma_n^{Y_j^n(\omega)}(\omega)$, in either case this is the non-zero top mass of γ. Recall also from (16) that $\texttt{BESQ}(-1)$ processes with converging initial states can be coupled to converge uniformly together with their lifetimes. In particular, we can use their convergence in distribution together with Lemma 21 for the convergence of the second top mass and interval partitions at level $y \vee Y_j^n(\omega) - Y_j^n(\omega) \to y \vee Y_j(\omega) - Y_j(\omega)$ to obtain for $\mathbb{P}$-a.e. ω

$$\mathbb{E}_{\Gamma_n^{Y_j^n(\omega)}(\omega),j}\left[f\left(\Gamma^{y \vee Y_j^n(\omega) - Y_j^n(\omega)}\right)\mathbf{1}\left\{y \vee Y_j^n(\omega) - Y_j^n(\omega) < Y_1\right\}\right]$$
$$\to \mathbb{E}_{\Gamma^{Y_j(\omega)}(\omega),j}\left[f\left(\Gamma^{y \vee Y_j(\omega) - Y_j(\omega)}\right)\mathbf{1}\left\{y \vee Y_j(\omega) - Y_j(\omega) < Y_1\right\}\right].$$

By Lemma 22(ii) and applying the previous convergences and dominated convergence, we find

$$\mathbb{E}\left(f(\Gamma_n^y)\mathbf{1}\{Y_j^n \leq y < Y_{j+1}^n\}\right) \to \mathbb{E}\left[f(\Gamma^y)\mathbf{1}\{Y_j \leq y < Y_{j+1}\}\right]. \tag{20}$$

A further application of the dominated convergence theorem yields $\mathbb{E}[f(\Gamma_n^y)] \to \mathbb{E}[f(\Gamma^y)]$, completing the proof in the $\mathcal{J}^\circ$-valued case.

We now consider the $\mathcal{I}^\circ$-valued case and suppose that $(0, a_n) \star (0, b_n) \star \beta_n \to (0, a) \star (0, b) \star \beta$ with $a > 0$ and $b \geq 0$, the convergence now being with respect to the $d_\mathcal{I}$-metric. We emphasize that this is weaker than convergence of the triples for the $d_\mathcal{J}$-metric and we could have, for example, $a_n \to 0$ and $b_n \to 0$ as sequences of real numbers. By Definition 14, there exist sequences $A_n, B_n \in (0, a_n) \star (0, b_n) \star \beta_n$, $\alpha_n^{(0)} \in \mathcal{I}^\circ$ and $\alpha_n^{(1)}, \alpha_n^{(2)} \in \mathcal{I}$ such that

$$(0, a_n) \star (0, b_n) \star \beta_n = \alpha_n^{(0)} \star A_n \star \alpha_n^{(1)} \star B_n \star \alpha_n^{(2)},$$

with $\mathrm{Leb}(A_n) \to a$, $\mathrm{Leb}(B_n) \to b$, $d_\mathcal{I}(\alpha_n^{(i)}, \emptyset) \to 0$ for $i \in \{0, 1\}$, and $d_\mathcal{I}(\alpha_n^{(2)}, \beta) \to 0$. Let $\check{\Psi}_n = (\mathbf{f}_{n,1}^{(0)}, \mathbf{f}_{n,2}^{(0)}, \overleftarrow{\mathbf{N}}_n^{(0)}, \mathbf{N}_n^{(0)}) \sim \mathbf{P}_{\alpha_n^{(0)}}^2$, $(\mathbf{f}_n^{(1)}, \overleftarrow{\mathbf{N}}_n^{(1)}, \mathbf{N}_n^{(1)}) \sim \mathbf{P}_{\mathrm{Leb}(A_n), \alpha_n^{(1)}}^1$, and $(\mathbf{f}_n^{(2)}, \overleftarrow{\mathbf{N}}_n^{(2)}, \mathbf{N}_n^{(2)}) \sim \mathbf{P}_{\mathrm{Leb}(B_n), \alpha_n^{(2)}}^1$ be independent. Observe that

$$\Psi_n := \left(\mathbf{f}_{n,1}^{(0)}, \mathbf{f}_{n,2}^{(0)}, \overleftarrow{\mathbf{N}}_n^{(0)}, \mathbf{N}_n^{(0)} \star \mathrm{CLADE}(\mathbf{f}_n^{(1)}, \overleftarrow{\mathbf{N}}_n^{(1)}) \star \mathbf{N}_n^{(1)} \star \mathrm{CLADE}(\mathbf{f}_n^{(2)}, \overleftarrow{\mathbf{N}}_n^{(2)}) \star \mathbf{N}_n^{(2)}\right) \sim \mathbf{P}_{a_n, b_n, \beta_n}^2,$$

and

$$\widetilde{\Psi}_n := \left(\mathbf{f}_n^{(1)}, \mathbf{f}_n^{(2)}, \overleftarrow{\mathbf{N}}_n^{(2)}, \mathbf{N}_n^{(2)}\right) \sim \mathbf{P}_{\mathrm{Leb}(A_n), \mathrm{Leb}(B_n), \alpha_n^{(2)}}^2.$$

Let $(\check{\gamma}_n^y, y \geq 0)$, $(\gamma_n^y, y \geq 0)$ and $(\widetilde{\gamma}_n^y, y \geq 0)$ be the IP-valued type-2 evolutions constructed from $\check{\Psi}_n$, Ψ_n and $\widetilde{\Psi}_n$ by deletion clocking as in Definition 17, concatenating top mass intervals as in Definition 1. Let $\tau_n = \inf\{y > 0 \colon \check{\gamma}_n^y = \emptyset\}$. Since $\mathrm{Leb}(A_n) \to a > 0$ and $(\|\check{\gamma}_n^y\|, y \geq 0) \sim \texttt{BESQ}_{\|\alpha_n^{(0)}\|}(-1)$ with $\|\alpha_n^{(0)}\| \to 0$, we have $\mathbb{P}(\tau_n < \zeta(\mathbf{f}_n^{(1)})) \to 1$. It is clear from the definitions

that the first block (taken from the clock spindle straddling level y) of γ_n^y is the first block of $\check{\gamma}_n^y$ for $0 \leq y < \tau_n$, whereas it is given by $\mathbf{f}_n^{(1)}(y)$ for $\tau_n \wedge \zeta(\mathbf{f}_n^{(1)}) \leq y < \zeta(\mathbf{f}_n^{(1)})$. Furthermore, the conditional distribution of $(\gamma_n^y, \; \tau_n \wedge \zeta(\mathbf{f}_n^{(1)}) \leq y < \zeta(\mathbf{f}_n^{(1)}))$ given $\check{\Psi}$ only depends on τ_n. It is the same as the conditional distribution given τ_n of

$$\left((0, \mathbf{f}_n^{(1)}(y)) \star \widehat{\alpha}_n^y \star \alpha_n^y, \; \tau_n \wedge \zeta(\mathbf{f}_n^{(1)}) \leq y < \zeta(\mathbf{f}_n^{(1)}) \right),$$

where the three processes are independent, $(\alpha_n^y, y \geq 0)$ is a type-1 evolution, and $(\widehat{\alpha}_n^y, y \geq 0)$ is a type-0 evolution up to level τ_n and then continues as a type-1 evolution. In particular, $(\|\widehat{\alpha}_n^y\|, y \geq 0)$ is a $\mathtt{BESQ}(1)$ starting from $\|\alpha_n^{(1)}\| \to 0$ up to level τ_n and then continues as $\mathtt{BESQ}(0)$. We conclude that for $\tau_n' = \inf\{y > \tau_n \colon \widehat{\alpha}_n^y = \emptyset\}$, we have

$$\mathbb{P}(\tau_n \leq \tau_n' < \zeta(\mathbf{f}_n^{(1)})) \to 1.$$

Since $\gamma_n^y = \widetilde{\gamma}_n^y$ for all $y \geq \tau_n' \geq \tau_n$ on the event $\{\tau_n \leq \tau_n' < \zeta(\mathbf{f}_n^{(1)})\}$ and $\mathbb{P}(\tau_n \leq \tau_n' \leq y) \to 1$, we find $\mathbb{P}(\gamma_n^y = \widetilde{\gamma}_n^y) \to 1$. This reduces the proof to the case when $a_n \to a$ and $b_n \to b$. The argument is now similar to the $\mathcal{J}^\circ$-valued case. We decompose as in (19) and then apply (20) to functions of the form

$$f(a, b, \beta) = \begin{cases} g((0, a) \star (0, b) \star \beta) & \text{for } j \text{ even,} \\ g((0, b) \star (0, a) \star \beta) & \text{for } j \text{ odd,} \end{cases}$$

which are continuous for all bounded continuous $g \colon \mathcal{I}^\circ \to \mathbb{R}$, by [11, Lemma 2.11]. $\qquad\square$

3.3. The total mass process.

In this section, we prove Theorem 3, that the total mass process of a type-2 evolution is a $\mathtt{BESQ}(-1)$. Our approach is to use the $\mathtt{BESQ}(-1)$ processes $\mathbf{f}^{(n)}$ and type-1 evolutions $(\mathbf{m}^{(n)}, \gamma^{(n)})$ with $\mathtt{BESQ}(0)$ total mass, $j \geq 0$. Since the type-2 total mass process is built from the sum of these, the following additivity lemma will be useful. This extends the well-known additivity of $\mathtt{BESQ}$ processes with nonnegative parameters.

Lemma 25. *Let $X \sim \mathtt{BESQ}_a(-1)$, $W \sim \mathtt{BESQ}_b(0)$ and $\underline{Z} \sim \mathtt{BESQ}_1(-1)$ be independent. Consider the times $T_X = \inf\{t \geq 0 \colon X_t = 0\}$, $T_W = \inf\{t \geq 0 \colon W_t = 0\}$ and $\tau = T_X \wedge T_W$. Define a process*

$$V_t = \begin{cases} X_t + W_t, & t \leq \tau, \\ Z_{t-\tau}, & t > \tau, \end{cases}$$

where $Z_s = (X_\tau + W_\tau)\underline{Z}_{s/(X_\tau + W_\tau)}$, $s \geq 0$. Then $V \sim \mathtt{BESQ}_{a+b}(-1)$.

Proof. Consider a probability space where all three processes $X, W, \underline{Z}$ are supported. On a standard extension of the sample space, there exist two independent Brownian motions $\beta, \underline{\beta}$ such that

$$\begin{aligned} d(X_t + W_t) &= -dt + 2\sqrt{X_t + W_t}\, d\beta_t, & 0 \leq t \leq \tau, \\ d\underline{Z}_s &= -ds + 2\sqrt{\underline{Z}_s}\, d\underline{\beta}_s, & s \geq 0. \end{aligned} \tag{21}$$

Consider the process

$$B_t = \begin{cases} \beta_t, & 0 \leq t \leq \tau, \\ \beta_\tau + \sqrt{X_\tau + W_\tau}\, \underline{\beta}_{(t-\tau)/(X_\tau + W_\tau)}, & t > \tau. \end{cases}$$

Then, it follows by Lévy's characterization of Brownian motion that B is a standard one-dimensional Brownian motion.

On this same probability space consider the strong solution of the stochastic differential equation (SDE)

$$dU_t = -dt + 2\sqrt{U_t}\, dB_t, \quad U_0 = a + b.$$

It is well-known [36, Chapter XI] that the above SDE has a strong solution that is pathwise unique. Obviously, $U \sim \mathtt{BESQ}_{a+b}(-1)$. However, it is clear from (21) that the process V also satisfies the relation $dV_t = -dt + 2\sqrt{V_t}\, dB_t$. Hence, by pathwise uniqueness, $V = U$, almost surely. Thus $V \sim \mathtt{BESQ}_{a+b}(-1)$. $\qquad\square$

Proof of Theorem 3. Consider a type-2 evolution $((m_1^y, m_2^y, \beta^y), y \geq 0)$ as constructed in Definition 1, from initial state $(a, b, \beta) \in \mathcal{J}^\circ$. If (a, b, β) equals $(a, 0, \emptyset)$ or $(0, b, \emptyset)$ then the result is trivial from the construction, so assume not. Then by Proposition 20, there is a.s. some finite $K \geq 0$ for which the evolution dies at time Y_{K+1}. During the interval $[Y_K, Y_{K+1})$, there is a *degeneration time* $D = \inf\{y \geq 0 : (m_1^y, \beta^y) = (0, \emptyset) \text{ or } (m_2^y, \beta^y) = (0, \emptyset)\}$ when the type-1 evolution $\gamma^{(K)}$ dies while the top block $\mathbf{f}^{(K)}$ continues to live until Y_{K+1}.

By the strong Markov property and Definition 1, after time D, the type-2 evolution comprises a single non-zero component m_i^y, with i being either 1 or 2, evolving as a $\mathtt{BESQ}(-1)$ until its absorption at zero. Let $\underline{Z}$ denote the $\mathtt{BESQ}_1(-1)$ process obtained by applying $\mathtt{BESQ}$ scaling to normalize mass of this component at degeneration: $\underline{Z}_y := (m_i^D)^{-1} m_i^{D + m_i^D y}$, $y \geq 0$. By the strong Markov property, $\underline{Z}$ is independent of the type-2 evolution run up until time D.

We define $D_n := \min\{Y_n, D\}$, $n \geq 0$, so that $D_n = D$ for n sufficiently large, and set

$$V_y := m_1^y + m_2^y + \|\beta^y\|, \quad V_y^{(n)} := \begin{cases} V_y & \text{if } y \leq D_n, \\ Z_{y-D_n}^{(n)} & \text{if } y > D_n, \end{cases} \quad \text{where } Z_s^{(n)} = V_{D_n} \underline{Z}_{s/V_{D_n}}, \ s \geq 0.$$

We will show inductively that all $V^{(n)}$, $n \geq 1$, and hence the a.s. limit $V = \lim_{n \to \infty} V^{(n)}$, are $\mathtt{BESQ}_{a+b+\|\beta\|}(-1)$.

For $n = 1$, we have $V_y = X_y + Y_y$, $0 \leq y \leq D_1$, where $X = \mathbf{f}^{(0)} \sim \mathtt{BESQ}_a(-1)$ and $Y = \mathbf{m}^{(0)} + \|\gamma^{(0)}\|$ independent, and with $D_1 = \min\{T_X, T_Y\}$ as in Lemma 25. Since $Y \sim \mathtt{BESQ}_{b+\|\beta\|}(0)$ by Proposition 10, Lemma 25 yields $V^{(1)} \sim \mathtt{BESQ}_{a+b+\|\beta\|}(-1)$.

Now, assume for induction that for some $n \geq 1$, $\widehat{V}^{(n)} \sim \mathtt{BESQ}_{\widehat{a}+\widehat{b}+\|\widehat{\beta}\|}(-1)$ for all type-2 evolutions $((\widehat{m}_1^y, \widehat{m}_2^y, \widehat{\beta}^y), y \geq 0)$ starting from any $(\widehat{a}, \widehat{b}, \widehat{\beta}) \in \mathcal{J}^\circ$. By the strong Markov property, we can apply the inductive hypothesis to $(\widehat{m}_1^y, \widehat{m}_2^y, \widehat{\beta}^y) := (m_1^{D_1+y}, m_2^{D_1+y}, \beta^{D_1+y})$, $y \geq 0$, on the event $\{Y_1 = D_1\} = \{D > Y_1\}$. Then $\widehat{D}_n = D_{n+1} - D_1$ and $\underline{\widehat{Z}} = \underline{Z}$. We see that

$$\begin{aligned} V_y^{(n+1)} &= \begin{cases} V_y & \text{if } y \leq D_{n+1}, \\ Z_{y-D_{n+1}}^{(n+1)} & \text{if } y > D_{n+1}, \end{cases} \\ &= \left. \begin{cases} V_y & \text{if } y \leq D_1, \\ \widehat{V}_{y-D_1} & \text{if } D_1 < y \leq D_1 + \widehat{D}_n, \\ \widehat{Z}_{y-D_1-\widehat{D}_n}^{(n)} & \text{if } y > D_1 + \widehat{D}_n, \end{cases} \right\} = \begin{cases} V_y & \text{if } y \leq D_1, \\ \widehat{V}_{y-D_1}^{(n)} & \text{if } y > D_1. \end{cases} \end{aligned}$$

By the inductive hypothesis, $\widehat{V}^{(n)} \sim \mathtt{BESQ}_{m_1^{D_1}+m_2^{D_1}+\|\beta^{D_1}\|}(-1)$, and by the strong Markov property and $\mathtt{BESQ}$ scaling, $((\widehat{V}_0^{(n)})^{-1} \widehat{V}_{s\widehat{V}_0^{(n)}}^{(n)}, s \geq 0) \sim \mathtt{BESQ}_1(-1)$ is unconditionally independent of $\mathcal{F}^{D_1}$, and hence of $((X_y, Y_y), 0 \leq y \leq D_1)$. Then, by the $n = 1$ case already established, we conclude that $V^{(n+1)} \sim \mathtt{BESQ}_{a+b+\|\beta\|}(-1)$, as required. $\qquad\square$

3.4. The Markov-like property of type-2 data quadruples.

We can extend the definition of cutoff data from type-1 evolutions, as seen before Lemma 8, to type-2 evolutions.

Definition 26. In the setting of Definition 17, for $j \geq 0$ even and $y \in [Y_j, Y_{j+1})$,

$$\left(\mathbf{f}_{\Psi,1}^y, \overleftarrow{\mathbf{N}}_\Psi^y, \mathbf{N}_\Psi^y\right) := \left(\mathbf{f}_{\Psi_j}^{y-Y_j}, \overleftarrow{\mathbf{N}}_{\Psi_j}^{y-Y_j}, \mathbf{N}_{\Psi_j}^{y-Y_j}\right), \quad \mathbf{f}_{\Psi,2}^y := \left(\mathbf{f}^{(j)}(y - Y_j + z), \ z \geq 0\right).$$

We make the same definition for $j \geq 1$ odd, but with subscripts '1' and '2' reversed. We also write $\Psi^y = (\mathbf{f}_{\Psi,1}^y, \mathbf{f}_{\Psi,2}^y, \overleftarrow{\mathbf{N}}_\Psi^y, \mathbf{N}_\Psi^y)$ for the cutoff data quadruple.

Recall notation $J(y)$, denoting the number of clock changes, and $I(y)$, denoting the index of the clock, for $y \geq 0$:

$$J(y) = \inf\{j \geq 0 : Y_{j+1} > y\}, \qquad I(y) = 1 \text{ if } J(y) + 1 \text{ is odd, or } I(y) = 2 \text{ if even.} \qquad (22)$$

In light of the previous definition, (14) can be rewritten as

$$m_{I(y)}^y = \mathbf{f}_{\Psi,I(y)}^y(0), \qquad \alpha^y = \mathrm{SKEWER}\left(0, \mathbf{N}_\Psi^y\right).$$

It should be clear from the independence of the $\mathtt{BESQ}(-1)$ top mass processes that $T_j^+ = \infty$ may happen for any $j \geq 1$. As a consequence of the argument of the proof of Lemma 20, it will, in fact, happen for some random finite $j \geq 1$, in such a way that the $(j-1)^{\mathrm{st}}$ type-1 evolution of Definition 17 vanishes at a level strictly below the last top mass process. We denote these extinction levels by $\zeta_i^+ = \inf\{y \geq 0 \colon m_i^z = 0 \text{ for all } z \geq y\}$, $i = 1, 2$. We write $D = \min\{\zeta_1^+, \zeta_2^+\}$ and $\zeta = \max\{\zeta_1^+, \zeta_2^+\}$. We call level ζ the *lifetime* of the type-2 evolution and level D its *degeneration time*.

Consider a type-2 data quadruple $(\mathbf{f}_1, \mathbf{f}_2, \overleftarrow{\mathbf{N}}, \mathbf{N}_\beta)$. Recall the definition above Lemma 7 of the point process $\mathbf{N}^{\leq y}$ of spindles below level y, based on the type-0 data pair $(\overleftarrow{\mathbf{N}}, \mathbf{N}_\beta)$. In the context of type-2 data, we denote the right-continuous natural filtration of $(\mathbf{f}_1(y), \mathbf{f}_2(y), \mathbf{N}^{\leq y})$, $y \geq 0$ by $(\mathcal{F}^y, y \geq 0)$, again abusing notation to suppress the dependence on type 2.

The cases when $b = 0$ and $\beta = \emptyset$, or when $a = 0$ and $\beta = \emptyset$, are one-dimensional since no non-trivial type-1 point data triple is ever formed in Definition 17. We therefore have $(m_1^y, m_2^y, \alpha^y) = (\mathbf{f}_1(y), 0, \emptyset)$, $y \geq 0$, or $(m_1^y, m_2^y, \alpha^y) = (0, \mathbf{f}_2(y), \emptyset)$, $y \geq 0$, respectively, and this degenerate type-2 evolution inherits the Markov property from $\mathtt{BESQ}(-1)$. For other initial states, we establish a Markov-like property of a form similar to Lemmas 7 and 8.

Throughout this section, $(\mathbf{f}_1, \mathbf{f}_2, \overleftarrow{\mathbf{N}}, \mathbf{N}_\beta) \sim \mathbf{P}_{a,b,\beta}^2$, where $a, b \geq 0$, $a + b > 0$, and $\beta \in \mathcal{I}$. Following Definition 17, let $\mathbf{N}_* := \mathrm{CLADE}(\mathbf{f}_2, \overleftarrow{\mathbf{N}}) \star \mathbf{N}_\beta$ and $\mathbf{X}_* := \xi(\mathbf{N}_*)$. From [11, proof of Proposition 5.11], the local time process associated with $\mathbf{X}_*$, denoted by $(\ell_{\mathbf{X}_*}^y(t); y, t \geq 0)$, is a.s. continuous in both level and time coordinates. Therefore, for the cutoff processes $\mathbf{N}_*^y$ and $\mathbf{N}_*^{\leq y}$ and their associated scaffolding processes $\mathbf{X}_*^y := \xi(\mathbf{N}_*^y)$ and $\mathbf{X}_*^{\leq y} := \xi(\mathbf{N}_*^{\leq y})$, we can define local times $(\ell_{\mathbf{X}_*^y}^0(t), t \geq 0)$ and $(\ell_{\mathbf{X}_*^{\leq y}}^0(t), t \geq 0)$ by extending continuously, approaching level y from above and below, respectively. Moreover,

$$\ell_{\mathbf{X}_*^{\leq y}}^y(\phi(t)) = \ell_{\mathbf{X}_*}^y(t) = \ell_{\mathbf{X}_*^y}^0(t - \phi(t)) \quad \text{where} \quad \phi(t) := \mathrm{Leb}\{s \leq t \colon \mathbf{X}_*(s) \leq y\}. \tag{23}$$

Proposition 27. *Let* $\Psi = (\mathbf{f}_1, \mathbf{f}_2, \overleftarrow{\mathbf{N}}, \mathbf{N}_\beta) \sim \mathbf{P}_{a,b,\beta}^2 = \mathtt{BESQ}_a(-1) \otimes \mathtt{BESQ}_b(-1) \otimes \mathbf{P}_\beta^0$ *for some* $a, b \in [0, \infty)$ *and* $\beta \in \mathcal{I}$, *so that at least two of* a, b *and* $\|\beta\|$ *are strictly positive. For* $y \geq 0$, *given* $\mathcal{F}^y$, $(\mathbf{f}_{\Psi,1}^y, \mathbf{f}_{\Psi,2}^y, \overleftarrow{\mathbf{N}}_\Psi^y, \mathbf{N}_\Psi^y)$ *has conditional distribution* $\mathbf{P}_{m_1^y, m_2^y, \alpha^y}^2 = \mathtt{BESQ}_{m_1^y}(-1) \otimes \mathtt{BESQ}_{m_2^y}(-1) \otimes \mathbf{P}_{\alpha^y}^0$.

We prove this by way of the following.

Lemma 28. *Fix* $y > 0$. *On* $\{\zeta > y\}$, *let*

$$J := J(y) = \inf\{j \geq 0 \colon Y_{j+1} > y\}, \quad S := T_J^+, \quad T := \inf\{t > T_{J+1}^- \colon \mathbf{X}_*(t) \geq y\}.$$

On $\{\zeta \leq y\}$, *let* $S = T = \infty$. *Then the local times* $\ell_{\mathbf{X}_*}^y(S)$ *and* $\ell_{\mathbf{X}_*}^y(T)$ *are measurable in* $\mathcal{F}^y$.

Proof. Consider the cutoff processes

$$\Psi^{\leq y} := ((\mathbf{f}_1(z), z \leq y), (\mathbf{f}_2(z), z \leq y), \overleftarrow{\mathbf{N}}^{\leq y}, \mathbf{N}_\beta^{\leq y}), \qquad \mathbf{X}_*^{\leq y} := \xi(\mathbf{N}_*^{\leq y}),$$

where $\mathbf{N}_\beta^{\leq y}$ is associated with the type-1 point measure $\mathbf{N}_\beta$, and $\overleftarrow{\mathbf{N}}^{\leq y}$ is such that $\overleftarrow{\mathbf{N}}^{\leq y} \star \mathbf{N}_\beta^{\leq y} = \mathbf{N}^{\leq y}$ with $\mathbf{N}^{\leq y}$ associated with the type-0 data $(\overleftarrow{\mathbf{N}}, \mathbf{N}_\beta)$ as in Section 2. Now, suppose we apply the construction of Definition 17 to $\Psi^{\leq y}$ in place of Ψ. For clarity, we refer to the times and levels associated with this construction on $\Psi^{\leq y}$ as U_j^+, U_j^- and Z_j, for $j \geq 0$, rather than T_j^+, T_j^- and Y_j, which are associated with the construction on Ψ.

Let ϕ be as in (23). We now show by induction that: (i) $U_{j-1}^+ = \phi(T_{j-1}^+)$, (ii) $Z_j = \min\{Y_j, y\}$, and (iii) $U_j^- = \phi(T_j^-)$ for all $1 \leq j \leq J+1$. By definition, $U_1^- = \phi(T_1^-) = U_0^+ = \phi(T_0^+) = 0$ and $Z_1 = \min\{Y_1, y\}$. If $J = 0$, then this completes the proof. Otherwise, assume the assertion holds up to some index $j \leq J$. Then $\mathbf{X}_*|_{[T_j^-, T_j^+)}^\leftarrow = \mathbf{X}_*^{\leq y}|_{[U_j^-, U_j^+)}^\leftarrow$, as, by our hypotheses, $\phi(T_j^-) = U_j^-$ and $\mathbf{X}_*$ is bounded above by $Y_j \leq y$ on this interval. This implies $Z_{j+1} = \min\{Y_{j+1}, y\}$ and $U_j^+ = \phi(T_j^+)$. Then T_{j+1}^- is the time of the next return of $\mathbf{X}_*$ to level $Y_j = Z_j$, so $\phi(T_{j+1}^-)$ is the time of next return of $\mathbf{X}_*^{\leq y}$ to this level, which equals U_{j+1}^-, as desired.

By (23), $\ell^y_{\mathbf{X}_*}(S) = \ell^y_{\mathbf{X}_*^{\leq y}}(\phi(T_J^+)) = \ell^y_{\mathbf{X}_*^{\leq y}}(U_J^+)$. Finally, $J = \inf\{j \geq 0 \colon Z_{j+1} = y\}$ a.s.. Thus, $\ell^y_{\mathbf{X}_*}(S)$ is measurable in $\mathcal{F}^y$. Similarly, $\phi(T) = \inf\{t > U_{J+1}^- \colon \mathbf{X}_*^{\leq y}(t) = y\} =: T'$. Thus, $\ell^y_{\mathbf{X}_*}(T) = \ell^y_{\mathbf{X}_*^{\leq y}}(T')$ is also measurable in $\mathcal{F}^y$. $\qquad\square$

Proof of Proposition 27. In the case $y = 0$, the assertion is trivial. Fix $y > 0$. Consider the type-1 data $(\mathbf{f}_2, \overleftarrow{\mathbf{N}}, \mathbf{N}_\beta)$ and associated point process $\mathbf{N}_* = \textsc{clade}(\mathbf{f}_2, \overleftarrow{\mathbf{N}}) \star \mathbf{N}_\beta$. Let $((m_*^z, \alpha_*^z), z \geq 0)$ denote the resulting type-1 evolution. Let $\mathbf{N}_*^y$ denote the cutoff process, as in Lemma 8. We restrict to the a.s. event that $\mathbf{N}_*$ behaves *nicely* about level y, in the sense that no two excursions about the level occur at the same local time.

Let J, S, and T be as in Lemma 28. We first work on the event $\{y < D,\ J \geq 1,\ J \text{ is even}\} \in \mathcal{F}^y$. Theorem 37 of [15] asserts that, if a spindle occurs at time t and survives to level y, then the corresponding block in the level y skewer occurs at diversity $\ell^y_{\mathbf{X}_*}(t)$. Thus, m_1^y and m_2^y are respectively the masses of the unique blocks $V_1, V_2 \in (0, m_*^y) \star \alpha_*^y$ for which $\mathscr{D}_{(0,m_*^y)\star\alpha_*^y}(V_1) = \ell^y(S)$ and $\mathscr{D}_{(0,m_*^y)\star\alpha_*^y}(V_2) = \ell^y(T)$. Finally, α^y corresponds to the set of blocks of α_*^y that have diversity greater than $\ell^y(T)$ to their left:

$$\alpha_*^y = \{U \in \alpha_*^y \colon \mathscr{D}_{\alpha_*^y}(U) \leq \ell^y(T)\} \star \alpha^y.$$

In particular, from Lemma 28 and the property that the type-1 evolution is adapted, we find that (m_1^y, m_2^y, α^y) is measurable in $\mathcal{F}^y$.

Analogously to the discussion of m_1^y and m_2^y, since J is even, $\mathbf{f}_{\Psi,1}^y = (\mathbf{f}^{(J)}(y - Y_J + z), z \geq 0)$ and $\mathbf{f}_{\Psi,2}^y$ are cut off the spindles that cross level y at times S and T, respectively. In other words, $\mathbf{f}_{\Psi,1}^y$ and $\mathbf{f}_{\Psi,2}^y$ are cut off the middle spindles of the excursions about y at local times $\ell^y_{\mathbf{X}_*}(S)$ and $\ell^y_{\mathbf{X}_*}(T)$, respectively. Let $R := \inf\{t > T \colon \mathbf{X}_*(t) = y\}$. Then, by Definition 26,

$$\overleftarrow{\mathbf{N}}_\Psi^y = \overleftarrow{\mathbf{N}}\big|^{\rightarrow}_{(-\infty, T_{\mathbf{X}_*(T)}(\overleftarrow{\mathbf{N}})]} \star \mathbf{N}_*^y\big|^{\rightarrow}_{(T - \phi(T), R - \phi(R))} \qquad \text{and} \qquad \mathbf{N}_\Psi^y = \mathbf{N}_*^y\big|^{\leftarrow}_{[R - \phi(R), \infty)},$$

where ϕ is as in (23). Proposition 5.6 of [11] implies that, given $\mathcal{F}^y$, the cutoff process $\mathbf{N}_*^y$ is conditionally distributed as $\bigstar_{U \in (0, m_*^y) \star \alpha_*^y} \mathbf{N}_U$, where each $\mathbf{N}_U$ is a clade distributed as $\textsc{clade}(\mathbf{f}_U, \overleftarrow{\mathbf{N}})$, with $\mathbf{f}_U \sim \textsc{besq}_{\mathrm{Leb}(U)}(-1)$ independent of $\overleftarrow{\mathbf{N}}$, and these clades are all conditionally independent given $(0, m_*^y) \star \alpha_*^y$. Similarly, for $\{y < D, J \geq 1, J \text{ is odd}\}$, the same argument applies, with roles of 1 and 2 swapped. On $\{J = 0\} = \{Y_1 > y\}$, we can just apply the Markov-like property for type-1 evolutions and the Markov property of $\textsc{besq}(-1)$. We conclude from this and the previous two paragraphs that $(\mathbf{f}_{\Psi,1}^y, \mathbf{f}_{\Psi,2}^y, \overleftarrow{\mathbf{N}}_\Psi^y, \mathbf{N}_\Psi^y)$ has the claimed conditional law. $\qquad\square$

3.5. **Type-2 evolutions via interweaving two type-1 point measures.** In this section we present another construction of type-2 evolutions from initial states in which the interval partition component is an independent multiple of a $\textsc{pdip}\left(\frac{1}{2}, \frac{1}{2}\right)$ random variable. Such interval partitions appear as pseudo-stationary distributions of type-0 and type-1 evolutions, and indeed, we will use this construction to study pseudo-stationarity properties of type-2 evolutions, including projections of type-2 evolutions to three-mass processes that only retain the evolution of the two top masses and the total mass of the interval partition.

Consider independent A and B for which $\mathbb{P}(A + B > 0) = 1$. Also consider independent $C_1, C_2 \sim \texttt{Gamma}(\frac{1}{2}, \gamma)$ and $\overline{\beta}_1, \overline{\beta}_2 \sim \textsc{pdip}(\frac{1}{2}, \frac{1}{2})$ independent of (A, B). Let $(\mathbf{f}_1, \overleftarrow{\mathbf{N}}_1, \mathbf{N}_{\beta_1})$ and $(\mathbf{f}_2, \overleftarrow{\mathbf{N}}_2, \mathbf{N}_{\beta_2})$ be two independent type-1 data triples with $\mathbf{f}_1(0) = A$, $\mathbf{f}_2(0) = B$, $\beta_1 = C_1\overline{\beta}_1$ and $\beta_2 = C_2\overline{\beta}_2$. Let $\mathbf{N}_1 := \textsc{clade}(\mathbf{f}_1, \overleftarrow{\mathbf{N}}_1) \star \mathbf{N}_{\beta_1}$, and correspondingly define $\mathbf{N}_2$. We will combine these to define a process $((\widetilde{m}_1^y, \widetilde{m}_2^y, \widetilde{\alpha}^y), y \geq 0)$ that we will show is a type-2 evolution. This *interweaving* construction is illustrated in Figure 10.

Let $\mathbf{X}_1 := \xi(\mathbf{N}_1)$ and $\mathbf{X}_2 := \xi(\mathbf{N}_2)$. We set $T_{-1} := T_0 := 0$, $Z_0 := 0$ and $Z_1 := \zeta(\mathbf{f}_1)$. For $i \geq 1$ we define

$$
\begin{aligned}
T_{2i-1} &:= \inf\{t \geq T_{2i-3} \colon \mathbf{X}_2(t) > Z_{2i-1}\}, & Z_{2i} &:= \mathbf{X}_2(T_{2i-1}), \\
T_{2i} &:= \inf\{t \geq T_{2i-2} \colon \mathbf{X}_1(t) > Z_{2i}\}, & Z_{2i+1} &:= \mathbf{X}_1(T_{2i}),
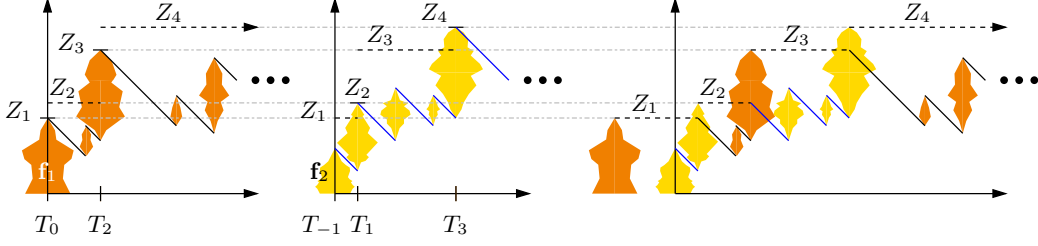\end{aligned}
\tag{24}
$$

FIGURE 10. Interweaving can be thought of as alternating intervals, $\mathbf{N}_i|_{(T_{j-2},T_j]}^{\leftarrow}$ from two type-1 scaffoldings with spindles, $(\mathbf{N}_1, \mathbf{N}_2)$. We begin with a single spindle, $\mathbf{f}_1$, from $\mathbf{N}_1$. Then, we include an interval from $\mathbf{N}_2$ until the time T_1 at which its scaffolding exceeds the death level $Z_1 = \zeta(\mathbf{f}_1)$, reaching some higher level Z_2. To this, we concatenate an interval from $\mathbf{N}_1$ until the time T_2 at which its scaffolding exceeds level Z_2, reaching some higher level Z_3, and so on.

with the conventions that $\inf(\emptyset) = \infty$ and $\mathbf{X}_1(\infty) = \infty$ and $\mathbf{X}_2(\infty) = \infty$. Also note that this includes setting $T_1 = 0$ if $\zeta(\mathbf{f}_2) > \zeta(\mathbf{f}_1)$. Let p denote the *parity* map, sending even numbers to 2 and odd numbers to 1. For $y \geq 0$ we define

$$\widetilde{I}(y) := p\big(\inf\{j \geq 0 \colon Z_{j+1} > y\}\big), \qquad \widetilde{J}(\infty) := \inf\{j \geq 1 \colon T_j = \infty\},$$

$$\big(0, \widetilde{m}_{3-\widetilde{I}(y)}^y\big) \star \big(0, \widetilde{m}_{\widetilde{I}(y)}^y\big) \star \widetilde{\alpha}^y := (0, \mathbf{f}_1(y)) \star (0, \mathbf{f}_2(y)) \star \widetilde{\theta}(y) \qquad (25)$$

where $\widetilde{\theta}(y) = \mathrm{SKEWER}\Big(y - \zeta(\mathbf{f}_2), \mathbf{N}_2\big|_{(0,T_1]\times\mathcal{E}}\Big) \star \underset{2 \leq j \leq \widetilde{J}(\infty)}{\bigstar} \mathrm{SKEWER}\Big(y - Z_{j-1}, \mathbf{N}_{p(j+1)}\big|_{(T_{j-2},T_j]\times\mathcal{E}}^{\leftarrow}\Big)$.

By this we mean that, (i) if the expression on the right of (25) has a leftmost block (note that this equals $(0, \mathbf{f}_1(y))$ if and only if $y < \zeta(\mathbf{f}_1)$), then we take $\widetilde{m}_{3-\widetilde{I}(y)}^y$ to denote the mass of this block, otherwise setting $\widetilde{m}_{3-\widetilde{I}(y)}^y := 0$; and (ii) if said expression has a second-to-leftmost block, then we denote its mass by $\widetilde{m}_{\widetilde{I}(y)}^y$, otherwise setting $\widetilde{m}_{\widetilde{I}(y)}^y := 0$. Then $\widetilde{\alpha}^y$ denotes what remains of $\widetilde{\theta}(y)$ after removing leftmost blocks as required to form $\widetilde{m}_1^y$ and $\widetilde{m}_2^y$ and, if necessary, shifting the remaining interval partition down to line up with 0 on its left end.

Proposition 29. *The process* $((\widetilde{m}_1^y, \widetilde{m}_2^y, \widetilde{\alpha}^y), y \geq 0)$ *defined in* (25) *is a type-2 evolution with initial state* $(\widetilde{m}_1^0, \widetilde{m}_2^0, \widetilde{\alpha}^0) = (A, B, C\overline{\beta})$, *where* A, B, C *and* $\overline{\beta}$ *are jointly independent, with* $C \sim \mathtt{Gamma}(\frac{1}{2}, \gamma)$ *and* $\overline{\beta} \sim \mathtt{PDIP}(\frac{1}{2}, \frac{1}{2})$.

Before we prove this proposition, we recall a simpler construction of pseudo-stationary type-1 data triples that does not require concatenating infinitely many clades.

Proposition 30 (Corollary 4.28(ii) of [11]). *Fix* $\gamma > 0$. *Let* $\mathbf{N}$ *denote a* $\mathtt{PRM}(\mathrm{Leb} \otimes \nu_{\mathrm{BES}})$ *independent of* $S \sim \mathtt{Exponential}(\gamma)$. *We define* $T := \inf\{t > 0 \colon M_{\mathbf{N}}^0(t) > S\}$, *where* $M_{\mathbf{N}}^0$ *is the aggregate mass process of* (7). *Then* $\beta := \mathrm{SKEWER}\big(0, \mathbf{N}|_{[0,T)}\big)$ *is a* $\mathtt{PDIP}\big(\frac{1}{2}, \frac{1}{2}\big)$ *scaled by an independent* $\mathtt{Gamma}\big(\frac{1}{2}, \gamma\big)$, *and, recalling the notation above Lemma 7,* $\big(\mathbf{N}|_{[0,T)}\big)^0$ *is a type-1 point measure with initial state* β. *Moreover,* $\big(\overleftarrow{\mathbf{N}}, \big(\mathbf{N}|_{[0,T)}\big)^0\big)$ *is pseudo-stationary type-0 data with* $\mathtt{Gamma}\big(\frac{1}{2}, \gamma\big)$ *initial mass.*

In the setting of this construction, we write $\mathbf{N}_\beta := \big(\mathbf{N}|_{[0,T)}\big)^0$. Now, let $\mathbf{f}$ denote a $\mathtt{BESQ}(-1)$ independent of the other objects, with any random initial mass, and define $\mathbf{N}_* := \mathrm{CLADE}(\mathbf{f}, \overleftarrow{\mathbf{N}}) \star \mathbf{N}_\beta$. In the special case that $\mathbf{f}(0) \sim \mathtt{Gamma}\big(\frac{1}{2}, \gamma\big)$, the measure $\mathbf{N}_*$ describes a pseudo-stationary type-1 evolution with $\mathtt{Exponential}(\gamma)$ initial mass, as in Proposition 12. For any distribution of $\mathbf{f}(0)$, this construction has the following consequence, by way of the Poisson property of $\mathbf{N}$ and the memorylessness of S.

Lemma 31 (Memorylessness for some type-1 point measures). *Fix* $\gamma > 0$ *and let* $\mathbf{N}_*$ *be as above. Let* R *be a stopping time in the right-continuous time filtration* $(\mathcal{F}_t, t \geq 0)$ *generated by* $\mathbf{N}_*$,

i.e. the least right-continuous filtration in which $\mathbf{N}_*|_{[0,t]}$ *is* $\mathcal{F}_t$*-measurable for every* $t \geq 0$. *Given* $\mathbf{N}_*|_{[0,R]\times\mathcal{E}}$ *with* $\xi_{\mathbf{N}_*}(R) = x$, *and further conditioning on* $\{\mathbf{N}_*|_{(R,\infty)\times\mathcal{E}} \neq 0\}$, *the conditional distribution of* $\mathbf{N}_*|_{(R,\infty)\times\mathcal{E}}^{\leftarrow}$ *equals the (unconditioned) distribution of* $\overleftarrow{\mathbf{N}}|_{[T_x(\overleftarrow{\mathbf{N}}),0)\times\mathcal{E}}^{\leftarrow} \star \mathbf{N}_\beta$.

Proof of Proposition 29. Let $(\mathbf{f}_1, \mathbf{f}_2, \overleftarrow{\mathbf{N}}, \mathbf{N}_\beta)$ be data for a type-2 evolution $((m_1^y, m_2^y, \alpha^y), y \geq 0)$ starting from the initial distribution as claimed. We follow the notation of Definition 17 and (22). Additionally, we define $J(\infty) := \inf\{j \geq 1 : T_j^+ = \infty\}$. We prove our assertion by showing

$$\left(\mathbf{f}_1, \mathbf{f}_2, \left(\mathbf{N}_{p(i+1)}|_{(T_{i-2},T_i]\times\mathcal{E}}^{\leftarrow}, Z_i\right), 1 \leq i \leq \widetilde{J}(\infty)\right) \stackrel{d}{=} \left(\mathbf{f}_1, \mathbf{f}_2, \left(\mathbf{N}_*|_{(T_i^-,T_i^+]\times\mathcal{E}}^{\leftarrow}, Y_i\right), 1 \leq i \leq J(\infty)\right) \quad (26)$$

$$\text{and} \quad \left(0, m_{I(y)}^y\right) \star \left(0, m_{3-I(y)}^y\right) \star \alpha^y = (0, \mathbf{f}_1(y)) \star (0, \mathbf{f}_2(y)) \star \theta(y) \quad \text{for all } y \geq 0, \quad (27)$$

where $\theta(y) := \text{SKEWER}\left(y - \zeta(\mathbf{f}_2), \mathbf{N}_*|_{(0,T_1^+]\times\mathcal{E}}\right) \star \bigstar_{2 \leq i \leq J^\infty} \text{SKEWER}\left(y - Y_{i-1}, \mathbf{N}_*|_{(T_i^-,T_i^+]\times\mathcal{E}}^{\leftarrow}\right)$.
These formulas, together with (25), complete the proof.

First, we prove (26). For $i \geq 1$, we note the equality of events

$$\left\{\widetilde{J}(\infty) = i\right\} = \left\{T_i = \infty; \widetilde{J}(\infty) \geq i\right\} = \left\{\sup_t \xi_{\mathbf{N}_{p(i+1)}|_{(T_{i-2},T_i]\times\mathcal{E}}^{\leftarrow}}(t) < Z_i - Z_{i-1}; \widetilde{J}(\infty) \geq i\right\}. \quad (28)$$

We conclude, by a recursive argument, that the indicator $\mathbf{1}\{\widetilde{J}(\infty) \leq j\}$ is a function of the $1 \leq i \leq j \wedge \widetilde{J}(\infty)$ terms on the left in (26). By a corresponding argument, the indicator $\mathbf{1}\{J(\infty) \leq j\}$ is a function of the $1 \leq i \leq j \wedge J(\infty)$ terms on the right.

We now establish the base case for an induction. By definition, $T_{-1} = T_1^- = 0$, $Z_1 = Y_1$, and $\mathbf{N}_2 \stackrel{d}{=} \mathbf{N}_*$. Recall from (24) that T_1 is the time when $\mathbf{X}_2$ first exceeds Z_1, while T_1^+ in (12) is the time when $\mathbf{X}_*$ exceeds Y_1. This proves equality in distribution for the $i = 1$ terms of (26).

Assume for induction that, for some $j \geq 1$, (26) holds when we substitute $j \wedge \widetilde{J}(\infty)$ for the $\widetilde{J}(\infty)$ bound on the left and substitute $j \wedge J(\infty)$ for $J(\infty)$ on the right. By the argument following (28), $\mathbb{P}\{\widetilde{J}(\infty) \leq j\} = \mathbb{P}\{J(\infty) \leq j\}$. We now show that the conditional distribution of the $(j+1)^{\text{st}}$ term on the left in (26), given the preceding terms and the event $\{j < \widetilde{J}(\infty)\}$, equals the conditional law of the corresponding term on the right given the preceding terms and the event $\{j < J(\infty)\}$.

Note that

$$Z_{j+1} = \mathbf{X}_{p(j+1)}(T_j) = Z_{j-1} + \xi_{\mathbf{N}_{p(j+1)}|_{(T_{j-2},T_j]\times\mathcal{E}}^{\leftarrow}}(T_j - T_{j-2}) =: G\left(\left(\mathbf{N}_{p(i+1)}|_{(T_{i-2},T_i]\times\mathcal{E}}^{\leftarrow}, Z_i\right), i \leq j\right)$$

and $Y_{j+1} = G\left(\left(\mathbf{N}_*|_{(T_i^-,T_i^+]\times\mathcal{E}}^{\leftarrow}, Y_i\right), i \leq j\right)$. Next, observe that $\mathbf{X}_{p(j+1)}(T_{j-2}) = Z_{j-1}$ while, correspondingly, $\mathbf{X}_*(T_j^-) = Y_{j-1}$. Since we have conditioned on $\{j < \widetilde{J}(\infty)\}$, which means $T_j < \infty$, we may apply Lemma 31 to $\mathbf{N}_{p(j+1)}$ at this time. In particular, by the independence of $\mathbf{N}_1$ and $\mathbf{N}_2$, and by this lemma, given Z_j, the restricted process $\mathbf{N}_{p(j+1)}|_{(T_{j-2},\infty)\times\mathcal{E}}^{\leftarrow}$ is conditionally independent of all preceding terms on the left in (26). Correspondingly, $\mathbf{N}_*|_{(T_j^-,\infty)\times\mathcal{E}}^{\leftarrow}$ is conditionally independent of all preceding terms on the right in (26), given Y_j, and these restricted point processes have the same conditional distribution. Finally, $T_j^+ - T_j^-$ is the first time that $\xi(\mathbf{N}_*|_{(T_j^-,\infty)\times\mathcal{E}}^{\leftarrow})$ exceeds $Y_j - Y_{j-1}$, and correspondingly for $\mathbf{N}_{p(j+1)}$. This completes our induction and proves (26).

We now prove (27). Recall the deletion clocking construction of the type-2 evolution in Definition 17.

Case 1: $y < \min\{\zeta(\mathbf{f}_1), \zeta(\mathbf{f}_2)\}$. Then $J(y) = 0$, $I(y) = 2$, so the two leftmost blocks on the left hand side of (27) are $(0, m_1^y) \star (0, m_2^y)$, which equal $(0, \mathbf{f}_1(y)) \star (0, \mathbf{f}_2(y))$, as claimed. By definition, $\mathbf{X}_*$ is bounded below by $Y_i \geq Y_1 > y$ on each interval $(T_i^+, T_{i+1}^-]$. Therefore,

$$\alpha^y = \text{SKEWER}\left(y - \zeta(\mathbf{f}_2), \mathbf{N}_*|_{(0,\infty)\times\mathcal{E}}\right) = \theta(y), \quad (29)$$

as desired. Indeed, $\theta(y)$, as defined following (27), simply skips over certain intervals of $\mathbf{N}_*$ that cannot contribute to the skewer.

Case 2: $\zeta(\mathbf{f}_2) \le y < \zeta(\mathbf{f}_1)$. Then, again, $J(y) = 0$ and $I(y) = 2$. As before, $(0, m_1^y) = (0, \mathbf{f}_1(y))$, in agreement with (27). However, now $\mathbf{f}_2(y) = 0$. Thus,

$$(0, m_2^y) \star \alpha^y = \text{SKEWER}(y, \mathbf{N}_*) = \text{SKEWER}\left(y - \zeta(\mathbf{f}_2), \mathbf{N}_*|_{(0,\infty) \times \mathcal{E}}\right) = \theta(y),$$

since, as in Case 1, $\theta(y)$ skips over intervals that do not contribute.

Case 3: $\zeta(\mathbf{f}_1) \le y < \zeta(\mathbf{f}_2)$. Then $J(y) = I(y) = 1$ and $T_1^+ = 0$. Then $(0, m_{3-I(y)}^y) = (0, m_2^y) = (0, \mathbf{f}_2(y))$, while $\mathbf{f}_1(y) = 0$, in agreement with (27). Moreover,

$$(0, m_1^y) \star \alpha^y = \text{SKEWER}(y - Y_1, \mathbf{N}_*|_{(T_2^-, \infty) \times \mathcal{E}}^{\leftarrow}).$$

In this case, since $T_1^- = T_1^+$, the first term in the formula for $\theta(y)$ is empty. Then, the concatenation of subsequent terms in $\theta(y)$ equals the above expression, since $\mathbf{X}_*$ is bounded below by $Y_i \ge Y_2 > y$ on each interval $(T_i^+, T_{i+1}^-]$ with $i \ge 2$.

Case 4: $\max\{\zeta(\mathbf{f}_1), \zeta(\mathbf{f}_2)\} \le y$. Then $J(y) \ge 1$ and $T_{J(y)+1}^- > 0$. Moreover, $\mathbf{f}_1(y) = \mathbf{f}_2(y) = 0$, so all that remains on the right in (27) is $\theta(y)$. Note that $\mathbf{X}_*$ is bounded above by $Y_{J(y)} \le y$ on each interval $(T_i^-, T_i^+]$ with $i < J(y)$, as well as on $(T_{J(y)}^-, T_{J(y)}^+)$. Then $\mathbf{X}_*$ jumps up across level y at time $T_{J(y)}^+$, giving rise to the broken spindle $\mathbf{f}_{3-I(y)}^{(J(y))}$. Thus, the terms in $\theta(y)$ with $i < J(y)$ do not contribute, and the $i = J(y)$ term contributes only a single block:

$$\left(0, m_{3-I(y)}^y\right) = \left(0, \mathbf{f}_{3-I(y)}^{(J(y))}(y - Y_{J(y)})\right) = \text{SKEWER}\left(y - Y_{J(y)-1}, \mathbf{N}_*|_{(T_{J(y)}^-, T_{J(y)}^+] \times \mathcal{E}}^{\leftarrow}\right).$$

Then

$$\left(0, m_{I(y)}^y\right) \star \alpha^y = \text{SKEWER}\left(y - Y_{J(y)}, \mathbf{N}_*|_{(T_{J(y)+1}^-, \infty) \times \mathcal{E}}^{\leftarrow}\right),$$

which equals the concatenation of terms in $\theta(y)$ over $i > J(y)$, since, similarly to the previous cases, this expression skips over intervals where $\mathbf{X}_*$ is bounded below by $Y_{J(y)+1} > y$. $\qquad \square$

After Definition 17 we interpreted the spindles removed during deletion clocking as emigration. Where is the emigration in the interweaving construction? The interweaving construction is based on two type-1 evolutions (without emigration). The one with the shorter lifetime is completely incorporated into the type-2 evolution, while the one with the longer lifetime will only be incorporated up to the clock spindle that exceeds that shorter lifetime. Following this clock spindle is a $\text{Stable}\left(\frac{3}{2}\right)$ process with excursions above the minimum that allow an analogous interpretation of emigration as in deletion clocking.

4. STATIONARITY AND CONNECTION TO WRIGHT-FISHER PROCESSES

In this section, we prove Theorem 4, which describes a stationary variant of the type-2 evolution, constructed by normalizing and time-changing (de-Poissonizing) the type-2 evolution and allowing it to jump back into stationarity (resample) instead of being absorbed in a single-block state at degeneration times.

4.1. Pseudo-stationarity.

Since type-2 evolutions degenerate to a single block of positive mass before reaching zero total mass, pseudo-stationarity results differ from Proposition 11 for types 0 and 1. Specifically, we obtain results conditionally given that degeneration has not yet happened. We furthermore identify the total mass at degeneration conditionally given the time of degeneration.

Proposition 32 (Pseudo-stationarity of type-2 evolution). *Consider $(A_1, A_2, A_3) \sim \text{Dir}\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ and an independent interval partition $\bar{\beta} \sim \text{PDIP}\left(\frac{1}{2}, \frac{1}{2}\right)$, with $M(0) > 0$ an independent random variable. Let $((m_1^y, m_2^y, \alpha^y), y \ge 0)$ denote a type-2 evolution initially distributed as $(M(0)A_1, M(0)A_2, M(0)A_3\bar{\beta})$. Let $(M(y), y \ge 0)$ denote its total mass process. For $y > 0$, given $\{D > y\}$, $M(y)$ is conditionally independent of $(m_1^y/M(y), m_2^y/M(y), \alpha^y/M(y))$. The latter is conditionally distributed according to the (unconditioned) law of $(A_1, A_2, A_3\bar{\beta})$.*

In light of this result, we refer to the law of $(M(0)A_1, M(0)A_2, M(0)A_3\overline{\beta})$ above as the *pseudo-stationary law for type-2 evolution* with mass $M(0)$. Following the proof in [11] of Proposition 11 above, we first prove this for $M(0) \sim \mathtt{Gamma}\left(\frac{3}{2}, \gamma\right)$, $\gamma > 0$ and then generalize via Laplace inversion.

Proposition 33. *Consider a type-2 evolution* $((m_1^y, m_2^y, \alpha^y), y \geq 0)$ *with initial blocks* (m_1^0, m_2^0) *independent of* $\alpha^0 = M\overline{\beta}$, *where* $M \sim \mathtt{Gamma}\left(\frac{1}{2}, \gamma\right)$ *and* $\overline{\beta} \sim \mathtt{PDIP}\left(\frac{1}{2}, \frac{1}{2}\right)$ *are independent. Then for* $y > 0$, *given* $\{D > y\}$, *the interval partition* α^y *is conditionally independent of* (m_1^y, m_2^y), *conditionally distributed according to the (unconditional) law of* $(2y\gamma + 1)M\overline{\beta}$.

If, additionally, m_1^0 *and* m_2^0 *are i.i.d.* $\mathtt{Gamma}\left(\frac{1}{2}, \gamma\right)$, *then given* $\{D > y\}$, m_1^y *and* m_2^y *are conditionally i.i.d.* $\mathtt{Gamma}\left(\frac{1}{2}, \gamma/(2y\gamma + 1)\right)$.

Proof. Let $(\mathbf{f}_1, \mathbf{f}_2, \overleftarrow{\mathbf{N}}, \mathbf{N}_\beta)$ be type-2 data for this evolution. From Proposition 30, we may assume $\mathbf{N}_\beta = \left(\mathbf{N}|_{[0,T)}\right)^0$, where $\mathbf{N}$ is a $\mathtt{PRM}(\mathrm{Leb} \otimes \nu_{\mathrm{BES}})$ on $[0, \infty) \times \mathcal{E}$ and T is the time at which the aggregate mass of spindles crossing level 0, as defined in (7), first exceeds an independent mass threshold $S \sim \mathtt{Exponential}\,(\gamma)$.

We follow the notation of (22), in which $I(y)$ denotes the index, 1 or 2, of the clock mass at level y. So $m_{I(y)}^y$ is the clock mass at that level, $m_{3-I(y)}^y$ is the non-clock top mass at that level, α^y is the interval partition of remaining, "spinal" masses, and $\mathbf{N}_* := \mathrm{CLADE}(\mathbf{f}_2, \overleftarrow{\mathbf{N}}) \star \mathbf{N}_\beta$. Let $((m_*^z, \alpha_*^z), z \geq 0)$ denote the type-1 evolution $\overline{\mathrm{SKEWER}}(\mathbf{N}_*)$.

It follows from Definition 17 that, on $\{D > y\}$, the non-clock top mass at level y is the mass of a spindle found in $\mathbf{N}_*|_{[0,T)}$ at the stopping time $R = \inf\{t > T_{J(y)+1}^- : \xi_{\mathbf{N}_*}(t) > y\} < T$, and the remaining interval partition α^y equals $\mathrm{SKEWER}(y - \xi_{\mathbf{N}_*}(R), \mathbf{N}_*|_{(R,T)})$.

Let $R' := \inf\{t > R : \xi_{\mathbf{N}_*}(t) = y\}$ and $T_y(\mathbf{N}_*) := \inf\{t \geq 0 : \xi_{\mathbf{N}_*}(t) = y\}$. By Lemma 31, the conditional law of $\mathbf{N}_*|_{(R',\infty)}^{\leftarrow}$ given $\{D > y\}$ and $(\mathbf{f}_1, \mathbf{N}_*|_{[0,R']})$ equals the conditional law of $\mathbf{N}_*|_{(T_y(\mathbf{N}_*),\infty)}^{\leftarrow}$ given $\{m_*^y + \|\alpha_*^y\| > 0\}$. Passing to the skewers, the correspondingly conditioned laws of α^y and α_*^y are equal. By Proposition 12, this is an independent $\mathtt{Gamma}\left(\frac{1}{2}, \gamma/(2y\gamma + 1)\right)$ multiple of a $\mathtt{PDIP}\left(\frac{1}{2}, \frac{1}{2}\right)$. This also implies that α^y is conditionally independent of (m_1^y, m_2^y) given $\{D > y\}$, proving the first assertion of the proposition.

To prove the second assertion, we apply Lemma 29. In the representation there, we have $D = \min\{\zeta_1, \zeta_2\}$. In particular, conditioning on $\{D > y\}$ is the same as conditioning on $\{\zeta_1 > y, \zeta_2 > y\}$. By Proposition 12 and the independence of the two pseudo-stationary type-1 evolutions in that construction, m_1^y and m_2^y are conditionally independent given $\{\zeta_1 > y, \zeta_2 > y\}$, with common distribution $\mathtt{Gamma}\left(\frac{1}{2}, \gamma/(2y\gamma + 1)\right)$. $\qquad\square$

Proposition 34. *For $a, b, c > 0$ and $\overline{\beta} \sim \mathtt{PDIP}\left(\frac{1}{2}, \frac{1}{2}\right)$, consider a type-2 evolution starting from $(a, b, c\overline{\beta})$. Let $\overline{\beta}'$ be an independent $\mathtt{PDIP}\left(\frac{1}{2}, \frac{1}{2}\right)$, and let $\widetilde{\alpha}^y$ denote $\alpha^y/\|\alpha^y\|$ when $\alpha^y \neq \emptyset$ (this holds a.s. given $y < D$), or $\overline{\beta}'$ otherwise. Then for $y > 0$, $\widetilde{\alpha}^y$ is independent of $(m_1^y, m_2^y, \|\alpha^y\|)$ and has law $\mathtt{PDIP}\left(\frac{1}{2}, \frac{1}{2}\right)$.*

Proof. For $\gamma > 0$, consider $B_\gamma \sim \mathtt{Gamma}\left(\frac{1}{2}, \gamma\right)$ independent of all other objects. By decomposing according to the events $\{D > y\}$ and $\{D < y\}$, and applying the first assertion of Proposition 33 in the former case, we see that for all continuous $f : \mathbb{R}^3 \to [0, \infty)$ and $g : \mathcal{I}^\circ \to [0, \infty)$,

$$\int_0^\infty \sqrt{\frac{\gamma}{\pi x}} e^{-\gamma x} \mathbf{E}_{a,b,x\overline{\beta}}^2 \left[f(m_1^y, m_2^y, \|\alpha^y\|)g(\widetilde{\alpha}^y)\right] dx = \mathbf{E}_{a,b,B_\gamma\overline{\beta}}^2 \left[f(m_1^y, m_2^y, \|\alpha^y\|)g(\widetilde{\alpha}^y)\right]$$

$$= \mathbb{E}\left[g(\overline{\beta})\right] \int_0^\infty \sqrt{\frac{\gamma}{\pi x}} e^{-\gamma x} \mathbf{E}_{a,b,x\overline{\beta}}^2 \left[f(m_1^y, m_2^y, \|\alpha^y\|)\right] dx.$$

We cancel factors of $\sqrt{\gamma}$ and appeal to the uniqueness of Laplace transforms to find that

$$\mathbf{E}_{a,b,x\overline{\beta}}^2 \left[f(m_1^y, m_2^y, \|\alpha^y\|)g(\widetilde{\alpha}^y)\right] = \mathbf{E}_{a,b,x\overline{\beta}}^2 \left[f(m_1^y, m_2^y, \|\alpha^y\|)\right] \mathbb{E}\left[g(\overline{\beta})\right]$$

for a.e. $x > 0$. By Proposition 24, the right hand side is continuous in x. Note that $(a, b, \beta) \mapsto f(a, b, \|\beta\|)g(\beta/\|\beta\|)\mathbf{1}\{\beta \neq \emptyset\}$ is $\mathbb{P}_{a,b,x\overline{\beta}}^2$-a.s. continuous at (m_1^y, m_2^y, α^y). Thus, the left hand

side is continuous in x as well; see e.g. [22, Theorem 4.27]. We conclude that the above formula holds for every x. □

Proof of Proposition 32. Let $((m_1^z, m_2^z, \alpha^z), z \geq 0)$ be as in the statement of the proposition, and fix $y > 0$. The conditional law of (m_1^y, m_2^y, α^y) given $D > y$ can be obtained as a mixture, over the law of the vector (MA_1, MA_2, MA_3) of initial masses, of the conditional laws described in Proposition 34. In particular, conditionally given $\{\alpha^y \neq \emptyset\}$, $\alpha^y/\|\alpha^y\| \sim \texttt{PDIP}\left(\frac{1}{2}, \frac{1}{2}\right)$, conditionally independent of $(m_1^y, m_2^y, \|\alpha^y\|)$. To prove that $(m_1^y, m_2^y, \|\alpha^y\|)/M(y)$ then has conditional law $\texttt{Dir}\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$, we make an argument similar to that in the proof of Proposition 34.

Recall the standard beta-gamma algebra that a $\texttt{Dir}(x_1, \ldots, x_n)$ vector, multiplied by an independent $\texttt{Gamma}(x_1 + \cdots + x_n, \gamma)$ scalar, gives rise to a vector of independent variables, with the jth having law $\texttt{Gamma}(x_j, \gamma)$. Let $(\widetilde{m}_1^y, \widetilde{m}_2^y, \widetilde{m}_3^y)$ denote $(m_1^y/M(y), m_2^y/M(y), \|\alpha^y\|/M(y))$ when $y < D$ or (A_1', A_2', A_3') otherwise, where the latter is an independent $\texttt{Dir}\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$. By the second assertion of Proposition 33, for $\gamma > 0$ and measurable $f \colon \mathbb{R}^3 \to [0, \infty)$ we have

$$\int_0^\infty 2\sqrt{\frac{x\gamma^3}{\pi}} e^{-\gamma x} \mathbf{E}^2_{A_1 x, A_2 x, A_3 x \overline{\beta}}\left[f(\widetilde{m}_1^y, \widetilde{m}_2^y, \widetilde{m}_3^y)\right] dx = \mathbb{E}\left[f(A_1, A_2, A_3)\right].$$

Multiplying the right hand side by $\int_0^\infty 2\sqrt{x\gamma^3/\pi} e^{-\gamma x} dx = 1$, canceling factors of $\gamma^{3/2}$, and appealing to uniqueness of Laplace transforms and Proposition 24, as in the previous proof, gives the desired result. □

For our next results, we require a scaling invariance property of type-2 evolutions. We recall the scaling map for point processes of spindles from [11, Equation 4.2]: for $c > 0$,

$$c \odot_{\text{cld}} N = \sum_{\text{points } (t,f) \text{ of } N} \delta\left(c^{3/2}t, c \odot_{\text{BES}} f\right), \quad \text{where } c \odot_{\text{BES}} f = (cf(y/c), y \in \mathbb{R}). \quad (30)$$

For type-2 data $\Psi = (\mathbf{f}_1, \mathbf{f}_2, \overleftarrow{\mathbf{N}}, \mathbf{N}_\beta)$, we will write $c \odot_{\text{cld}} \Psi$ to denote $(c \odot_{\text{BES}} \mathbf{f}_1, c \odot_{\text{BES}} \mathbf{f}_2, c \odot_{\text{cld}} \overleftarrow{\mathbf{N}}, c \odot_{\text{cld}} \mathbf{N}_\beta)$. We adopt the convention that $0 \odot_{\text{cld}} \Psi = (0, 0, \overleftarrow{\mathbf{N}}, 0)$, where the first two zeros on the right denote zero functions, and the last a zero measure on $\mathcal{E} \times \mathbb{R}$.

Lemma 35. *Suppose $\Psi = (\mathbf{f}_1, \mathbf{f}_2, \overleftarrow{\mathbf{N}}, \mathbf{N}_\beta)$ is a type-2 data quadruple and $M \geq 0$ is a real-valued random variable, conditionally independent of Ψ given the initial state $(\mathbf{f}_1(0), \mathbf{f}_2(0), \beta)$ of the type-2 evolution associated with Ψ. Then $M \odot_{\text{cld}} \Psi$ is also a type-2 data quadruple.*

Proof. This follows from the well-known scaling invariance of squared Bessel processes (see e.g. [19, A.3]), as well as that of clades of spindles [11, Lemma 4.5] and the law $\nu_{\text{BES}}(c \odot_{\text{BES}} A) = c^{-3/2} \nu_{\text{BES}}(A)$ for $A \subseteq \mathcal{E}$, from [11, Lemma 3.9]. □

For type-2 data Ψ with $M_\Psi(0) = \mathbf{f}_1(0) + \mathbf{f}_2(0) + \|\textsc{skewer}(0, \mathbf{N})\| \neq 0$, we adopt the notation $\overline{\Psi} := M_\Psi(0)^{-1} \odot_{\text{cld}} \Psi$. This is data for a type-2 evolution scaled to have unit initial mass. Going in the other direction, if we begin with data $\overline{\Psi}$ for a type-2 evolution with pseudo-stationary initial distribution and unit initial mass, then for any independent random $M \geq 0$, the quadruple $M \odot_{\text{cld}} \overline{\Psi}$ is data for a type-2 evolution with pseudo-stationary initial distribution and initial mass M. Denote by μ_m the pseudo-stationary distribution on $\mathcal{I}^\circ$ with total mass m.

We will also denote by $\mu_{a,b,c}$ the distribution on $\mathcal{I}^\circ$ of $(a, b, c\overline{\beta})$, with $\overline{\beta} \sim \texttt{PDIP}\left(\frac{1}{2}, \frac{1}{2}\right)$, for all $(a, b, c) \in [0, \infty)$ with either $a + b > 0$ or $a = b = c = 0$. We will use notation $\mathbf{P}^2_\mu$ for the distribution on $\mathcal{E}^2 \times \overleftarrow{\mathcal{N}} \times \mathcal{N}$ of type-2 data with random μ-distributed initial data, for any distribution on $\mathcal{I}^\circ$.

Lemma 36 (Strong pseudo-stationarity). *(i) Let $\overline{\beta} \sim \texttt{PDIP}\left(\frac{1}{2}, \frac{1}{2}\right)$, and let (A, B, C) be an independent vector for which, with probability 1, at least two components are positive. Consider type-2 data Ψ with initial state $(A, B, C\overline{\beta})$. Denote by $(A(y), B(y), C(y)) = (m_1^y, m_2^y, \|\alpha^y\|)$ the associated 3-mass process and by $(\mathcal{F}^y_{3-\text{mass}}, y \geq 0)$ the right-continuous*

filtration it generates. Let Y be a stopping time in this filtration. Then for all $\mathcal{F}^Y_{3-\mathrm{mass}}$-measurable $\eta\colon \Omega \to [0,\infty)$ and all measurable $H\colon \mathcal{E}^2 \times \overline{\mathcal{N}} \times \mathcal{N} \to [0,\infty)$,

$$\mathbb{E}\left(\eta H(\Psi^Y)\right) = \mathbb{E}\left(\eta \mathbf{P}^2_{\mu_{A(Y),B(Y),C(Y)}}[H]\right).$$

I.e. the cutoff data above level Y are type-2 data. Conditionally given $\mathcal{F}^Y_{3-\mathrm{mass}}$, the initial data of Ψ^Y are distributed as $(A(Y), B(Y), C(Y)\overline{\beta}')$ for independent $\overline{\beta}' \sim \mathtt{PDIP}\left(\frac{1}{2},\frac{1}{2}\right)$.

(ii) *Now consider instead type-2 data of the form $\Psi = M(0) \odot_{\mathrm{cld}} \overline{\Psi}$, where $\overline{\Psi}$ is data for a pseudo-stationary type-2 evolution with unit initial mass, independent of $M(0)$. Denote by $M(y) = m^y_1 + m^y_2 + \|\alpha^y\|$, $y \geq 0$, the associated total mass process and by $(\mathcal{F}^y_{\mathrm{mass}}, y \geq 0)$ the right-continuous filtration it generates. Let Y be a stopping time in this filtration. Then for all $\mathcal{F}^Y_{\mathrm{mass}}$-measurable $\eta\colon \Omega \to [0,\infty)$ and measurable $H\colon \mathcal{E}^2 \times \overline{\mathcal{N}} \times \mathcal{N} \to [0,\infty)$,*

$$\mathbb{E}\left(\eta H(M(Y)^{-1} \odot_{\mathrm{cld}} \Psi^Y)\,\big|\, D > Y\right) = \mathbb{E}\left(\eta \,|\, D > Y\right)\mathbb{E}\left(H(\overline{\Psi})\right).$$

Proof. For the first assertion, if we further condition on $\{D \leq Y\}$ then the statements follow trivially from Proposition 27, as $\|\alpha^Y\| = 0$. For fixed $Y = y$, conditional on $\{D > y\}$, the first assertion follows from the pseudo-stationarity of the interval partition in Proposition 34 and the Markov-like property of Proposition 27 for type-2 data, in the same manner as in the proof of [11, Lemma 6.8] for type-0 and type-1 evolutions. The generalization to stopping times is standard; see e.g. the proof of [11, Theorem 6.9]. The proof of the second assertion is the same, using Proposition 32 instead of Proposition 34.

We point out that the cited results, [11, Lemma 6.8, Theorem 6.9], were stated in terms of interval-partition evolutions. However, those results, and the methods used to prove them, extend to results like those stated in this lemma, in terms of (type-0, type-1 or) type-2 data. $\qquad\square$

Proposition 37. *Consider a type-2 evolution $((m^y_1, m^y_2, \alpha^y), y \geq 0)$ starting from $(x_1, x_2, x_3\overline{\beta})$ for $\overline{\beta} \sim \mathtt{PDIP}(\frac{1}{2},\frac{1}{2})$. Then the associated 3-mass process $((m^y_1, m^y_2, \|\alpha^y\|), y \geq 0)$ is a Markov process starting from (x_1, x_2, x_3).*

Proof. We check the Rogers–Pitman intertwining criterion [37, Theorem 2 and Remarks (i)-(ii)]. Consider the map $\phi(x_1, x_2, \alpha) = (x_1, x_2, \|\alpha\|)$ and the stochastic kernel $\Lambda((x_1, x_2, x_3), A) = \mathbb{P}((x_1, x_2, x_3\overline{\beta}) \in A)$, where $\overline{\beta} \sim \mathtt{PDIP}(\frac{1}{2},\frac{1}{2})$. Then clearly $\Lambda((x_1, x_2, x_3), \phi^{-1}(x_1, x_2, x_3)) = 1$ and by Lemma 36(i), we also have

$$\mathbb{P}_\mu\left((m^y_1, m^y_2, \alpha^y) \in A \,\big|\, m^y_1, m^y_2, \|\alpha^y\|\right) = \Lambda((m^y_1, m^y_2, \|\alpha^y\|), A) \qquad a.s.,$$

for all initial distributions μ of the form $\Lambda((x_1, x_2, x_3), \,\cdot\,)$, as required. $\qquad\square$

The semi-group of the 3-mass process can be described as "replace the third component by a scaled $\mathtt{PDIP}(\frac{1}{2},\frac{1}{2})$, make type-2 evolution transitions, and then project the interval partition onto its mass."

4.2. Degeneration in pseudo-stationarity.

In the interweaving construction in pseudo-stationarity, it is easy to describe the degeneration time.

Proposition 38. *Fix $\gamma > 0$. Let D be the degeneration time of a type-2 evolution starting from $(A, B, C\overline{\beta})$, where A, B, C and $\overline{\beta}$ are jointly independent, with $C \sim \mathtt{Gamma}(\frac{1}{2}, \gamma)$ and $\overline{\beta} \sim \mathtt{PDIP}(\frac{1}{2},\frac{1}{2})$. Then $\mathbb{P}(D > y) = \mathbb{P}(\zeta_1 > y)\mathbb{P}(\zeta_2 > y)$ for all $y > 0$, where ζ_1 and ζ_2 are the lifetimes of type-1 evolutions starting from $(A, C\overline{\beta})$ and $(B, C\overline{\beta})$, respectively.*

If also $A, B \sim \mathtt{Gamma}(\frac{1}{2}, \gamma)$, then $\mathbb{P}\{D > y\} = (2y\gamma + 1)^{-2}$ for all $y > 0$.

Proof. The interweaving construction is such that on $\{\widetilde{J}(\infty) \text{ even}\}$

$$(0, \widetilde{m}^y_1) \star \underset{2 \leq j \leq \widetilde{J}(\infty) \text{ even}}{\bigstar} \mathrm{SKEWER}\left(y - Z_{j-1}, \mathbf{N}_{p(j+1)}\big|^{\leftarrow}_{(T_{j-2}, T_j] \times \mathcal{E}}\right) = \mathrm{SKEWER}\left(y, \mathbf{N}_1\right),$$

and on $\{\widetilde{J}(\infty) \text{ odd}\}$,

$$(0, \widetilde{m}^y_2) \star \mathrm{SKEWER}\left(y - \zeta(\mathbf{f}_2), \mathbf{N}_2\big|_{(0, T_1] \times \mathcal{E}}\right) \star \underset{2 \leq j \leq \widetilde{J}(\infty) \text{ odd}}{\bigstar} \mathrm{SKEWER}\left(y - Z_{j-1}, \mathbf{N}_{p(j+1)}\big|^{\leftarrow}_{(T_{j-2}, T_j] \times \mathcal{E}}\right)$$

equals $\mathrm{SKEWER}\,(y, \mathbf{N}_2)$, while on the respective opposite event the expression on the LHS yields $\mathrm{SKEWER}\left(y, \mathbf{N}_1|_{[0, T_{\widetilde{J}(\infty)-1}]}\right)$ and $\mathrm{SKEWER}\left(y, \mathbf{N}_2|_{[0, T_{\widetilde{J}(\infty)-1}]}\right)$, respectively.

On $\{\widetilde{J}(\infty)\text{ even}\}$, the definitions of $Z_{\widetilde{J}(\infty)}$ and $T_{\widetilde{J}(\infty)}$ imply that $\zeta_2 \geq \zeta = Z_{\widetilde{J}(\infty)} > \zeta_1 = D$, where ζ_i is the death level of the type-1 evolution $\overline{\mathrm{SKEWER}}(\mathbf{N}_i)$, $i = 1, 2$, and ζ is the death level of the type-2 evolution $((\widetilde{m}_1^y, \widetilde{m}_2^y, \widetilde{\alpha}^y), y \geq 0)$. Together with corresponding observations on $\{\widetilde{J}(\infty)\text{ odd}\}$, we see that D is the minimum of the lifetimes ζ_1 and ζ_2 of the two pseudo-stationary type-1 evolutions used in the construction. Hence

If we apply the interweaving construction to independent A and B with $\mathtt{Gamma}\left(\frac{1}{2}, \gamma\right)$ distribution, we obtain the type-1 pseudo-stationary initial distribution. As these are i.i.d. with $\mathtt{Exponential}(\gamma)$ initial mass, from [11, equation (6.3)] they each have lifetime at least y with probability $(2y\gamma + 1)^{-1}$. The minimum of two i.i.d. variables with this law has probability $(2y\gamma + 1)^{-2}$ of exceeding y, as claimed. $\qquad\square$

Proposition 39. *Consider a type-2 evolution $((m_1^y, m_2^y, \alpha^y), y \geq 0)$ starting from the initial condition of Proposition 32 with $M \sim \mathtt{Gamma}(\frac{3}{2}, \gamma)$, with degeneration time D. Let $A = \{I(D) = 1\}$; this is the event that $(m_1^y, y \geq 0)$ is the surviving clock spindle at the time of degeneration. In this event, $(m_1^D, m_2^D, \alpha^D) = (M^D, 0, \emptyset)$; in the complementary event, $(m_1^D, m_2^D, \alpha^D) = (0, M^D, \emptyset)$. Then $\mathbf{P}(A) = \frac{1}{2}$, A is independent of (D, M^D), and $\mathtt{Gamma}\left(\frac{1}{2}, \gamma/(2\gamma y + 1)\right)$ is a regular conditional distribution for M^D given $D = y$.*

Proof. We are interested in the joint distribution of $(D, m_1^D, m_2^D, \alpha^D)$. Using the construction of Lemma 29 from two independent type-1 evolutions with extinction times ζ_1 and ζ_2 and top mass processes m_1 and m_2, we have $D = \min\{\zeta_1, \zeta_2\}$, $A = \{\zeta_1 > \zeta_2\}$, and

$$(D, M^D) = (\zeta_2, m_1^{\zeta_2})\mathbf{1}_A + (\zeta_1, m_2^{\zeta_1})\mathbf{1}_{A^c}.$$

Under the stated initial conditions, these two type-1 evolutions are in fact i.i.d.. From this, it is clear by symmetry that $\mathbf{P}(A) = \frac{1}{2}$ and A is independent of (D, M^D), as claimed.

For all nonnegative measurable f and g on $\mathbb{R}$,

$$\mathbb{E}(f(D)g(M^D)) = \mathbb{E}(f(\zeta_2)g(m_1^{\zeta_2})\mathbf{1}_A) + \mathbb{E}(f(\zeta_1)g(m_2^{\zeta_1})\mathbf{1}_{A^c}).$$

We use Proposition 12 to rewrite the first term on the right hand side as

$$\int_0^\infty f(y)\mathbb{E}(g(m_1^y)\mathbf{1}\{\zeta_1 > y\})\mathbb{P}(\zeta_2 \in dy)$$

$$= \int_0^\infty f(y)\mathbb{E}(\mathbf{1}\{\zeta_1 > y\})\int_0^\infty g(x)\frac{1}{\sqrt{\pi x}}\sqrt{\frac{\gamma}{2\gamma y + 1}}\exp\left(-\frac{\gamma}{2\gamma y + 1}x\right)dx\,\mathbb{P}(\zeta_2 \in dy)$$

$$= \mathbb{E}\left(f(\zeta_2)\mathbf{1}\{\zeta_1 > \zeta_2\}\int_0^\infty g(x)\frac{1}{\sqrt{\pi x}}\sqrt{\frac{\gamma}{2\gamma\zeta_2 + 1}}\exp\left(-\frac{\gamma}{2\gamma\zeta_2 + 1}x\right)dx\right).$$

The second term can be written similarly, by symmetry, and together they give

$$\mathbb{E}(f(D)g(M^D)) = \mathbb{E}\left(f(D)\int_0^\infty g(x)\frac{1}{\sqrt{\pi x}}\sqrt{\frac{\gamma}{2\gamma D + 1}}\exp\left(-\frac{\gamma}{2\gamma D + 1}x\right)dx\right).$$

This proves the claimed regular conditional distribution for M^D. $\qquad\square$

Note that this result (and proof) formalizes an extension of the second part of Proposition 33 to the random time $y = D$, the degeneration time, and yields the same conditional distribution for the surviving top mass as for the surviving top masses when conditioning on $y < D$.

4.3. De-Poissonization. Consider a type-2 evolution $\mathbf{T} = (T^y, y \geq 0) = ((m_1^y, m_2^y, \alpha^y), y \geq 0)$ constructed from independent $\mathbf{f}_1 \sim \mathtt{BESQ}_a(-1)$ and $\mathbf{N}_* \sim \mathbf{P}^1_{(0,b)\star\beta}$ as in Definition 17. We now consider the distribution $\mathbb{P}^2_{a,b,\beta}$ of $\mathbf{T}$ on the space $\mathbb{D}([0, \infty), \mathcal{J}^\circ)$ of càdlàg functions from $[0, \infty)$ to $\mathcal{J}^\circ$.

For $T = (a, b, \beta) \in \mathcal{J}^\circ$, we consider the total mass $\|T\| = a + b + \|\beta\|$. For $\mathbf{T} = (T^y, y \geq 0) \in \mathbb{D}([0, \infty), \mathcal{J}^\circ)$, we define a time-change function

$$\rho_{\mathbf{T}} \colon [0, \infty) \to [0, \infty], \qquad \rho_{\mathbf{T}}(u) = \inf \left\{ y \geq 0 \colon \int_0^y \|T^x\|^{-1} dx > u \right\}, \quad u \geq 0, \qquad (31)$$

which is continuous and strictly increasing until a potential absorption at ∞. Recall from Theorem 3 that for a type-2 evolution $\mathbf{T}$ starting from $(a, b, \beta) \in \mathcal{J}^\circ \setminus \{(0, 0, \emptyset)\}$, we have $\|\mathbf{T}\| = (\|T^y\|, y \geq 0) \sim \mathtt{BESQ}_{a+b+\|\beta\|}(-1)$. By [19, p.314-5], $\rho_{\mathbf{T}}$ is bijective onto $[0, \zeta)$ a.s., where $\zeta = \inf\{y \geq 0 \colon \|T^y\| = 0\}$. Let

$$\mathcal{J}_1^\circ := \{T \in \mathcal{J}^\circ \colon \|T\| = 1\}, \qquad \mathcal{I}_1^\circ := \{\beta \in \mathcal{I}^\circ \colon \|\beta\| = 1\}. \qquad (32)$$

Definition 40. Let $\bar{\nu}$ be a distribution on $\mathcal{J}_1^\circ$. Given a type-2 evolution $\mathbf{T} \sim \mathbb{P}_{\bar{\nu}}^2$ starting according to $\bar{\nu}$, we associate the *de-Poissonized type-2 evolution* $\overline{T}^u = T^{\rho_{\mathbf{T}}(u)}/\|T^{\rho_{\mathbf{T}}(u)}\|$, $u \geq 0$. We denote its distribution on $\mathbb{D}([0, \infty), \mathcal{J}_1^\circ)$ by $\overline{\mathbb{P}}_{\bar{\nu}}^{2,-}$.

We define the *IP-valued de-Poissonized type-2 evolution* to be

$$\left\{ \left(0, m_{I(\rho_{\mathbf{T}}(u))}^{\rho_{\mathbf{T}}(u)}\right) \right\} \star \left\{ \left(0, m_{3-I(\rho_{\mathbf{T}}(u))}^{\rho_{\mathbf{T}}(u)}\right) \right\} \star \alpha^{\rho_{\mathbf{T}}(u)}, \quad u \geq 0.$$

Theorem 41. *De-Poissonized type-2 evolutions are Borel right Markov processes absorbed in finite time in either $(1, 0, \emptyset)$ or $(0, 1, \emptyset)$. The IP-valued variants of these processes are also Borel right Markov processes, but are additionally path-continuous, and are absorbed in finite time in the degenerate interval partition $\{(0, 1)\}$ of the interval $(0, 1)$.*

Proof. $\mathcal{J}_1^\circ$ (likewise $\mathcal{I}_1^\circ$) is a Borel subset of a Lusin space, and is therefore Lusin. Both continuous time changes and normalization on $\mathcal{J}^\circ \setminus \{(0, 0, \emptyset)\}$ (respectively, $\mathcal{I}^\circ \setminus \{\emptyset\}$) preserve the property of sample paths being càdlàg (resp. continuous). The strong Markov property of Theorem 2 and the continuity in the initial state of Proposition 24 transfer to the de-Poissonized processes as in [11, Proposition 6.7, proof of Theorem 1.6].

For both processes, absorption occurs at the time $\overline{D}$ that satisfies $\rho_{\mathbf{T}}(\overline{D}) = D$ since $D < \zeta$ a.s.; note that all states $(m, 0, \emptyset)$, $m \in (0, \infty)$, are normalized to $(1, 0, \emptyset)$, and similarly for $(0, 1, \emptyset)$. For the IP-valued process, all states $\{(0, m)\}$ are normalized to $\{(0, 1)\}$. Finally, the time-change is such that $\rho_{\mathbf{T}}(u) < \zeta$ for all $u \in [0, \infty)$. $\qquad \square$

Pal [28, 29] studied Wright–Fisher diffusions with positive and negative real parameters $\theta_1, \ldots, \theta_n$ as de-Poissonized processes associated with vectors of independent $Z_i \sim \mathtt{BESQ}(2\theta_i)$, $1 \leq i \leq n$. Combining the arguments of [28, Proposition 11] and [29, Theorem 4], we may define generalized Wright–Fisher diffusions (running at 4 times the speed of [28, 29]) as either weak solutions to certain systems of stochastic differential equations or, as is relevant for us, as

$$\overline{Z}_i(u) = \frac{Z_i(\rho(u))}{Z_+(\rho(u))}, \quad 1 \leq i \leq n, \quad 0 \leq u \leq \overline{\tau} = \inf\{s \geq 0 \colon \exists i \text{ s.t. } \overline{Z}_i(s) = 0 \text{ and } \theta_i \leq 0\}, \quad (33)$$

where $Z_+(y) := \sum_{i=1}^n Z_i(y)$ and $\rho(u)$ is as in (31), but with $Z_+(x)$ in place of $\|T^x\|$ inside the integral. See also [11, pp. 60–61].

Proposition 42. *Let $\overline{\mathbf{T}} = ((X_1(u), X_2(u), \beta(u)), u \geq 0) \sim \overline{\mathbb{P}}_{a,b,\beta}^{2,-}$ be a de-Poissonized type-2 evolution starting from any $(a, b, \beta) \in \mathcal{J}_1^\circ$. Let $U = \inf\{u \geq 0 \colon X_1(u) = 0 \text{ or } X_2(u) = 0\}$. Then the 3-mass process $((X_1(u), X_2(u), 1 - X_1(u) - X_2(u)), 0 \leq u \leq U)$ is a generalized Wright–Fisher process with parameter vector $(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$.*

If furthermore the initial state is taken as $\beta = (1 - a - b)\overline{\beta}$ for $\overline{\beta} \sim \mathtt{PDIP}(\frac{1}{2}, \frac{1}{2})$, then the 3-mass process $((X_1(u), X_2(u), 1 - X_1(u) - X_2(u)), u \geq 0)$ is a Markovian extension of the generalized Wright–Fisher process.

Proof. For the first claim, we assume without loss of generality that $\overline{\mathbf{T}}$ is constructed as in Definition 40 from a type-2 evolution $\mathbf{T} = (T^y, y \geq 0)$ arising from type-2 data $(\mathbf{f}_1, \mathbf{f}_2, \overleftarrow{\mathbf{N}}, \mathbf{N}_\beta) \sim \mathbf{P}_{a,b,\beta}^2$ as in Definition 17. By Proposition 10, we have $\|\overline{\mathrm{SKEWER}}(\overleftarrow{\mathbf{N}} \star \mathbf{N}_\beta)\| \sim \mathtt{BESQ}_{\|\beta\|}(1)$. This process, together with $\mathbf{f}_1 \sim \mathtt{BESQ}_a(-1)$ and $\mathbf{f}_2 \sim \mathtt{BESQ}_b(-1)$ forms a triple of $\mathtt{BESQ}$ processes,

as in the paragraph above the proposition. Thus, we can construct a generalized Wright–Fisher process from $Z_1 = \mathbf{f}_1$, $Z_2 = \mathbf{f}_2$ and $Z_3 = \|\overline{\mathrm{SKEWER}}(\overleftarrow{\mathbf{N}} \star \mathbf{N}_\beta)\|$. Since $Z_+(y) = \|T^y\|$ for $0 \le y \le \tau := \inf\{y \ge 0 : \exists i \text{ s.t. } Z_i(y) = 0\}$, we have $\rho(u) = \rho_{\mathbf{T}}(u)$, and hence $X_i(u) = \overline{Z}_i(u)$, $i = 1, 2$, and $X_3(u) = 1 - X_1(u) - X_2(u) = 1 - \overline{Z}_1(u) - \overline{Z}_2(u) = \overline{Z}_3(u)$, $0 \le u \le U = \overline{\tau}$. This completes the proof.

The second claim follows from Proposition 37 and the observation that the (Poissonized) 3-mass process of that proposition can be de-Poissonized by the same scaling/time-change operation as the type-2 evolution, as the scaling and time change only depend on the common total mass process. $\qquad\square$

4.4. Resampling and stationarity.
As we have seen in Theorem 41, de-Poissonized type-2 evolutions degenerate at a finite random time $\overline{D} < \infty$ in one of the two absorbing states $(1, 0, \emptyset)$ and $(0, 1, \emptyset)$. In this section we will restart the process instead of entering the absorbing states. Informally and with the conjectured stationary Aldous diffusion in mind, we take the opportunity to sample afresh from the Brownian CRT at each degeneration time. Recall the state space $\mathcal{J}_1^*$ of (4). Formally, we sample from the distribution $\overline{\mu}$ on $\mathcal{J}_1^*$ of $(X_1, X_2, (1 - X_1 - X_2)\overline{\beta})$, where $(X_1, X_2, 1 - X_1 - X_2) \sim \mathrm{Dirichlet}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ is independent of the interval partition $\overline{\beta} \sim \mathrm{PDIP}\left(\frac{1}{2}, \frac{1}{2}\right)$, as discussed in the introduction.

Definition 43. Let $(a, b, \beta) \in \mathcal{J}_1^*$. Let $(\overline{T}_{(j)}^u, 0 \le u < \overline{D}_{(j)})$, $j \ge 0$, be a sequence of independent copies of $(\overline{T}_-^u, 0 \le u < \overline{D})$, where $\overline{\mathbf{T}}_- \sim \overline{\mathbb{P}}_{a,b,\beta}^{2,-}$ for $j = 0$ and $\overline{\mathbf{T}}_- \sim \overline{\mathbb{P}}_{\overline{\mu}}^{2,-}$ for $j \ge 1$, see Definition 40. Set $V_0 = 0$ and denote the cumulative lifetimes inductively by $V_j = V_{j-1} + \overline{D}_{(j)}$, $j \ge 1$. Then the concatenation

$$\overline{T}_+^{V_j + u} = \overline{T}_{(j)}^u, \qquad 0 \le u < \overline{D}_{(j)}, \ j \ge 0,$$

is called a *(resampling) 2-tree evolution* starting from (a, b, β). We denote its distribution on $\mathbb{D}([0, \infty), \mathcal{J}_1^*)$ by $\overline{\mathbb{P}}_{a,b,\beta}^{2,+}$. For clarity, we continue to use notation $(\overline{T}_+^u, u \ge 0)$ for the canonical process on $\mathbb{D}([0, \infty), \mathcal{J}_1^*)$ when working under $\overline{\mathbb{P}}_{a,b,\beta}^{2,+}$.

Proof of Theorem 4. To confirm that the 2-tree evolution is a Borel right Markov process, we only need to check the strong Markov property. Given this construction, this can be seen in the context of general results about resurrecting Markov processes [26]. However, since our setting is much more elementary than the general theory, we sketch an elementary proof. Consider a stopping time Y in the right-continuous filtration $(\mathcal{F}^y, y \ge 0)$ generated by the canonical process on $\mathbb{D}([0, \infty), \mathcal{J}_1^*)$. For $j \ge 1$, let $Y_j = \min\{Y, V_j\}$. Denote by $(\theta_y, y \ge 0)$ the canonical shift operators on $\mathbb{D}([0, \infty), \mathcal{J}_1^*)$. Then for all probability measures $\overline{\nu}$ on $\mathcal{J}_1^*$, for all $0 \le j \le k$, for all $\mathcal{F}^{V_1}$-measurable $\eta_i, f \colon \mathbb{D}([0, \infty), \mathcal{J}_1^*) \to [0, \infty)$, with $\mathcal{F}^Y$-measurable $\eta_j \circ \theta_{V_j}$, we check that

$$\overline{\mathbb{P}}_{\overline{\nu}}^{2,+}\left[\prod_{i=0}^k \eta_i \circ \theta_{V_i} \mathbf{1}_{\{V_j \le Y < V_{j+1}\}} f \circ \theta_Y\right] = \overline{\mathbb{P}}_{\overline{\nu}}^{2,+}\left[\prod_{i=0}^j \eta_i \circ \theta_{V_i} \mathbf{1}_{\{V_j \le Y < V_{j+1}\}} \overline{\mathbb{P}}_{\overline{T}_+^Y}^{2,+}\left[f \prod_{i=j+1}^k \eta_i \circ \theta_{V_{j-i}}\right]\right]$$

holds, by inductively applying the strong Markov property under $\overline{\mathbb{P}}_{\overline{\nu}}^{2,+}$ and $\overline{\mathbb{P}}_{\overline{\mu}}^{2,+}$ at V_i, which holds by construction, and the strong Markov property under $\overline{\mathbb{P}}_{\overline{\mu}}^{2,-}$ (or, for $j = 0$ under $\overline{\mathbb{P}}_{\overline{\nu}}^{2,-}$) at the $Y - V_j$. Summing over $j \ge 0$ and applying a monotone class theorem proves the strong Markov property.

Now, we prove that $\overline{\mu}$, defined before Definition 43, is the unique stationary distribution, and that the process converges to it. Applying Lemma 36(ii) to the $(\mathcal{F}_{\mathrm{mass}}^y, y \ge 0)$-stopping time $Y = \rho_{\mathbf{T}}(u)$, we find

$$\mathbb{E}_{\overline{\mu}}^2\left(g(T^{\rho_{\mathbf{T}}(u)}/\|T^{\rho_{\mathbf{T}}(u)}\|)\mathbf{1}_{\{D > \rho_{\mathbf{T}}(u)\}}\right) = \mathbb{P}_{\overline{\mu}}^2(D > \rho_{\mathbf{T}}(u))\overline{\mu}(g). \tag{34}$$

Now consider a resampling 2-tree evolution $(\overline{T}_+^u, u \ge 0)$ with initial distribution $\overline{\mu}$. We use the notation of Definition 43. Let $U \sim \mathrm{Exponential}(\lambda)$ independent of the resampling 2-tree

evolution. Then (34) yields $\mathbb{E}\left(g\left(\overline{T}_+^U\right)1_{\{U<V_1\}}\right)=\mathbb{P}(U<V_1)\bar{\mu}(g)$. For $m\geq 1$,

$$\mathbb{E}\left(g\left(\overline{T}_+^U\right)1_{\{V_m\leq U<V_{m+1}\}}\right)=\int_0^\infty \lambda e^{-\lambda u}\mathbb{E}\left(g\left(\overline{T}_+^u\right)1_{\{V_m\leq u<V_{m+1}\}}\right)du$$

$$=\mathbb{E}\left(e^{-\lambda V_m}\int_0^\infty \lambda e^{-\lambda s}g\left(\overline{T}_+^{V_m+s}\right)1_{\{V_m+s<V_{m+1}\}}ds\right)$$

$$=\int_0^\infty \lambda e^{-\lambda s}\mathbb{E}\left(e^{-\lambda V_m}g\left(\overline{T}_+^{V_m+s}\right)1_{\{V_m+s<V_{m+1}\}}\right)ds$$

$$=\int_0^\infty \lambda e^{-\lambda s}\mathbb{E}\left(e^{-\lambda V_m}\mathbb{E}\left(g(\overline{T}_+^s)1_{\{s<V_1\}}\right)\right)ds$$

$$=\mathbb{E}\left(e^{-\lambda V_m}\mathbb{E}\left(g\left(\overline{T}_+^U\right)1_{\{U<V_1\}}\right)\right)$$

$$=\mathbb{E}\left(e^{-\lambda V_m}\right)\mathbb{P}(U<V_1)\bar{\mu}(g)=\mathbb{P}(V_m\leq U<V_{m+1})\bar{\mu}(g).$$

Summing over m and inverting Laplace transforms in λ, we find that $\overline{\mathbb{E}}_{\bar{\mu}}^{2,+}(g(T^u))=\bar{\mu}(g)$ for all $u\geq 0$, i.e. $\bar{\mu}$ is stationary for $\overline{\mathbf{T}}_+$. Furthermore, since resampling is according to the stationary distribution $\bar{\mu}$, we have for any other initial distribution $\bar{\nu}$ that for all bounded measurable $g\colon \mathcal{J}_1^*\to[0,\infty)$

$$\overline{\mathbb{E}}_{\bar{\nu}}^{2,+}(g(T^u))=\overline{\mathbb{E}}_{\bar{\nu}}^{2,+}\left(g(T^u)1_{\{u<V_1\}}\right)+\overline{\mathbb{E}}_{\bar{\nu}}^{2,+}(V_1\leq u)\bar{\mu}(g)\to\bar{\mu}(g),$$

since $V_1=\overline{D}$ is finite $\overline{\mathbb{P}}_{\bar{\nu}}^{2,+}$-a.s.. In particular, the stationary distribution is unique. □

As in Propositions 37 and 42, we can project a 2-tree evolution $((\overline{m}_1^y,\overline{m}_2^y,\overline{\beta}^y),\,y\geq 0)$ down to a *resampling 3-mass process* $((\overline{m}_1^y,\overline{m}_2^y,\overline{\beta}^y),\,y\geq 0)$. We can now prove Theorem 5.

Proof of Theorem 5. Proposition 42 and Theorem 4 imply that the resampling 3-mass process is a Borel right Markov process that extends the generalized Wright–Fisher process to a recurrent process on the simplex $\{(a,b,c)\in[0,1)^3\colon a+b+c=1\}$, which has $\mathtt{Dir}\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$ stationary distribution, and converges to stationarity. □

Note that the Wright–Fisher diffusion with parameters $\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$ has this same invariant law. Recall the definition in (2) of the diversity $\mathscr{D}_\beta$ of an interval partition $\beta\in\mathcal{I}$.

Corollary 44. *Under* $\overline{\mathbb{P}}_{\bar{\nu}}^{2,+}$, *let* $(\bar{\beta}^u,\,u\geq 0)$ *denote the evolution of the interval partition component. Then the total diversity process* $(\mathscr{D}_{\bar{\beta}^u}(\infty),\,u\geq 0)$ *is continuous except at the resampling times* V_m, $m\geq 1$.

Proof. Consider de-Poissonized type-2 evolution, its IP-valued variant as in Definition 40, and the extension of the triple-valued process to a 2-tree evolution via resampling. Up until the first resampling time, the interval partition component of the resampling process differs from the IP-valued evolution by at most two leftmost blocks, removed from the latter to construct the former. As noted in Theorem 41, the latter process is path-continuous under $d_\mathcal{I}$, so its total diversity process is continuous. Removing a finite number of blocks does not change its total diversity. This same argument proves continuity between any two consecutive resampling times. □

4.5. Hölder estimates.
Let us denote by $\widetilde{\mu}$ the distribution of the $\mathcal{I}^\circ$-valued interval partition

$$(0,\overline{A})\star(0,\overline{B})\star\overline{G}\,\overline{\beta}$$

for independent $(\overline{A},\overline{B},\overline{G})\sim\mathtt{Dir}(\frac{1}{2},\frac{1}{2},\frac{1}{2})$ and $\overline{\beta}\sim\mathtt{PDIP}(\frac{1}{2},\frac{1}{2})$. This distribution is not pseudo-stationary in the strong sense that the distribution of an $\mathcal{I}^\circ$-valued type-2 evolution starting from $\widetilde{\mu}$ has as marginal distributions the distributions of random multiples of this interval partition – intuitively, the leftmost block is stochastically larger than the second block. However, we will be able to appeal to the pseudo-stationarity of $\mathcal{J}_1^\circ$-valued type-2 evolutions starting from $(\overline{A},\overline{B},\overline{G}\,\overline{\beta})$ in situations that treat the two top masses symmetrically.

Proposition 45. *Let $(\widetilde{\beta}^y, y \geq 0)$ be an $\mathcal{I}^\circ$-valued type-2 evolution starting according to $\widetilde{\mu}$. Let $\theta \in (0, 1/4)$ and $y > 0$. Then there is a random Hölder constant $L = L_{\theta,y}$ with moments of all orders such that*

$$d_{\mathcal{I}}(\widetilde{\beta}^a, \widetilde{\beta}^b) \leq L|b - a|^\theta \qquad \text{for all } 0 \leq a < b \leq y.$$

The remainder of this subsection is devoted to the proof of this proposition. We begin by some preliminary considerations. Let us first consider the type-2 evolution $\beta^y = M\widetilde{\beta}^{y/M}$, $y \geq 0$, with $\texttt{Gamma}\left(\frac{3}{2}, \gamma\right)$ initial mass M for some $\gamma > 0$, cf. Proposition 33. Recall that

- the evolution of $(\beta^y, y \geq 0)$ can be constructed by interweaving two independent pseudo-stationary type-1 evolutions of initial mass $\texttt{Exponential}(\gamma)$ (Propositions 29 and 32);
- the pseudo-stationary type-1 evolution consists of a type-1 evolution starting from a single interval $(0, A)$ with $A \sim \texttt{Gamma}\left(\frac{1}{2}, \gamma\right)$ concatenated left-to-right with an independent type-1 evolution starting from a $\texttt{PDIP}\left(\frac{1}{2}, \frac{1}{2}\right)$ scaled by an independent $\texttt{Gamma}\left(\frac{1}{2}, \gamma\right)$ mass (Proposition 12);
- a type-1 evolution starting from a single $\texttt{Gamma}\left(\frac{1}{2}, \gamma\right)$-distributed interval $(0, A)$ can be constructed from a $\texttt{BESQ}_A(-1)$ process, with death level ζ and an independent $\texttt{Stable}\left(\frac{3}{2}\right)$ process with $\texttt{BESQ}(-1)$ excursions in its jumps and run until it hits $-\zeta$, then shifted up by ζ to descend from ζ to 0;
- a type-1 evolution starting from $\texttt{PDIP}\left(\frac{1}{2}, \frac{1}{2}\right)$ scaled by mass $\texttt{Gamma}\left(\frac{1}{2}, \gamma\right)$ can be constructed from a $\texttt{Stable}\left(\frac{3}{2}\right)$ process $\widetilde{\mathbf{X}}$ starting from 0, with $\texttt{BESQ}(-1)$ excursions in its jumps stopped at a time $\widetilde{T}$, which is the left endpoint of the excursion away from 0 where the mass at level 0 exceeds an independent $\texttt{Exponential}(\gamma)$ threshold (Proposition 30);
- the local times $(\widetilde{\ell}^y(t), 0 \leq t \leq \widetilde{T}, y \geq 0)$ of $(\widetilde{\mathbf{X}}(t), 0 \leq t \leq \widetilde{T})$ have the property that for each $a \geq 0$ and $\theta \in (0, 1/4)$, the random variable

$$\widetilde{D}_\theta^a = \sup_{0 \leq t \leq \widetilde{T}, 0 \leq x < y \leq a} \frac{|\widetilde{\ell}^x(t) - \widetilde{\ell}^y(t)|}{|y - x|^\theta}$$

  has moments of all orders ([15, Theorem 3]);
- for a type-1 evolution $(\alpha^y, y \geq 0)$ arising from some scaffolding $\mathbf{X}$ marked by $\texttt{BESQ}(-1)$ spindles, it is a.s. the case that for every y and every block $U \in \alpha^y$, the diversity $\mathscr{D}_{\alpha^y}(U)$ equals the local time $\ell^y(t)$ in $\mathbf{X}$, up to the time t at which the spindle corresponding to block U arises ([15, Theorem 1] or [11, Theorem 4.15]).

Consider the $\texttt{Stable}\left(\frac{3}{2}\right)$ process starting from ζ obtained by concatenating the descent from ζ to 0 before $\widetilde{\mathbf{X}}$. Denote this process by $(\widehat{\mathbf{X}}(t), 0 \leq t \leq \widehat{T})$ and its local times by $(\hat{\ell}^y(t), 0 \leq t \leq \widehat{T}, y \geq 0)$. In this context, [15, Theorem 3] has the following consequence.

Lemma 46. *The following random variable has moments of all orders:*

$$\widehat{D}_\theta^a = \sup_{0 \leq t \leq \widehat{T}, 0 \leq x < y \leq a} \frac{|\hat{\ell}^x(t) - \hat{\ell}^y(t)|}{|y - x|^\theta}.$$

Proof. Let $H_\zeta = \inf\{t \geq 0 \colon \widetilde{\mathbf{X}}(t) = \zeta\}$. Then the event $\{H_\zeta < \widetilde{T}\}$ has positive probability and by the strong Markov property and by the property of Poisson processes of marked excursions, the conditional distribution given $H_\zeta < \widetilde{T}$ of the process $(\widetilde{\mathbf{X}}(H_\zeta + s), 0 \leq s \leq \widetilde{T} - H_\zeta)$ is the same as the unconditional distribution of $(\widehat{\mathbf{X}}(t), 0 \leq t \leq T)$; c.f. Lemma 31. But then the associated local times $(\widetilde{\ell}^y(H_\zeta + s) - \widetilde{\ell}^y(H_\zeta), 0 \leq s \leq \widetilde{T} - H_\zeta, y \geq 0)$ have as their conditional distribution the distribution of $(\hat{\ell}^y(t), 0 \leq t \leq \widehat{T}, y \geq 0)$. By the triangle inequality,

$$\mathbb{E}\left[(\widehat{D}_\theta^a)^p\right] \leq \mathbb{E}\left[(2\widetilde{D}_\theta^a)^p \big| H_\zeta < \widetilde{T}\right] < \infty. \qquad \square$$

This allows us to bound terms (i) and (ii) of Definition 14 of $d_{\mathcal{I}}$, which deal with diversity.

Lemma 47. *There is a random variable L_θ with moments of all orders such that uniformly over all those matchings $((U_j, U_j'), 1 \le j \le m)$ of intervals of β^0 and β^y that are taken from the same $\mathtt{BESQ}(-1)$ excursion, we have*

$$\left|\mathscr{D}_{\beta^y}(\infty) - \mathscr{D}_{\beta^0}(\infty)\right| \le L_\theta y^\theta \quad and \quad \max_{1 \le j \le m} \left|\mathscr{D}_{\beta^y}(U_j') - \mathscr{D}_{\beta^0}(U_j)\right| \le L_\theta y^\theta.$$

Proof. Think of $(\beta^y, y \ge 0)$ as arising from an interweaving construction, as in Section 3.5, so for each $y \ge 0$, β^y is formed as in (25), by concatenating alternating intervals of the skewers of two i.i.d. copies $(\widehat{\mathbf{X}}_{(1)}, \widehat{\mathbf{X}}_{(2)})$ of $\widehat{\mathbf{X}}$ with jumps marked by $\mathtt{BESQ}(-1)$ spindles. Now, consider a block $U \in \beta^0$; this corresponds to one such spindle, marking a jump at some time t in either $\widehat{\mathbf{X}}_1$ or $\widehat{\mathbf{X}}_2$. Suppose, for example, that this spindle appears in $\mathbf{X}_1$ with $t \in [T_2, T_4)$, in the notation of (25). Then by [11, Theorem 4.15], $\mathscr{D}_{\beta^0}(U) = \hat{\ell}_{(1)}^0(t) + \hat{\ell}_{(2)}^0(T_3)$, and if $U' \in \beta^y$ corresponds to the same spindle, then $\mathscr{D}_{\beta^y}(U') = \hat{\ell}_{(1)}^y(t) + \hat{\ell}_{(2)}^y(T_3)$. Such comparisons can be made for spindles coming from any interval $[T_{j-2}, T_j)$ in either $\widehat{\mathbf{X}}_{(1)}$ or $\widehat{\mathbf{X}}_{(2)}$. Thus, the claimed bounds follow from Lemma 46 by the triangle inequality, with the pth moment of L_θ being bounded by twice that of $\widehat{D}_\theta^y$. $\qquad\square$

It remains to bound terms (iii) and (iv) in Definition 14, which deal with mass. Consider a sequence of m distinct size-biased picks among the blocks of β^0, and match these with the blocks arising from the same spindle at time y, $((U_j, U_j'), 1 \le j \le m)$, allowing that $\mathrm{Leb}(U_j')$ may equal zero for some j if the spindle does not survive. We can separately control

- total discrepancy between matched beads $\sum_{1 \le j \le m} \left|\mathrm{Leb}(U_j) - \mathrm{Leb}(U_j')\right|$,
- unmatched level-0 mass $\|\beta^0\| - \sum_{1 \le j \le m} \mathrm{Leb}(U_j)\mathbf{1}\{\mathrm{Leb}(U_j') > 0\}$,
- and unmatched level-y mass $\|\beta^y\| - \sum_{1 \le j \le m} \mathrm{Leb}(U_j')$.

Denote by $\widetilde{\mu}_\gamma$ the distribution of $M\widetilde{\beta}^0$ for independent $\widetilde{\beta}^0 \sim \widetilde{\mu}$ and $M \sim \mathtt{Gamma}\left(\frac{3}{2}, \gamma\right)$.

Lemma 48. *Let $(\beta^y, y \ge 0)$ be an $\mathcal{I}^\circ$-valued type-2 evolution starting according to $\widetilde{\mu}_\gamma$. Let $\theta \in (0, 1/4)$ and $p > 0$. Then there is a constant $C = C_{\gamma, \theta, p}$ such that*

$$\mathbb{E}\left[(d_\mathcal{I}(\beta^0, \beta^y))^p\right] \le Cy^{\theta p} \qquad for\ all\ 0 \le y \le 1. \tag{35}$$

Proof. Consider such a process $(\beta^y, y \ge 0)$. Its initial state is of the form $\beta^0 = (0, A) \star (0, B) \star G\bar{\beta}$, where A, B, G are i.i.d. $\mathtt{Gamma}\left(\frac{1}{2}, \gamma\right)$ random variables, independent of $\bar{\beta} \sim \mathtt{PDIP}\left(\frac{1}{2}, \frac{1}{2}\right)$. Further let $\beta^* = (\beta_1^*, \beta_2^*, \dots) \in [0, 1]^\infty$ denote a size-biased random ordering of the masses of $\bar{\beta}$.

To construct the matching, we take the blocks $U_1 = A$, $U_2 = B$, together with the blocks U_i for $3 \le i \le m = \lfloor 3y^{-1/4}\rfloor$, where U_i is the block corresponding to $G\beta_{i-2}^*$ and match them with the blocks $U_A', U_B', U_1', \dots$ that arise from the corresponding spindles at level y. Consequently, $\mathrm{Leb}(U_i') = \mathbf{g}_i(y)$ where $\mathbf{g}_i \sim \mathtt{BESQ}_{\mathrm{Leb}(U_i)}(-1)$ given $\mathrm{Leb}(U_i)$. Note that this means that some of our blocks will be matched with empty blocks and should thus be accounted for in the remaining mass component of the metric. We will handle this later.

Assuming $p \ge 2$, $y \in (0, 1]$, and using [15, Lemma 33] and the fact that M has finite moments of all orders, there are constants C_1, C_2, C_3, C_4 and C_5, depending only on p, such that

$$\mathbb{E}\left[\left(\sum_{1 \le j \le m} |\mathrm{Leb}(U_j) - \mathrm{Leb}(U_j')|\right)^p\right] \le m^{p-1}\mathbb{E}\left[\sum_{1 \le j \le m} |\mathrm{Leb}(U_j) - \mathrm{Leb}(U_j')|^p\right]$$

$$\le m^{p-1} y^{p/2} \sum_{1 \le j \le m} \mathbb{E}\left[\left(C_1 + \sqrt{C_2\mathrm{Leb}(U_j) + C_3}\right)^p\right]$$

$$\le m^{p-1} y^{p/2} \sum_{1 \le j \le m} \mathbb{E}\left[\left(C_1 + \sqrt{C_2\mathrm{Leb}(M) + C_3}\right)^p\right]$$

$$\le C_4 m^p y^{p/2} \le C_5 y^{p/4},$$

absorbing 3^p into the constant at the last step.

The unmatched mass at level 0 is $G \sum_{j=m-1}^{\infty} \beta_i^*$. Let $(Y_n)_{n\geq 1}$ be a sequence of independent random variables, also independent of G, such that Y_n has $\texttt{Beta}(1/2, (n+1)/2)$ distribution. Using the stick-breaking construction of the Poisson–Dirichlet distribution, we see that for all $y \in (0, 1]$

$$\mathbb{E}\left[\left(G \sum_{j=m-1}^{\infty} \beta_i^*\right)^p\right] = \mathbb{E}[G^p]\mathbb{E}\left[\left(1 - \sum_{j=1}^{m-2} \beta_i^*\right)^p\right] = \mathbb{E}[G^p]\mathbb{E}\left[\left(\prod_{j=1}^{m-2}(1 - Y_j)\right)^p\right]$$

$$= \mathbb{E}[G^p] \prod_{j=1}^{m-2} \mathbb{E}\left[(1 - Y_j)^p\right] = \mathbb{E}[G^p]\Gamma(1+p) \frac{\Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{m}{2}+p\right)}$$

$$\sim \mathbb{E}[G^p]\Gamma(1+p) m^{-p} = \mathbb{E}[G^p]\Gamma(1+p)\lfloor 3y^{-1/4}\rfloor^{-p} \leq C_6 y^{p/4},$$

for some $C_6 \geq C_5$.

We are left with estimating the unmatched mass at time y. By the triangle inequality,

$$\|\beta^y\| - \sum_{j=1}^{m} \mathrm{Leb}(U_j') \leq \left| \|\beta^y\| - M\right| + \left|G \sum_{j=m-1}^{\infty} \beta_j^*\right| + \sum_{1\leq j\leq m} \left|\mathrm{Leb}(U_j) - \mathrm{Leb}(U_j')\right|.$$

Furthermore, by Theorem 3, $(\|\beta^y\|)_{y\geq 0}$ is a $\texttt{BESQ}_M(-1)$ process to which [15, Lemma 33] applies, as above. Consequently, we have for some $C_7 > 0$ that

$$\mathbb{E}\left|\|\beta^y\| - \sum_{j=1}^{m} \mathrm{Leb}(U_j')\right|^p$$

$$\leq 3^{p-1}\left(\mathbb{E}\left|\|\beta^y\| - M\right|^p + \mathbb{E}\left|G \sum_{j=m-1}^{\infty} \beta_j^*\right|^p + \mathbb{E}\left[\left(\sum_{1\leq j\leq m}\left|\mathrm{Leb}(U_j) - \mathrm{Leb}(U_j')\right|\right)^p\right]\right)$$

$$\leq C_7(y^{p/2} + y^{p/4} + y^{p/4}) \leq 3C_7 y^{p/4}.$$

To account for the fact that some $\mathrm{Leb}(U_i')$ may be 0, and thus the corresponding U_i should count towards unmatched mass at time 0, we bound the metric $d_{\mathcal{I}}$ using the correspondence defined above (and bounding the maximum in the definition of $d_{\mathcal{I}}$ by a sum) to see that

$$d_{\mathcal{I}}(\beta^0, \beta^y) \leq \sum_{1\leq j\leq m} |\mathrm{Leb}(U_j) - \mathrm{Leb}(U_j')|\mathbf{1}\{\mathrm{Leb}(U_j') > 0\}$$

$$+ \|\beta^y\| - \sum_{j=1}^{m} \mathrm{Leb}(U_j')\mathbf{1}\{\mathrm{Leb}(U_j') > 0\} + M - \sum_{j=1}^{m} \mathrm{Leb}(U_j)\mathbf{1}\{\mathrm{Leb}(U_j') > 0\}$$

$$+ |\ell^0(T) - \ell^y(T)| + \max_{1\leq j\leq m}|\ell^0(U_j) - \ell^y(U_j')|\mathbf{1}\{\mathrm{Leb}(U_j') > 0\}$$

$$= \sum_{1\leq j\leq m} |\mathrm{Leb}(U_j) - \mathrm{Leb}(U_j')| + \|\beta^y\| - \sum_{j=1}^{m} \mathrm{Leb}(U_j') + M - \sum_{j=1}^{m} \mathrm{Leb}(U_j)$$

$$+ |\ell^0(T) - \ell^y(T)| + \max_{1\leq j\leq m}|\ell^0(U_j) - \ell^y(U_j')|\mathbf{1}\{\mathrm{Leb}(U_j') > 0\}.$$

Dropping the indicator on the last term and combining this with our calculations above and Lemma 47 shows that for $0 < \theta < 1/4$ there exists some constant $C_{\gamma,\theta,p}$ depending only on γ, θ and p that satisfies (35). $\qquad\square$

Proof of Proposition 45. Let $(\beta^y, y \geq 0)$ be a type-2 evolution with initial distribution μ_γ. Denote the total mass evolution by $Z(y) = \|\beta^y\|$, $y \geq 0$. Let $0 \leq a < b \leq 1$. Then

$$\mathbb{E}_{\mu_\gamma}\left[(d_{\mathcal{I}}(\beta^a, \beta^b))^p\right] = \mathbb{E}_{\mu_\gamma}\left[\mathbf{1}\{D > a\}\mathbb{E}_{\beta^a}\left[(d_{\mathcal{I}}(\beta^0, \beta^{b-a}))^p\right]\right] + \mathbb{E}_{\mu_\gamma}\left[|Z(a) - Z(b)|^p\mathbf{1}\{D < a\}\right].$$

For the first term, we condition on $D > a$ and apply the pseudo-stationarity of Proposition 32. While the distribution of β^a given $D > a$ may not be $\mu_{\gamma/(2\gamma a+1)}$, it is $\mu_{\gamma/(2\gamma a+1)}$ up to a potential swap of the two leftmost blocks, and the matching set up in the proof of Lemma 48

is unaffected by such a swap so that scaling by $2\gamma a + 1$ and applying the bound of Lemma 48 yields the upper bound:

$$\mathbb{E}_{\mu_{\gamma/(2\gamma a+1)}}\left[\left(d_{\mathcal{I}}\left(\beta^0, \beta^{b-a}\right)\right)^p\right] = (2\gamma a + 1)^p \mathbb{E}_{\mu_\gamma}\left[\left(d_{\mathcal{I}}\left(\beta^0, \beta^{(b-a)/(2\gamma a+1)}\right)\right)^p\right]$$

$$\leq (2\gamma a + 1)^p C(b-a)^{\theta p}/(2\gamma a + 1)^{\theta p} \leq (2\gamma + 1)^{p(1-\theta)}C|b-a|^{\theta p}.$$

For the second term, we apply [15, Lemma 33] to find the upper bound

$$|b-a|^{p/2}\mathbb{E}_{\mu_\gamma}\left[\left(1 + 2(p-1) + 2\sqrt{p-1}\sqrt{Z(0) + 2(p-1)}\right)^p\right],$$

which is easily seen to be a finite multiple of $|b-a|^{p/2} \leq |b-a|^{\theta p}$.

By the Kolmogorov-Chentsov theorem [36, Theorem I.(2.1)], this shows that for all $0 < \theta < 1/4$ and $p > 0$,

$$\mathbb{E}_{\mu_\gamma}\left[\left(\sup_{0 \leq a < b \leq 1} \frac{d_{\mathcal{I}}(\beta^a, \beta^b)}{|b-a|^\theta}\right)^p\right] < \infty.$$

We can write the LHS by integrating out the random initial mass. Cancelling $\gamma^{3/2}/\Gamma(3/2)$ gives

$$\int_0^\infty e^{-\gamma x}\sqrt{x}\mathbb{E}_x\left[\left(\sup_{0 \leq a < b \leq 1} \frac{d_{\mathcal{I}}(\beta^a, \beta^b)}{|b-a|^\theta}\right)^p\right]dx < \infty.$$

By Fubini's theorem, this yields for a.e. $x \in (0, \infty)$ that $\mathbb{E}_x\left[\left(\sup_{0 \leq a < b \leq 1} \frac{d_{\mathcal{I}}(\beta^a, \beta^b)}{|b-a|^\theta}\right)^p\right] < \infty.$

But for any $x, y \in (0, \infty)$, we can find $c < 1/y$ so that this expectation is finite for initial mass xc. By scaling,

$$\infty > \mathbb{E}_{cx}\left[\sup_{0 \leq a < b \leq 1}\left(\frac{d_{\mathcal{I}}(\beta^a, \beta^b)}{|b-a|^\theta}\right)^p\right] = c^p \mathbb{E}_x\left[\sup_{0 \leq a < b \leq 1}\left(\frac{d_{\mathcal{I}}(\beta^{a/c}, \beta^{b/c})}{|b-a|^\theta}\right)^p\right]$$

$$= c^{p(1-\theta)}\mathbb{E}_x\left[\sup_{0 \leq a' < b' \leq 1/c}\left(\frac{d_{\mathcal{I}}(\beta^{a'}, \beta^{b'})}{|b'-a'|^\theta}\right)^p\right] \geq c^{p(1-\theta)}\mathbb{E}_x\left[\sup_{0 \leq a' < b' \leq y}\left(\frac{d_{\mathcal{I}}(\beta^{a'}, \beta^{b'})}{|b'-a'|^\theta}\right)^p\right],$$

so the expectation is finite for any initial mass, including unit initial mass $x = 1$, and for any $y \in (0, \infty)$. $\qquad\square$

References

[1] D. Aldous. The continuum random tree. I. *Ann. Probab.*, 19(1):1–28, 1991.

[2] D. Aldous. Recursive self-similarity for random trees, random triangulations and Brownian excursion. *Ann. Probab.*, 22(2):527–545, 1994.

[3] D. Aldous. Problem. Give a rigorous construction of this "diffusion on continuum trees" http://www.stat.berkeley.edu/ aldous/Research/OP/fw.html, 1999.

[4] D. J. Aldous. Exchangeability and related topics. In *École d'été de probabilités de Saint-Flour, XIII—1983*, volume 1117 of *Lecture Notes in Math.*, pages 1–198. Springer, Berlin, 1985.

[5] D. J. Aldous. Mixing time for a Markov chain on cladograms. *Combin. Probab. Comput.*, 9(3):191–204, 2000.

[6] A.-L. Basdevant. Ruelle's probability cascades seen as a fragmentation process. *Markov Proc. Rel. Fields*, 12(3):447–474, 2006.

[7] J. Bect. *Processus de Markov diffusifs par morceaux: outils analytiques et numériques*. Thèse de doctorat, Université Paris-Sud XI, 171 p., also available at https://tel.archives-ouvertes.fr/tel-00169791, 2007.

[8] J. Bertoin. *Lévy processes*, volume 121 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1996.

[9] K. L. Chung and J. B. Walsh. *Markov processes, Brownian motion, and time symmetry*, volume 249. Springer Science & Business Media, 2006.

[10] N. Forman, G. Brito, Y. Chou, A. Forney, and C. Li. WXML final report: Chinese restaurant process. 2017.

[11] N. Forman, S. Pal, D. Rizzolo, and M. Winkel. Diffusions on a space of interval partitions with Poisson-Dirichlet stationary distributions. arXiv: 1609.06706v2 [math.PR], 2017.

[12] N. Forman, S. Pal, D. Rizzolo, and M. Winkel. A projective system of k-tree evolutions related to the Aldous diffusion. Work in progress, 2018.

[13] N. Forman, S. Pal, D. Rizzolo, and M. Winkel. Existence of the Aldous diffusion on continuum trees. Work in progress, 2018.

[14] N. Forman, S. Pal, D. Rizzolo, and M. Winkel. Projections of the Aldous chain on binary trees: Intertwining and consistency. arXiv:1802.00862 [math.PR], 2018.

[15] N. Forman, S. Pal, D. Rizzolo, and M. Winkel. Uniform control of local times of spectrally positive stable processes. *To appear in Ann. Appl. Probab.*, 2018. arXiv: 1609.06707.

[16] J. Geiger. Contour processes of random trees. In *Stochastic partial differential equations (Edinburgh, 1994)*, volume 216 of *London Math. Soc. Lecture Note Ser.*, pages 72–96. Cambridge Univ. Press, Cambridge, 1995.

[17] J. Geiger and G. Kersting. Depth-first search of random trees, and Poisson point processes. In *Classical and modern branching processes (Minneapolis, MN, 1994)*, volume 84 of *IMA Vol. Math. Appl.*, pages 111–126. Springer, New York, 1997.

[18] A. Gnedin and J. Pitman. Regenerative composition structures. *Ann. Probab.*, 33(2):445–479, 2005.

[19] A. Göing-Jaeschke and M. Yor. A survey and some generalizations of Bessel processes. *Bernoulli*, 9(2):313–349, 2003.

[20] B. Haas, J. Pitman, and M. Winkel. Spinal partitions and invariance under re-rooting of continuum random trees. *Ann. Probab.*, 37(4):1381–1411, 2009.

[21] P. Jagers. A general stochastic model for population development. *Skand. Aktuarietidskr.*, pages 84–103, 1969.

[22] O. Kallenberg. *Foundations of modern probability.* Probability and its Applications (New York). Springer-Verlag, New York, second edition, 2002.

[23] A. Lambert. The contour of splitting trees is a Lévy process. *Ann. Probab.*, 38(1):348–395, 2010.

[24] Z. Li. *Measure-valued branching Markov processes.* Probability and its Applications. Springer-Verlag, Berlin Heidelberg, 2011.

[25] P. McCullagh, J. Pitman, and M. Winkel. Gibbs fragmentation trees. *Bernoulli*, 14(4):988–1002, 2008.

[26] P. A. Meyer. Renaissance, recollements, mélanges, ralentissement de processus de Markov. *Ann. Inst. Fourier*, 25(3-4):465–497, 1975.

[27] G. Miermont. Self-similar fragmentations derived from the stable tree. I. Splitting at heights. *Probab. Theory Related Fields*, 127(3):423–454, 2003.

[28] S. Pal. Analysis of market weights under volatility-stabilized market models. *Ann. Appl. Probab.*, 21(3):1180–1213, 2011.

[29] S. Pal. Wright-Fisher diffusion with negative mutation rates. *Ann. Probab.*, 41(2):503–526, 2013.

[30] M. Perman, J. Pitman, and M. Yor. Size-biased sampling of Poisson point processes and excursions. *Probab. Theory Related Fields*, 92(1):21–39, 1992.

[31] J. Pitman. Poisson-Kingman partitions. In *Statistics and science: a Festschrift for Terry Speed*, volume 40 of *IMS Lecture Notes Monogr. Ser.*, pages 1–34. Inst. Math. Statist., Beachwood, OH, 2003.

[32] J. Pitman. *Combinatorial stochastic processes*, volume 1875 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2006. Lectures from the 32nd Summer School on Probability Theory held in Saint-Flour, July 7–24, 2002, With a foreword by Jean Picard.

[33] J. Pitman and M. Winkel. Regenerative tree growth: binary self-similar continuum random trees and Poisson-Dirichlet compositions. *Ann. Probab.*, 37(5):1999–2041, 2009.

[34] J. Pitman and M. Yor. A decomposition of Bessel bridges. *Z. Wahrsch. Verw. Gebiete*, 59(4):425–457, 1982.

[35] J. Pitman and M. Yor. Arcsine laws and interval partitions derived from a stable subordinator. *Proc. London Math. Soc. (3)*, 65(2):326–356, 1992.

[36] D. Revuz and M. Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition, 1999.

[37] L. C. G. Rogers and J. W. Pitman. Markov functions. *The Annals of Probability*, 9(4):573–582, 1981.

[38] M. Sharpe. *General theory of Markov processes*, volume 133 of *Pure and Applied Mathematics*. Academic Press, Inc., Boston, MA, 1988.

[39] J. Warren and M. Yor. The Brownian burglar: conditioning Brownian motion by its local time process. In *Séminaire de Probabilités, XXXII*, volume 1686 of *Lecture Notes in Math.*, pages 328–342. Springer, Berlin, 1998.

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