

# An Introduction to Bayesian Nonparametric Modelling

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# Outline

Some Examples of Parametric Models

Bayesian Nonparametric Modelling

Infinite Mixture Models

Some Measure Theory

Dirichlet Processes

Indian Buffet and Beta Processes

Hierarchical Dirichlet Processes

Pitman-Yor Processes

Summary

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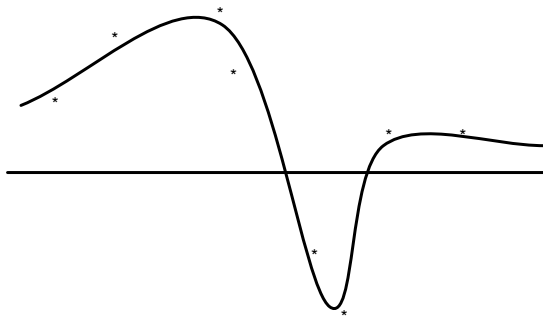
Hierarchical Dirichlet Processes

Pitman-Yor Processes

Summary

# Regression with Basis Functions

- ▶ Supervised learning of a function  $f^* : \mathbb{X} \rightarrow \mathbb{Y}$  from training data  $\{x_i, y_i\}_{i=1}^n$ .



# Regression with Basis Functions

- ▶ Assume a set of basis functions  $\phi_1, \dots, \phi_K$  and parametrize a function:

$$f(x; \mathbf{w}) = \sum_{k=1}^K w_k \phi_k(x)$$

Parameters  $\mathbf{w} = \{w_1, \dots, w_K\}$ .

- ▶ Find optimal parameters

$$\operatorname{argmin}_{\mathbf{w}} \sum_{i=1}^n \left| y_i - f(x_i; \mathbf{w}) \right|^2 = \operatorname{argmin}_{\mathbf{w}} \sum_{i=1}^n \left| y_i - \sum_{k=1}^K w_k \phi_k(x_i) \right|^2$$

- ▶ We will be Bayesian in this lecture, so we need to rephrase using probabilistic model with priors on parameters:

$$y_i | x_i, \mathbf{w} = f(x_i; \mathbf{w}) + \epsilon_i \quad \epsilon_i \sim \mathcal{N}(0, \sigma^2)$$
$$w_k \sim \mathcal{N}(0, \tau^2)$$

- ▶ Computer posterior  $p(\mathbf{w} | \{x_i, y_i\})$ .

# Regression with Basis Functions

$$f(x; \mathbf{w}) = \sum_{k=1}^K w_k \phi_k(x)$$

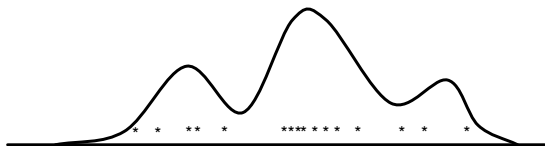
- ▶ What basis functions to use?
- ▶ How many basis functions to use?
- ▶ Do we really believe that the true  $f^*(x)$  can be expressed as  $f^*(x) = f(x; \mathbf{w}^*)$  for some  $\mathbf{w}^*$ ?

$$\epsilon_j \sim \mathcal{N}(0, \sigma^2)$$

- ▶ Do we believe the noise process is Gaussian?

# Density Estimation with Mixture Models

- ▶ Unsupervised learning of a density  $f^*(x)$  from training samples  $\{x_i\}$ .



- ▶ Perhaps use an exponential family distribution, e.g. Gaussian?

$$\mathcal{N}(x; \mu, \Sigma) = |2\pi\Sigma|^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(x - \mu)^\top \Sigma^{-1}(x - \mu)\right)$$

Unimodal, restrictive shape, light tail...

- ▶ Use a mixture model instead,

$$f(x) = \sum_{k=1}^K \pi_k \mathcal{N}(x; \mu_k, \Sigma_k)$$

- ▶ Do we believe that the true density is a mixture of  $K$  components?
- ▶ How many mixture components to use?

# Latent Variable Modelling

- ▶ Say we have  $n$  vector observations  $x_1, \dots, x_n$ .
- ▶ Model each observation as a linear combination of  $K$  latent sources:

$$x_i = \sum_{k=1}^K \Lambda_k y_{ik} + \epsilon_i$$

$y_{ik}$ : activity of source  $k$  in datum  $i$ .

$\Lambda_k$ : basis vector describing effect of source  $k$ .

- ▶ Examples include principle components analysis, factor analysis, independent components analysis.
- ▶ How many sources are there?
- ▶ Do we believe that  $K$  sources is sufficient to explain all our data?
- ▶ What prior distribution should we use for sources?



# Topic Modelling with Latent Dirichlet Allocation

- ▶ Infer topics from a document corpus, topics being sets of words that tend to co-occur together.
- ▶ Using (Bayesian) latent Dirichlet allocation:

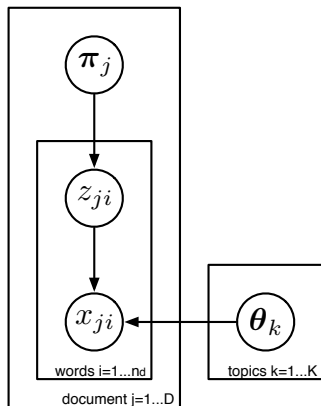
$$\pi_j \sim \text{Dirichlet}\left(\frac{\alpha}{K}, \dots, \frac{\alpha}{K}\right)$$

$$\theta_k \sim \text{Dirichlet}\left(\frac{\beta}{W}, \dots, \frac{\beta}{W}\right)$$

$$z_{ji} | \pi_j \sim \text{Multinomial}(\pi_j)$$

$$x_{ji} | z_{ji}, \theta_{z_{ji}} \sim \text{Multinomial}(\theta_{z_{ji}})$$

- ▶ How many topics can we find from the corpus?



# Outline

Some Examples of Parametric Models

**Bayesian Nonparametric Modelling**

Infinite Mixture Models

Some Measure Theory

Dirichlet Processes

Indian Buffet and Beta Processes

Hierarchical Dirichlet Processes

Pitman-Yor Processes

Summary

# Modelling Data

- ▶ Models are almost never correct for real world data.
- ▶ How do we deal with model misfit?
  - ▶ Quantify closeness to true model, and optimality of fitted model;
  - ▶ Model selection or averaging;
  - ▶ Increase the flexibility of your model class.
- ▶ Bayesian nonparametrics are good solutions from the second and third perspectives.

# Model Selection and Model Averaging

- ▶ Data  $\mathbf{x} = \{x_1, x_2, \dots, x_n\}$ .
- ▶ Model  $M_k$  parametrized by  $\theta_k$ , for  $k = 1, 2, \dots$
- ▶ Marginal likelihood:

$$p(\mathbf{x}|M_k) = \int p(\mathbf{x}|\theta_k, M_k)p(\theta_k, M_k)d\theta_k$$

- ▶ Model selection and averaging:

$$M = \underset{M_k}{\operatorname{argmax}} p(\mathbf{x}|M_k) \quad \text{or} \quad p(k, \theta_k|\mathbf{x}) = \frac{p(k)p(\theta_k|M_k)p(\mathbf{x}|\theta_k, M_k)}{\sum_{k'} p(k')p(\theta_{k'}|M_{k'})p(\mathbf{x}|\theta_{k'}, M_{k'})}$$

- ▶ Model selection and averaging is to prevent overfitting and underfitting, and are usually expensive to compute.
- ▶ But reasonable and proper Bayesian methods should not overfit anyway [Rasmussen and Ghahramani 2001].

# Nonparametric Modelling

- ▶ What is a nonparametric model?
  - ▶ A really large parametric model;
  - ▶ A parametric model where the number of parameters increases with data;
  - ▶ A model over infinite dimensional function or measure spaces.
  - ▶ A family of distributions that is dense in some large space.
- ▶ Why nonparametric models in Bayesian theory of learning?
  - ▶ broad class of priors that allows data to “speak for itself”;
  - ▶ side-step model selection and averaging.
- ▶ How do we deal with the very large parameter spaces?
  - ▶ Marginalize out all but a finite number of parameters;
  - ▶ Define infinite space implicitly (akin to the kernel trick) using either Kolmogorov Consistency Theorem or de Finetti’s Theorem.

# Gaussian Processes

- ▶ A *Gaussian process* (GP) is a random function  $f : \mathbb{X} \rightarrow \mathbb{R}$  such that for any finite set of input points  $x_1, \dots, x_n$ ,

$$\begin{bmatrix} f(x_1) \\ \vdots \\ f(x_n) \end{bmatrix} \sim \mathcal{N} \left( \begin{bmatrix} m(x_1) \\ \vdots \\ m(x_n) \end{bmatrix}, \begin{bmatrix} c(x_1, x_1) & \dots & c(x_1, x_n) \\ \vdots & \ddots & \vdots \\ c(x_n, x_1) & \dots & c(x_n, x_n) \end{bmatrix} \right)$$

where the parameters are the mean function  $m(x)$  and covariance kernel  $c(x, y)$ .

- ▶ Note: a random function  $f$  is a stochastic process. It is a collection of random variables  $\{f(x)\}_{x \in \mathbb{X}}$  one for each possible input value  $x$ .
- ▶ Can also be expressed as

$$f(x) = \sum_{k=1}^K w_k \phi_k(x) \quad \text{as } K \rightarrow \infty.$$

# Posterior and Predictive Distributions

- ▶ How do we compute the posterior and predictive distributions?
- ▶ Training set  $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$  and test input  $x_{n+1}$ .
- ▶ Out of the (uncountably infinitely) many random variables  $\{f(x)\}_{x \in \mathbb{X}}$  making up the GP only  $n + 1$  has to do with the data:

$$f(x_1), f(x_2), \dots, f(x_{n+1})$$

- ▶ Training data gives observations  $f(x_1) = y_1, \dots, f(x_n) = y_n$ . The predictive distribution of  $f(x_{n+1})$  is simply

$$p(f(x_{n+1}) | f(x_1) = y_1, \dots, f(x_n) = y_n)$$

which is easy to compute since  $f(x_1), \dots, f(x_{n+1})$  is Gaussian.

- ▶ This can be generalized to noisy observations  $y_i = f(x_i) + \epsilon_i$  or non-linear effects  $y_i \sim D(f(x_i))$  where  $D(\theta)$  is a distribution parametrized by  $\theta$ .

# Consistency and Existence

- ▶ The definition of Gaussian processes only give finite dimensional marginal distributions of the stochastic process.
- ▶ Fortunately these marginal distributions are *consistent*.
  - ▶ For every finite set  $\mathbf{x} \subset \mathbb{X}$  we have a distinct distribution  $p_{\mathbf{x}}([f(x)]_{x \in \mathbf{x}})$ . These distributions are said to be consistent if

$$p_{\mathbf{x}}([f(x)]_{x \in \mathbf{x}}) = \int p_{\mathbf{x} \cup \mathbf{y}}([f(x)]_{x \in \mathbf{x} \cup \mathbf{y}}) d[f(x)]_{x \in \mathbf{y}}$$

for disjoint and finite  $\mathbf{x}, \mathbf{y} \subset \mathbb{X}$ .

- ▶ The marginal distributions for the GP are consistent because *Gaussians are closed under marginalization*.
- ▶ The *Kolmogorov Consistency Theorem* guarantees existence of GPs, i.e. the whole stochastic process  $\{f(x)\}_{x \in \mathbb{X}}$ .
  - ▶ Further information in Peter Orbanz' Bayesian nonparametric tutorial.



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# Bayesian Mixture Models

- ▶ Let's be Bayesian about mixture models, and place priors over our parameters (and to compute posteriors).
- ▶ First, introduce variable  $z_i$  indicator which component  $x_i$  belongs to.

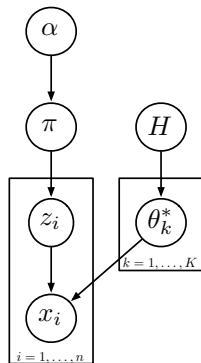
$$z_i | \pi \sim \text{Multinomial}(\pi)$$

$$x_i | z_i = k, \mu, \Sigma \sim \mathcal{N}(\mu_k, \Sigma_k)$$

- ▶ Second, introduce conjugate priors for parameters:

$$\pi \sim \text{Dirichlet}\left(\frac{\alpha}{K}, \dots, \frac{\alpha}{K}\right)$$

$$\mu_k, \Sigma_k = \theta_k^* \sim H = \mathcal{N}\text{-}\mathcal{IW}(0, \mathbf{s}, \mathbf{d}, \Phi)$$



# Gibbs Sampling for Bayesian Mixture Models

- ▶ All conditional distributions are simple to compute:

$$p(z_i = k | \text{others}) \propto \pi_k \mathcal{N}(x_i; \mu_k, \Sigma_k)$$

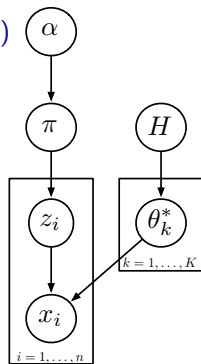
$$\pi | \mathbf{z} \sim \text{Dirichlet}\left(\frac{\alpha}{K} + n_1(\mathbf{z}), \dots, \frac{\alpha}{K} + n_K(\mathbf{z})\right)$$

$$\mu_k, \Sigma_k | \text{others} \sim \mathcal{N}\text{-IW}(\nu', \mathbf{s}', \mathbf{d}', \Phi')$$

- ▶ Not as efficient as collapsed Gibbs sampling which integrates out  $\pi, \mu, \Sigma$ :

$$p(z_i = k | \text{others}) \propto \frac{\frac{\alpha}{K} + n_k(\mathbf{z}_{-i})}{\alpha + n - 1} \times$$

$$p(x_i | \{x_{i'} : i' \neq i, z_{i'} = k\})$$



- ▶ Demo: `fm_demointeractive`.

# Infinite Bayesian Mixture Models

- ▶ We will take  $K \rightarrow \infty$ .
- ▶ Imagine a very large value of  $K$ .
- ▶ There are at most  $n < K$  occupied components, so most components are *empty*. We can lump these empty components together:

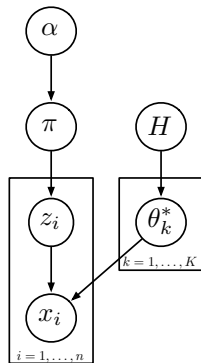
Occupied clusters:

$$p(z_i = k | \text{others}) \propto \frac{\alpha + n_k(\mathbf{z}_{-i})}{n - 1 + \alpha} p(x_i | \mathbf{x}_k^{-i})$$

Empty clusters:

$$p(z_i = k_{\text{empty}} | \mathbf{z}^{-i}) \propto \frac{\alpha \frac{K - K^*}{K}}{n - 1 + \alpha} p(x_i | \{\})$$

- ▶ Demo: `dpm_demointeractive`.



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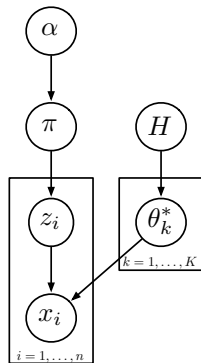
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Empty clusters:

$$p(z_i = k_{\text{empty}} | \mathbf{z}^{-i}) \propto \frac{\alpha}{n - 1 + \alpha} p(x_i | \{\})$$

- ▶ Demo: `dpm_demointeractive`.



# Infinite Bayesian Mixture Models

- ▶ The actual infinite limit of finite mixture models does not make sense: any particular component will get a mixing proportion of 0.
- ▶ In the Gibbs sampler we bypassed this by lumping empty clusters together.
- ▶ Other better ways of making this infinite limit precise:
  - ▶ Look at the prior clustering structure induced by the Dirichlet prior over mixing proportions—*Chinese restaurant process*.
  - ▶ Re-order components so that those with larger mixing proportions tend to occur first, before taking the infinite limit—*stick-breaking construction*.
- ▶ Both are different views of the *Dirichlet process* (DP).
- ▶ DPs can be thought of as infinite dimensional Dirichlet distributions.
- ▶ The  $K \rightarrow \infty$  Gibbs sampler is for DP mixture models.

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# A Tiny Bit of Measure Theoretic Probability Theory

- ▶ A  $\sigma$ -*algebra*  $\Sigma$  is a family of subsets of a set  $\Theta$  such that
  - ▶  $\Sigma$  is not empty;
  - ▶ If  $A \in \Sigma$  then  $\Theta \setminus A \in \Sigma$ ;
  - ▶ If  $A_1, A_2, \dots \in \Sigma$  then  $\bigcup_{i=1}^{\infty} A_i \in \Sigma$ .
- ▶  $(\Theta, \Sigma)$  is a *measure space* and  $A \in \Sigma$  are the *measurable sets*.
- ▶ A *measure*  $\mu$  over  $(\Theta, \Sigma)$  is a function  $\mu : \Sigma \rightarrow [0, \infty]$  such that
  - ▶  $\mu(\emptyset) = 0$ ;
  - ▶ If  $A_1, A_2, \dots \in \Sigma$  are disjoint then  $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$ .
  - ▶ Everything we consider here will be measurable.
  - ▶ A probability measure is one where  $\mu(\Theta) = 1$ .
- ▶ Given two measure spaces  $(\Theta, \Sigma)$  and  $(\Delta, \Phi)$ , a function  $f : \Theta \rightarrow \Delta$  is *measurable* if  $f^{-1}(A) \in \Sigma$  for every  $A \in \Phi$ .



# A Tiny Bit of Measure Theoretic Probability Theory

- ▶ If  $p$  is a probability measure on  $(\Theta, \Sigma)$ , a *random variable*  $X$  taking values in  $\Delta$  is simply a measurable function  $X : \Theta \rightarrow \Delta$ .
  - ▶ Think of the probability space  $(\Theta, \Sigma, p)$  as a black-box random number generator, and  $X$  as a function taking random samples in  $\Theta$  and producing random samples in  $\Delta$ .
  - ▶ The probability of an event  $A \in \Phi$  is  $p(X \in A) = p(X^{-1}(A))$ .
- ▶ A *stochastic process* is simply a collection of random variables  $\{X_i\}_{i \in \mathbb{I}}$  over the same measure space  $(\Theta, \Sigma)$ , where  $\mathbb{I}$  is an index set.
  - ▶ What distinguishes a stochastic process from, say, a graphical model is that  $\mathbb{I}$  can be infinite, even uncountably so.
  - ▶ This raises issues of how do you even define them and how do you ensure that they can even exist (mathematically speaking).
- ▶ Stochastic processes form the core of many Bayesian nonparametric models.
  - ▶ Gaussian processes, Poisson processes, gamma processes, Dirichlet processes, beta processes...

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# Dirichlet Distributions

- ▶ A *Dirichlet distribution* is a distribution over the  $K$ -dimensional probability simplex:

$$\Delta_K = \{(\pi_1, \dots, \pi_K) : \pi_k \geq 0, \sum_k \pi_k = 1\}$$

- ▶ We say  $(\pi_1, \dots, \pi_K)$  is Dirichlet distributed,

$$(\pi_1, \dots, \pi_K) \sim \text{Dirichlet}(\lambda_1, \dots, \lambda_K)$$

with parameters  $(\lambda_1, \dots, \lambda_K)$ , if

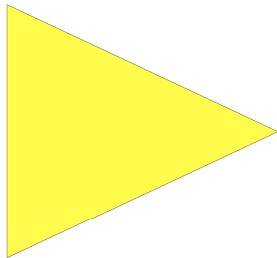
$$p(\pi_1, \dots, \pi_K) = \frac{\Gamma(\sum_k \lambda_k)}{\prod_k \Gamma(\lambda_k)} \prod_{k=1}^n \pi_k^{\lambda_k - 1}$$

- ▶ Equivalent to normalizing a set of independent gamma variables:

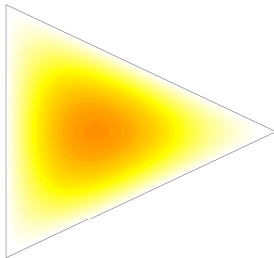
$$(\pi_1, \dots, \pi_K) = \frac{1}{\sum_k \gamma_k} (\gamma_1, \dots, \gamma_K)$$
$$\gamma_k \sim \text{Gamma}(\lambda_k) \quad \text{for } k = 1, \dots, K$$

# Dirichlet Distributions

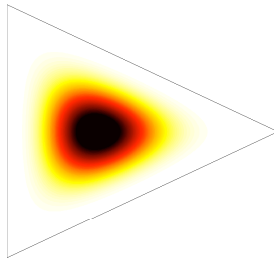
$\text{Dir}(1,0,1,0,1,0)$



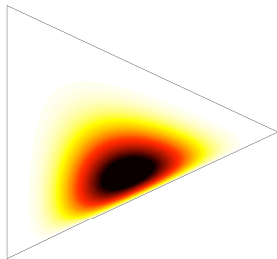
$\text{Dir}(2,0,2,0,2,0)$



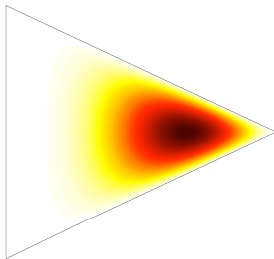
$\text{Dir}(5,0,5,0,5,0)$



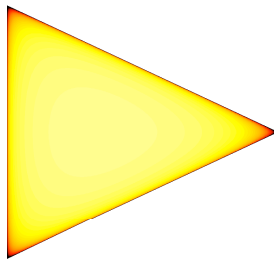
$\text{Dir}(5,0,5,0,2,0)$



$\text{Dir}(5,0,2,0,2,0)$



$\text{Dir}(0,7,0,7,0,7)$

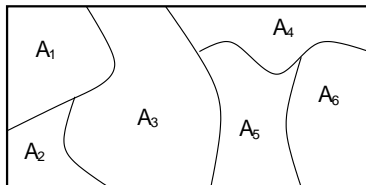


# Dirichlet Processes

- ▶ A *Dirichlet Process* (DP) is a random probability measure  $G$  over  $(\Theta, \Sigma)$  such that for any finite set of measurable partitions  $A_1 \dot{\cup} \dots \dot{\cup} A_K = \Theta$ ,

$$(G(A_1), \dots, G(A_K)) \sim \text{Dirichlet}(\lambda(A_1), \dots, \lambda(A_K))$$

where  $\lambda$  is a base measure.



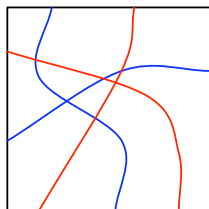
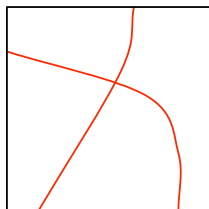
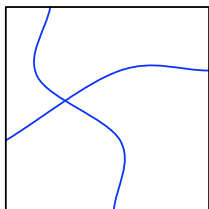
- ▶ The above family of distributions is consistent (next slide), and *Kolmogorov Consistency Theorem* can be applied to show existence (but there are technical conditions restricting the generality of the definition).

[Ferguson 1973, Blackwell and MacQueen 1973]

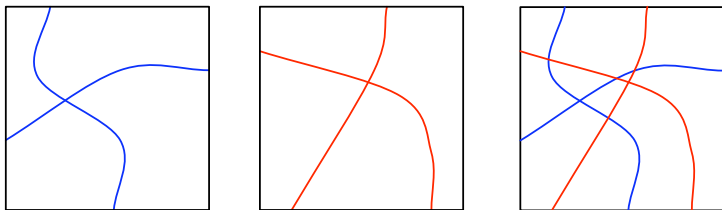
# Consistency of Dirichlet Marginals

- ▶ If we have two partitions  $(A_1, \dots, A_K)$  and  $(B_1, \dots, B_J)$  of  $\Theta$ , how do we see if the two Dirichlets are consistent?
- ▶ Because Dirichlet variables are normalized gamma variables and sums of gammas are gammas, if  $(l_1, \dots, l_j)$  is a partition of  $(1, \dots, K)$ ,

$$\left( \sum_{i \in l_1} \pi_i, \dots, \sum_{i \in l_j} \pi_i \right) \sim \text{Dirichlet} \left( \sum_{i \in l_1} \lambda_i, \dots, \sum_{i \in l_j} \lambda_i \right)$$



# Consistency of Dirichlet Marginals



- ▶ Form the common refinement  $(C_1, \dots, C_L)$  where each  $C_\ell$  is the intersection of some  $A_k$  with some  $B_j$ . Then:

By definition,  $(G(C_1), \dots, G(C_L)) \sim \text{Dirichlet}(\lambda(C_1), \dots, \lambda(C_L))$

$$\begin{aligned} (G(A_1), \dots, G(A_K)) &= (\sum_{C_\ell \subset A_1} G(C_\ell), \dots, \sum_{C_\ell \subset A_K} G(C_\ell)) \\ &\sim \text{Dirichlet}(\lambda(A_1), \dots, \lambda(A_K)) \end{aligned}$$

Similarly,  $(G(B_1), \dots, G(B_J)) \sim \text{Dirichlet}(\lambda(B_1), \dots, \lambda(B_J))$

so the distributions of  $(G(A_1), \dots, G(A_K))$  and  $(G(B_1), \dots, G(B_J))$  are consistent.

- ▶ Demonstration: DPgenerate.

# Parameters of Dirichlet Processes

- ▶ Usually we split the  $\lambda$  base measure into two parameters  $\lambda = \alpha H$ :
  - ▶ *Base distribution*  $H$ , which is like the *mean* of the DP.
  - ▶ *Strength parameter*  $\alpha$ , which is like an *inverse-variance* of the DP.
- ▶ We write:

$$G \sim \text{DP}(\alpha, H)$$

if for any partition  $(A_1, \dots, A_K)$  of  $\Theta$ :

$$(G(A_1), \dots, G(A_K)) \sim \text{Dirichlet}(\alpha H(A_1), \dots, \alpha H(A_K))$$

- ▶ The first and second moments of the DP:

$$\text{Expectation:} \quad \mathbb{E}[G(A)] = H(A)$$

$$\text{Variance:} \quad \mathbb{V}[G(A)] = \frac{H(A)(1 - H(A))}{\alpha + 1}$$

where  $A$  is any measurable subset of  $\Theta$ .



# Representations of Dirichlet Processes

- ▶ Draws from Dirichlet processes will always place all their mass on a countable set of points:

$$G = \sum_{k=1}^{\infty} \pi_k \delta_{\theta_k^*}$$

where  $\sum_k \pi_k = 1$  and  $\theta_k^* \in \Theta$ .

- ▶ What is the joint distribution over  $\pi_1, \pi_2, \dots$  and  $\theta_1^*, \theta_2^*, \dots$ ?
- ▶ Since  $G$  is a (random) probability measure over  $\Theta$ , we can treat it as a distribution and draw samples from it. Let

$$\theta_1, \theta_2, \dots \sim G$$

be random variables with distribution  $G$ .

- ▶ What is the marginal distribution of  $\theta_1, \theta_2, \dots$  with  $G$  integrated out?
- ▶ There is positive probability that sets of  $\theta_i$ 's can take on the same value  $\theta_k^*$  for some  $k$ , i.e. the  $\theta_i$ 's cluster together. How do these clusters look like?
- ▶ For practical modelling purposes this is sufficient. But is this sufficient to tell us all about  $G$ ?

# Stick-breaking Construction

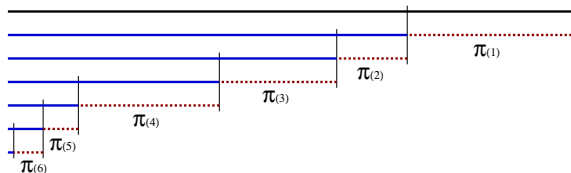
$$G = \sum_{k=1}^{\infty} \pi_k \delta_{\theta_k^*}$$

- ▶ There is a simple construction giving the joint distribution of  $\pi_1, \pi_2, \dots$  and  $\theta_1^*, \theta_2^*, \dots$  called the *stick-breaking construction*.

$$\theta_k^* \sim H$$

$$\pi_k = v_k \prod_{i=1}^{k-1} (1 - v_i)$$

$$v_k \sim \text{Beta}(1, \alpha)$$



- ▶ Also known as the *GEM* distribution, write  $\pi \sim \text{GEM}(\alpha)$ .

[Sethuraman 1994]

# Posterior of Dirichlet Processes

- ▶ Since  $G$  is a probability measure, we can draw samples from it,

$$G \sim \text{DP}(\alpha, H)$$
$$\theta_1, \dots, \theta_n | G \sim G$$

What is the posterior of  $G$  given observations of  $\theta_1, \dots, \theta_n$ ?

- ▶ The usual Dirichlet-multinomial conjugacy carries over to the nonparametric DP as well:

$$G | \theta_1, \dots, \theta_n \sim \text{DP}\left(\alpha + n, \frac{\alpha H + \sum_{i=1}^n \delta_{\theta_i}}{\alpha + n}\right)$$

# Pólya Urn Scheme

$$\theta_1, \theta_2, \dots \sim G$$

- ▶ The marginal distribution of  $\theta_1, \theta_2, \dots$  has a simple generative process called the *Pólya urn scheme*.

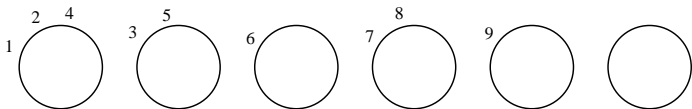
$$\theta_n | \theta_{1:n-1} \sim \frac{\alpha H + \sum_{i=1}^{n-1} \delta_{\theta_i}}{\alpha + n - 1}$$

- ▶ Picking balls of different colors from an urn:
  - ▶ Start with no balls in the urn.
  - ▶ with probability  $\propto \alpha$ , draw  $\theta_n \sim H$ , and add a ball of color  $\theta_n$  into urn.
  - ▶ With probability  $\propto n - 1$ , pick a ball at random from the urn, record  $\theta_n$  to be its color and return two balls of color  $\theta_n$  into urn.
- ▶ Pólya urn scheme is like a “representer” for the DP—a finite projection of an infinite object  $G$ .
- ▶ Also known as the *Blackwell-MacQueen urn scheme*.

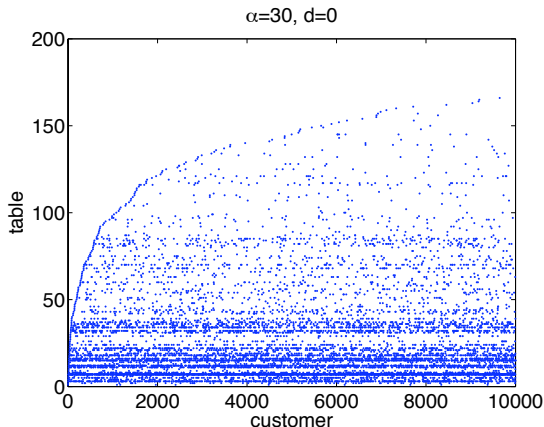
[Blackwell and MacQueen 1973]

# Chinese Restaurant Process

- ▶  $\theta_1, \dots, \theta_n$  take on  $K < n$  distinct values, say  $\theta_1^*, \dots, \theta_K^*$ .
- ▶ This defines a partition of  $(1, \dots, n)$  into  $K$  clusters, such that if  $i$  is in cluster  $k$ , then  $\theta_i = \theta_k^*$ .
- ▶ The distribution over partitions is a *Chinese restaurant process* (CRP).
- ▶ Generating from the CRP:
  - ▶ First customer sits at the first table.
  - ▶ Customer  $n$  sits at:
    - ▶ Table  $k$  with probability  $\frac{n_k}{\alpha + n - 1}$  where  $n_k$  is the number of customers at table  $k$ .
    - ▶ A new table  $K + 1$  with probability  $\frac{\alpha}{\alpha + n - 1}$ .
  - ▶ Customers  $\Leftrightarrow$  integers, tables  $\Leftrightarrow$  clusters.



# Chinese Restaurant Process



- ▶ The CRP exhibits the *clustering property* of the DP.
  - ▶ *Rich-gets-richer* effect implies small number of large clusters.
  - ▶ Expected number of clusters is  $K = O(\alpha \log n)$ .

# Exchangeability

- ▶ Instead of deriving the Pólya urn scheme by marginalizing out a DP, consider starting directly from the conditional distributions:

$$\theta_n | \theta_{1:n-1} \sim \frac{\alpha H + \sum_{i=1}^{n-1} \delta_{\theta_i}}{\alpha + n - 1}$$

- ▶ For any  $n$ , the joint distribution of  $\theta_1, \dots, \theta_n$  is:

$$p(\theta_1, \dots, \theta_n) = \frac{\alpha^K \prod_{k=1}^K h(\theta_k^*) (m_{nk} - 1)!}{\prod_{i=1}^n i - 1 + \alpha}$$

where  $h(\theta)$  is density of  $\theta$  under  $H$ ,  $\theta_1^*, \dots, \theta_K^*$  are the unique values, and  $\theta_k^*$  occurred  $m_{nk}$  times among  $\theta_1, \dots, \theta_n$ .

- ▶ The joint distribution is *exchangeable* wrt permutations of  $\theta_1, \dots, \theta_n$ .
- ▶ *De Finetti's Theorem* says that there must be a random probability measure  $G$  making  $\theta_1, \theta_2, \dots$  iid. This is the DP.

## De Finetti's Theorem

Let  $\theta_1, \theta_2, \dots$  be an infinite sequence of random variables with joint distribution  $p$ . If for all  $n \geq 1$ , and all permutations  $\sigma \in \Sigma_n$  on  $n$  objects,

$$p(\theta_1, \dots, \theta_n) = p(\theta_{\sigma(1)}, \dots, \theta_{\sigma(n)})$$

That is, the sequence is *infinitely exchangeable*. Then there exists a (unique) latent random parameter  $G$  such that:

$$p(\theta_1, \dots, \theta_n) = \int p(G) \prod_{i=1}^n p(\theta_i | G) dG$$

where  $p$  is a joint distribution over  $G$  and  $\theta_i$ 's.

- ▶  $\theta_i$ 's are *independent* given  $G$ .
- ▶ Sufficient to define  $G$  through the conditionals  $p(\theta_n | \theta_1, \dots, \theta_{n-1})$ .
- ▶  $G$  can be *infinite dimensional* (indeed it is often a *random measure*).
- ▶ The set of infinitely exchangeable sequences is convex and it is an important theoretical topic to study the set of extremal points.
- ▶ Partial exchangeability: Markov, group, arrays,...



# Outline

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**Indian Buffet and Beta Processes**

Hierarchical Dirichlet Processes

Pitman-Yor Processes

Summary

# Binary Latent Variable Models

- ▶ Consider a latent variable model with binary sources/features,

$$z_{ik} = \begin{cases} 1 & \text{with probability } \mu_k; \\ 0 & \text{with probability } 1 - \mu_k. \end{cases}$$

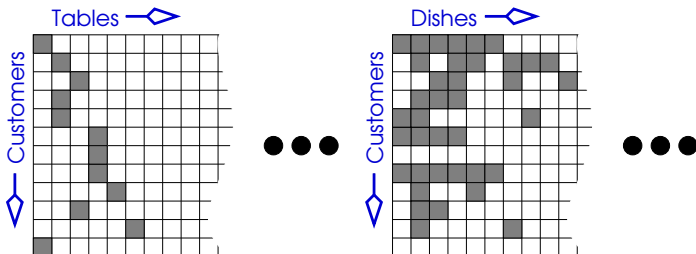
- ▶ Example: Data items could be movies like “Terminator 2”, “Shrek” and “Lord of the Rings”, and features could be “science fiction”, “fantasy”, “action” and “Arnold Schwarzenegger”.
- ▶ Place beta prior over the probabilities of features:

$$\mu_k \sim \text{Beta}\left(\frac{\alpha}{K}, 1\right)$$

- ▶ We will again take  $K \rightarrow \infty$ .

# Indian Buffet Processes

- ▶ The *Indian Buffet Process* (IBP) is akin to the Chinese restaurant process but describes each customer with a binary vector instead of cluster.
- ▶ Generating from an IBP:
  - ▶ Parameter  $\alpha$ .
  - ▶ First customer picks  $\text{Poisson}(\alpha)$  dishes to eat.
  - ▶ Subsequent customer  $i$  picks dish  $k$  with probability  $\frac{m_k}{i}$ ; and picks  $\text{Poisson}(\frac{\alpha}{i})$  new dishes.



# Indian Buffet Processes and Exchangeability

- ▶ The IBP is infinitely exchangeable. For this to make sense, we need to “forget” the ordering of the dishes.
  - ▶ “Name” each dish  $k$  with a  $\Lambda_k^*$  drawn iid from  $H$ .
  - ▶ Each customer now eats a set of dishes:  $\Psi_i = \{\Lambda_k^* : z_{ik} = 1\}$ .
  - ▶ The joint probability of  $\Psi_1, \dots, \Psi_n$  can be calculated:

$$p(\Psi_1, \dots, \Psi_n) = \exp\left(-\alpha \sum_{i=1}^n \frac{1}{i}\right) \alpha^K \prod_{k=1}^K \frac{(m_k - 1)!(n - m_k)!}{n!} h(\Lambda_k^*)$$

$K$ : total number of dishes tried by  $n$  customers.

$\Lambda_k^*$ : Name of  $k$ th dish tried.

$m_k$ : number of customers who tried dish  $\Lambda_k^*$ .

- ▶ De Finetti's Theorem again states that there is some random measure underlying the IBP.
- ▶ This random measure is the beta process.

[Griffiths and Ghahramani 2006, Thibaux and Jordan 2007]

# Beta Processes

- ▶ A *beta process*  $B \sim \text{BP}(c, \alpha H)$  is a random discrete measure with form:

$$B = \sum_{k=1}^{\infty} \mu_k \delta_{\theta_k^*}$$

where the points  $P = \{(\theta_1^*, \mu_1), (\theta_2^*, \mu_2), \dots\}$  are spikes in a 2D Poisson process with rate measure:

$$c\mu^{-1}(1 - \mu)^{c-1} d\mu \alpha H(d\theta)$$

- ▶ The beta process with  $c = 1$  is the de Finetti measure for the IBP. When  $c \neq 1$  we have a two parameter generalization of the IBP.
- ▶ This is an example of a *completely random measure*.
- ▶ A beta process *does not* have Beta distributed marginals.

[Hjort 1990, Ghahramani et al. 2007]

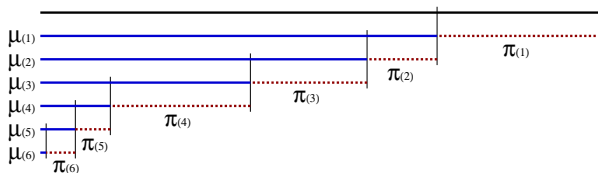
# Stick-breaking Construction for Beta Processes

- ▶ When  $c = 1$  it was shown that the following generates a draw of  $B$ :

$$v_k \sim \text{Beta}(1, \alpha) \quad \mu_k = (1 - v_k) \prod_{i=1}^{k-1} (1 - v_i) \quad \theta_k^* \sim H$$

$$B = \sum_{k=1}^{\infty} \mu_k \delta_{\theta_k^*}$$

- ▶ The above is the complement of the stick-breaking construction for DPs!



# Applications of Indian Buffet Processes

- ▶ The IBP can be used in concert with different likelihood models in a variety of applications.

$$Z \sim \text{IBP}(\alpha)$$

$$X \sim F(Z, Y)$$

$$Y \sim H$$

$$p(Z, Y|X) = \frac{p(Z, Y)p(X|Z, Y)}{p(X)}$$

- ▶ Latent factor models for distributed representation [Griffiths and Ghahramani 2005].
- ▶ Matrix factorization for collaborative filtering [Meeds et al 2007].
- ▶ Latent causal discovery for medical diagnostics [Wood et al 2006].
- ▶ Protein complex discovery [Chu et al 2006].
- ▶ Psychological choice behaviour [Görür and Rasmussen 2006].
- ▶ Independent Components Analysis [Knowles and Ghahramani 2007].

# Infinite Independent Components Analysis

- ▶ Each image  $X_i$  is a linear combination of sparse features:

$$X_i = \sum_k \Lambda_k^* y_{ik}$$

where  $y_{ik}$  is activity of feature  $k$  with sparse prior. One possibility is a mixture of a Gaussian and a point mass at 0:

$$y_{ik} = z_{ik} a_{ik} \quad a_{ik} \sim \mathcal{N}(0, 1) \quad Z \sim \text{IBP}(\alpha)$$

- ▶ An ICA model with infinite number of features.

[Knowles and Ghahramani 2007]



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Summary

# Topic Modelling with Latent Dirichlet Allocation

- ▶ Infer topics from a document corpus, topics being sets of words that tend to co-occur together.
- ▶ Using (Bayesian) latent Dirichlet allocation:

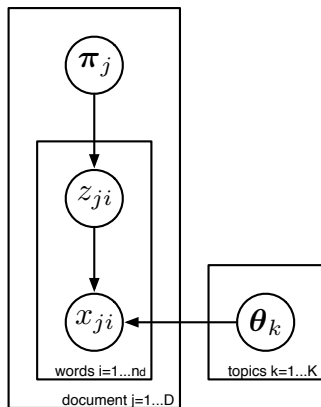
$$\pi_j \sim \text{Dirichlet}\left(\frac{\alpha}{K}, \dots, \frac{\alpha}{K}\right)$$

$$\theta_k \sim \text{Dirichlet}\left(\frac{\beta}{W}, \dots, \frac{\beta}{W}\right)$$

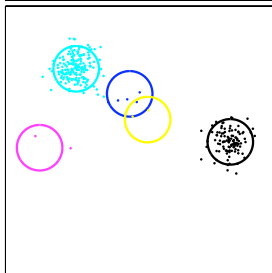
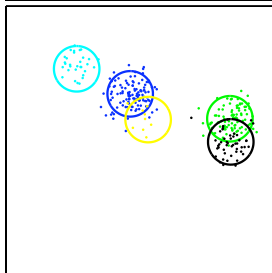
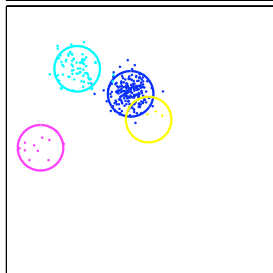
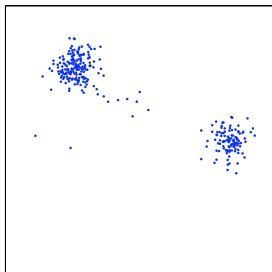
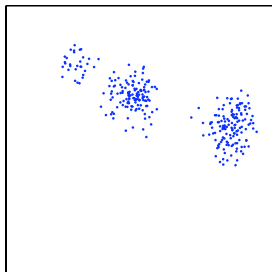
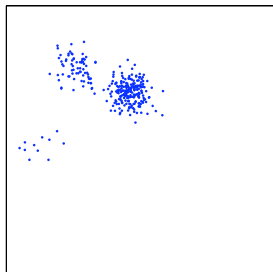
$$z_{ji} | \pi_j \sim \text{Multinomial}(\pi_j)$$

$$x_{ji} | z_{ji}, \theta_{z_{ji}} \sim \text{Multinomial}(\theta_{z_{ji}})$$

- ▶ Can we take  $K \rightarrow \infty$ ?

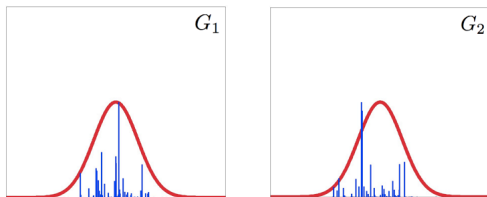


# Hierarchical Dirichlet Processes



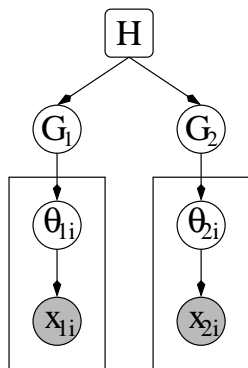
# Hierarchical Dirichlet Processes

- ▶ Use a DP mixture for each group.



- ▶ Unfortunately there is no sharing of clusters across different groups because  $H$  is smooth.
- ▶ Solution: make the base distribution  $H$  discrete.
- ▶ Put a DP prior on the common base distribution.

[Teh et al. 2006]



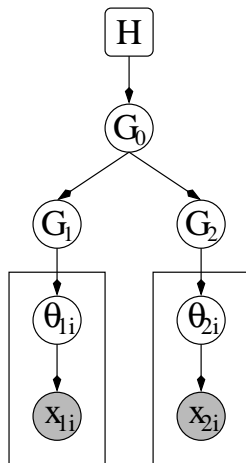
# Hierarchical Dirichlet Processes

- ▶ A hierarchical Dirichlet process:

$$G_0 \sim \text{DP}(\alpha_0, H)$$

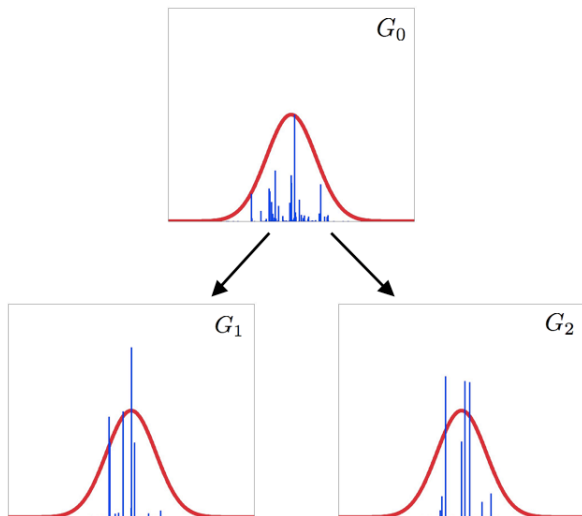
$$G_1, G_2 | G_0 \sim \text{DP}(\alpha, G_0) \text{ iid}$$

- ▶ Extension to larger hierarchies is straightforward.



# Hierarchical Dirichlet Processes

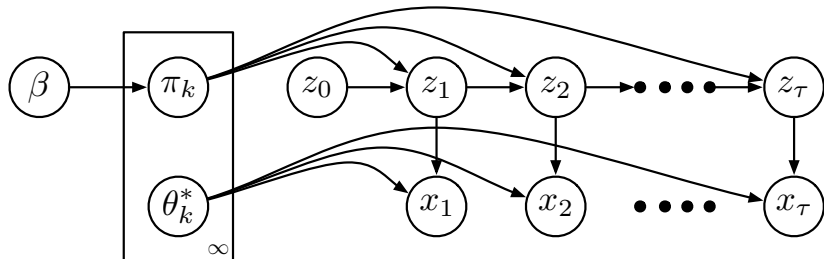
- ▶ Making  $G_0$  discrete forces shared cluster between  $G_1$  and  $G_2$ .



# Hierarchical Dirichlet Processes

- ▶ Document topic modelling:
  - ▶ Allows documents to be modelled with DP mixtures of topics, with topics shared across corpora.
- ▶ Infinite hidden Markov modelling:
  - ▶ Allows HMMs with an infinite number of states, with transitions from each allowable state to every other allowable state.
- ▶ Learning discrete structures from data:
  - ▶ Determining number of objects, nonterminals, states etc.

# Infinite Hidden Markov Models



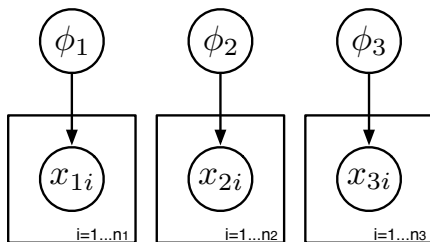
$$\beta \sim \text{GEM}(\gamma) \quad \pi_k | \beta \sim \text{DP}(\alpha, \beta) \quad z_i | z_{i-1}, \pi_{z_{i-1}} \sim \text{Multinomial}(\pi_{z_{i-1}})$$
$$\theta_k^* \sim H \quad x_i | z_i, \theta_{z_i}^* \sim F(\theta_{z_i}^*)$$

- ▶ Hidden Markov models with an infinite number of states.
- ▶ Hierarchical DPs used to share information among transition probability vectors prevents “run-away” states.

[Beal et al. 2002, Teh et al. 2006]

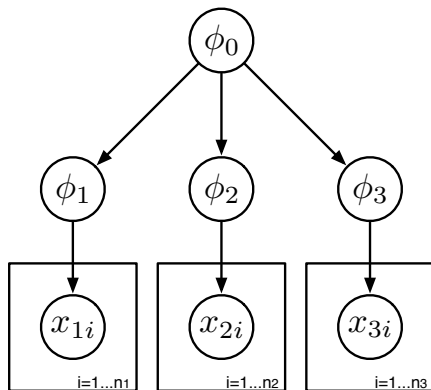


# Hierarchical Modelling



- ▶ Better estimation of parameters.
- ▶ Multitask learning, learning to learn: generalizing across related tasks.

# Hierarchical Modelling



- ▶ Better estimation of parameters.
- ▶ Multitask learning, learning to learn: generalizing across related tasks.

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Summary

# Pitman-Yor Processes

- ▶ Two-parameter generalization of the Chinese restaurant process:

$$p(\text{customer } n \text{ sat at table } k | \text{past}) = \begin{cases} \frac{n_k - \beta}{n - 1 + \alpha} & \text{if occupied table} \\ \frac{\alpha + \beta K}{n - 1 + \alpha} & \text{if new table} \end{cases}$$

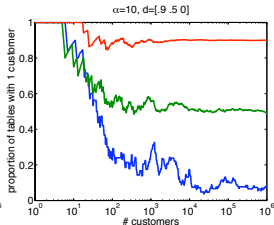
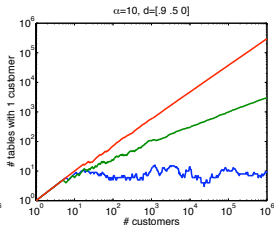
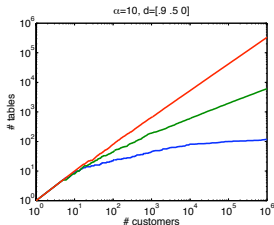
- ▶ Associating each cluster  $k$  with a unique draw  $\theta_k^* \sim H$ , the corresponding Pólya urn scheme is also exchangeable.
- ▶ De Finetti's Theorem states that there is a random measure underlying this two-parameter generalization.
  - ▶ This is the *Pitman-Yor process*.
- ▶ The Pitman-Yor process also has a stick-breaking construction:

$$\pi_k = v_k \prod_{i=1}^{k-1} (1 - v_i) \quad \beta_k \sim \text{Beta}(1 - \beta, \alpha + \beta k) \quad \theta_k^* \sim H \quad G = \sum_{k=1}^{\infty} \pi_k \delta_{\theta_k^*}$$

[Pitman and Yor 1997, Perman et al. 1992]

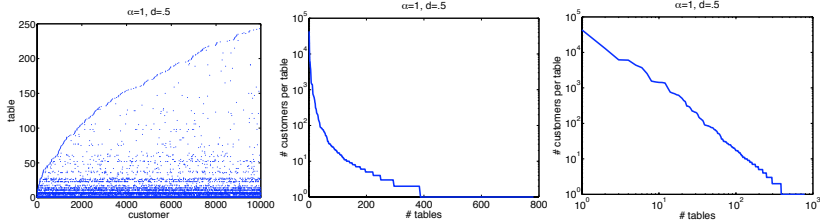
# Pitman-Yor Processes

- ▶ Two salient features of the Pitman-Yor process:
  - ▶ With more occupied tables, the chance of even more tables becomes higher.
  - ▶ Tables with smaller occupancy numbers tend to have lower chance of getting new customers.
- ▶ The above means that Pitman-Yor processes produce Zipf's Law type behaviour, with  $K = O(\alpha n^\beta)$ .

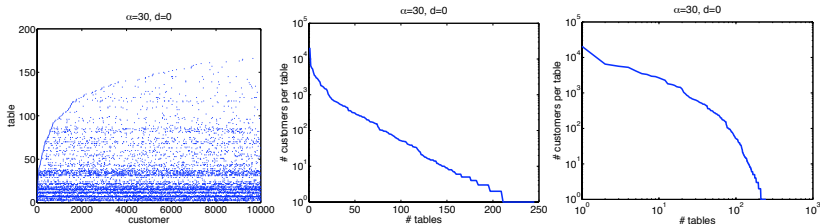


# Pitman-Yor Processes

## Draw from a Pitman-Yor process



## Draw from a Dirichlet process



# Hierarchical Pitman-Yor Language Models

- ▶ Pitman-Yor processes can be suitable models for many natural phenomena with power-law statistics.
- ▶ Language modelling with Markov assumption:

$$\begin{aligned} & p(\text{Mary has a little lamb}) \\ & \approx p(\text{Mary})p(\text{has}|\text{Mary})p(\text{a}|\text{Mary has})p(\text{little}|\text{has a})p(\text{lamb}|\text{a little}) \end{aligned}$$

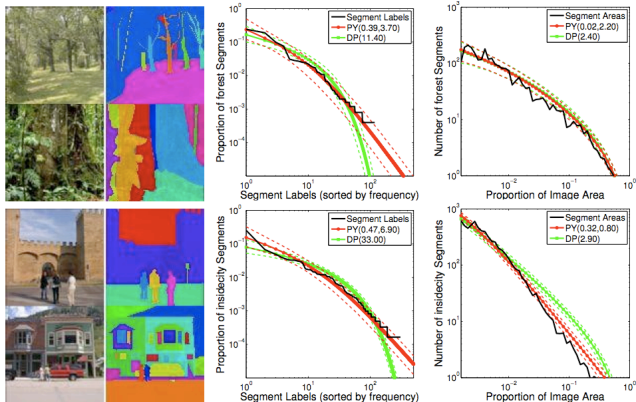
- ▶ Parameterize with  $p(w_3|w_1, w_2) = G_{w_1, w_2}[w_3]$  and use a hierarchical Pitman-Yor process prior:

$$\begin{aligned} G_{w_1, w_2} | G_{w_2} & \sim \text{PY}(\alpha_2, \beta_2, G_{w_2}) \\ G_{w_2} | G_{\emptyset} & \sim \text{PY}(\alpha_1, \beta_1, G_{\emptyset}) \\ G_{\emptyset} | U & \sim \text{PY}(\alpha_0, \beta_0, U) \end{aligned}$$

- ▶ State-of-the-art results, connection to Kneser-Ney smoothing.

[Goldwater et al. 2006a, Teh 2006b, Wood et al. 2009]

# Image Segmentation with Pitman-Yor Processes



- ▶ Human segmentations of images also seem to follow power-law.
- ▶ An unsupervised image segmentation model based on dependent hierarchical Pitman-Yor processes achieves state-of-the-art results.

[Sudderth and Jordan 2009]



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Summary

# Summary

- ▶ Motivation for Bayesian nonparametrics:
  - ▶ Allows practitioners to define and work with models with large support, sidesteps model selection.
  - ▶ New models with useful properties.
  - ▶ Large variety of applications.
- ▶ Various standard Bayesian nonparametric models:
  - ▶ Dirichlet processes
  - ▶ Hierarchical Dirichlet processes
  - ▶ Infinite hidden Markov models
  - ▶ Indian buffet and beta processes
  - ▶ Pitman-Yor processes
- ▶ Touched upon two important theoretical tools:
  - ▶ Consistency and Kolmogorov's Consistency Theorem
  - ▶ Exchangeability and de Finetti's Theorem
- ▶ Described a number of applications of Bayesian nonparametrics.
- ▶ Missing: Inference methods based on MCMC, variational etc, consistency and convergence.

## Other Introductions to Bayesian Nonparametrics

- ▶ Zoubin Ghahramani, UAI 2005 Tutorial.
- ▶ Michael Jordan, NIPS 2005 Tutorial.
- ▶ Volker Tresp, ICML nonparametric Bayes workshop 2006.
- ▶ Peter Orbanz, Foundations of Nonparametric Bayesian Methods, 2009.
- ▶ I have given a number myself (check webpage).
- ▶ I have an introduction to Dirichlet processes [Teh 2007], and another to hierarchical Bayesian nonparametric models [Teh and Jordan 2010].

# Bayesian Nonparametric Software

- ▶ Hierarchical Bayesian Compiler (HBC). Hal Daume III.  
<http://www.cs.utah.edu/hal/HBC/>
- ▶ DPpackage. Alejandro Jara.  
<http://cran.r-project.org/web/packages/DPpackage/index.html>
- ▶ Hierarchical Pitman Yor Language Model. Songfang Huang.  
<http://homepages.inf.ed.ac.uk/s0562315/progs/index.html>
- ▶ Nonparametric Bayesian Mixture Models. Yee Whye Teh.  
<http://www.gatsby.ucl.ac.uk/ywteh/research/software.html>
- ▶ Others...

# AISTATS 2010



May 13-15, 2010. Chia Laguna Resort, Sardinia, Italy

<http://www.aistats.org>

Deadline: November 6, 2009

# Outline

Relating Different Representations of Dirichlet Processes

Representations of Hierarchical Dirichlet Processes

Extended Bibliography

# Representations of Dirichlet Processes

- ▶ Posterior Dirichlet process:

$$\begin{array}{l} G \sim \text{DP}(\alpha, H) \\ \theta | G \sim G \end{array} \iff \begin{array}{l} \theta \sim H \\ G | \theta \sim \text{DP} \left( \alpha + 1, \frac{\alpha H + \delta_\theta}{\alpha + 1} \right) \end{array}$$

- ▶ Pólya urn scheme:

$$\theta_n | \theta_{1:n-1} \sim \frac{\alpha H + \sum_{i=1}^{n-1} \delta_{\theta_i}}{\alpha + n - 1}$$

- ▶ Chinese restaurant process:

$$p(\text{customer } n \text{ sat at table } k | \text{past}) = \begin{cases} \frac{n_k}{n-1+\alpha} & \text{if occupied table} \\ \frac{\alpha}{n-1+\alpha} & \text{if new table} \end{cases}$$

- ▶ Stick-breaking construction:

$$\pi_k = \beta_k \prod_{i=1}^{k-1} (1 - \beta_i) \quad \beta_k \sim \text{Beta}(1, \alpha) \quad \theta_k^* \sim H \quad G = \sum_{k=1}^{\infty} \pi_k \delta_{\theta_k^*}$$

# Posterior Dirichlet Processes

- ▶ Suppose  $G$  is DP distributed, and  $\theta$  is  $G$  distributed:

$$G \sim \text{DP}(\alpha, H)$$

$$\theta|G \sim G$$

- ▶ We are interested in:
  - ▶ The marginal distribution of  $\theta$  with  $G$  integrated out.
  - ▶ The posterior distribution of  $G$  conditioning on  $\theta$ .



# Posterior Dirichlet Processes

Conjugacy between Dirichlet Distribution and Multinomial.

- ▶ Consider:

$$(\pi_1, \dots, \pi_K) \sim \text{Dirichlet}(\alpha_1, \dots, \alpha_K)$$

$$z | (\pi_1, \dots, \pi_K) \sim \text{Discrete}(\pi_1, \dots, \pi_K)$$

$z$  is a multinomial variate, taking on value  $i \in \{1, \dots, n\}$  with probability  $\pi_i$ .

- ▶ Then:

$$z \sim \text{Discrete} \left( \frac{\alpha_1}{\sum_i \alpha_i}, \dots, \frac{\alpha_K}{\sum_i \alpha_i} \right)$$

$$(\pi_1, \dots, \pi_K) | z \sim \text{Dirichlet}(\alpha_1 + \delta_1(z), \dots, \alpha_K + \delta_K(z))$$

where  $\delta_j(z) = 1$  if  $z$  takes on value  $i$ , 0 otherwise.

- ▶ Converse also true.

# Posterior Dirichlet Processes

- ▶ Fix a partition  $(A_1, \dots, A_K)$  of  $\Theta$ . Then

$$(G(A_1), \dots, G(A_K)) \sim \text{Dirichlet}(\alpha H(A_1), \dots, \alpha H(A_K))$$

$$P(\theta \in A_i | G) = G(A_i)$$

- ▶ Using Dirichlet-multinomial conjugacy,

$$P(\theta \in A_i) = H(A_i)$$

$$(G(A_1), \dots, G(A_K)) | \theta \sim \text{Dirichlet}(\alpha H(A_1) + \delta_\theta(A_1), \dots, \alpha H(A_K) + \delta_\theta(A_K))$$

- ▶ The above is true for every finite partition of  $\Theta$ . In particular, taking a really fine partition,

$$p(d\theta) = H(d\theta)$$

i.e.  $\theta \sim H$  with  $G$  integrated out.

- ▶ Also, the posterior  $G | \theta$  is also a Dirichlet process:

$$G | \theta \sim \text{DP} \left( \alpha + 1, \frac{\alpha H + \delta_\theta}{\alpha + 1} \right)$$

# Posterior Dirichlet Processes

$$\begin{array}{l} G \sim \text{DP}(\alpha, H) \\ \theta | G \sim G \end{array} \iff \begin{array}{l} \theta \sim H \\ G | \theta \sim \text{DP}\left(\alpha + 1, \frac{\alpha H + \delta_\theta}{\alpha + 1}\right) \end{array}$$

# Pólya Urn Scheme

- ▶ First sample:

$$\begin{aligned} \theta_1 | G &\sim G & G &\sim \text{DP}(\alpha, H) \\ \iff \theta_1 &\sim H & G | \theta_1 &\sim \text{DP}(\alpha + 1, \frac{\alpha H + \delta_{\theta_1}}{\alpha + 1}) \end{aligned}$$

- ▶ Second sample:

$$\begin{aligned} \theta_2 | \theta_1, G &\sim G & G | \theta_1 &\sim \text{DP}(\alpha + 1, \frac{\alpha H + \delta_{\theta_1}}{\alpha + 1}) \\ \iff \theta_2 | \theta_1 &\sim \frac{\alpha H + \delta_{\theta_1}}{\alpha + 1} & G | \theta_1, \theta_2 &\sim \text{DP}(\alpha + 2, \frac{\alpha H + \delta_{\theta_1} + \delta_{\theta_2}}{\alpha + 2}) \end{aligned}$$

- ▶  $n^{\text{th}}$  sample

$$\begin{aligned} \theta_n | \theta_{1:n-1}, G &\sim G & G | \theta_{1:n-1} &\sim \text{DP}(\alpha + n - 1, \frac{\alpha H + \sum_{i=1}^{n-1} \delta_{\theta_i}}{\alpha + n - 1}) \\ \iff \theta_n | \theta_{1:n-1} &\sim \frac{\alpha H + \sum_{i=1}^{n-1} \delta_{\theta_i}}{\alpha + n - 1} & G | \theta_{1:n} &\sim \text{DP}(\alpha + n, \frac{\alpha H + \sum_{i=1}^n \delta_{\theta_i}}{\alpha + n}) \end{aligned}$$

# Stick-breaking Construction

- ▶ Returning to the posterior process:

$$\begin{array}{l} G \sim \text{DP}(\alpha, H) \\ \theta | G \sim G \end{array} \quad \Leftrightarrow \quad \begin{array}{l} \theta \sim H \\ G | \theta \sim \text{DP}(\alpha + 1, \frac{\alpha H + \delta_\theta}{\alpha + 1}) \end{array}$$

- ▶ Consider a partition  $(\theta, \Theta \setminus \theta)$  of  $\Theta$ . We have:

$$\begin{aligned} (G(\theta), G(\Theta \setminus \theta)) | \theta &\sim \text{Dirichlet}((\alpha + 1) \frac{\alpha H + \delta_\theta}{\alpha + 1}(\theta), (\alpha + 1) \frac{\alpha H + \delta_\theta}{\alpha + 1}(\Theta \setminus \theta)) \\ &= \text{Dirichlet}(1, \alpha) \end{aligned}$$

- ▶  $G$  has a point mass located at  $\theta$ :

$$G = \beta \delta_\theta + (1 - \beta) G' \quad \text{with} \quad \beta \sim \text{Beta}(1, \alpha)$$

and  $G'$  is the (renormalized) probability measure with the point mass removed.

- ▶ What is  $G'$ ?

# Stick-breaking Construction

- ▶ Currently, we have:

$$\begin{array}{l} G \sim \text{DP}(\alpha, H) \\ \theta \sim G \end{array} \Rightarrow \begin{array}{l} \theta \sim H \\ G|\theta \sim \text{DP}(\alpha + 1, \frac{\alpha H + \delta_\theta}{\alpha + 1}) \\ G = \beta \delta_\theta + (1 - \beta)G' \\ \beta \sim \text{Beta}(1, \alpha) \end{array}$$

- ▶ Consider a further partition  $(\theta, A_1, \dots, A_K)$  of  $\Theta$ :

$$\begin{aligned} & (G(\theta), G(A_1), \dots, G(A_K)) \\ &= (\beta, (1 - \beta)G'(A_1), \dots, (1 - \beta)G'(A_K)) \\ &\sim \text{Dirichlet}(1, \alpha H(A_1), \dots, \alpha H(A_K)) \end{aligned}$$

- ▶ The agglomerative/decimative property of Dirichlet implies:

$$\begin{aligned} (G'(A_1), \dots, G'(A_K))|\theta &\sim \text{Dirichlet}(\alpha H(A_1), \dots, \alpha H(A_K)) \\ G' &\sim \text{DP}(\alpha, H) \end{aligned}$$

# Stick-breaking Construction

► We have:

$$G \sim \text{DP}(\alpha, H)$$

$$G = \beta_1 \delta_{\theta_1^*} + (1 - \beta_1) G_1$$

$$G = \beta_1 \delta_{\theta_1^*} + (1 - \beta_1)(\beta_2 \delta_{\theta_2^*} + (1 - \beta_2) G_2)$$

⋮

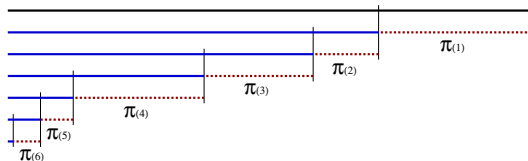
$$G = \sum_{k=1}^{\infty} \pi_k \delta_{\theta_k^*}$$

where

$$\pi_k = \beta_k \prod_{i=1}^{k-1} (1 - \beta_i)$$

$$\beta_k \sim \text{Beta}(1, \alpha)$$

$$\theta_k^* \sim H$$



# Outline

Relating Different Representations of Dirichlet Processes

**Representations of Hierarchical Dirichlet Processes**

Extended Bibliography



# Stick-breaking Construction

- ▶ We shall assume the following HDP hierarchy:

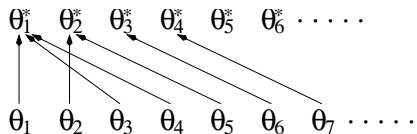
$$\begin{aligned}G_0 &\sim \text{DP}(\gamma, H) \\G_j | G_0 &\sim \text{DP}(\alpha, G_0) \quad \text{for } j = 1, \dots, J\end{aligned}$$

- ▶ The stick-breaking construction for the HDP is:

$$\begin{aligned}G_0 &= \sum_{k=1}^{\infty} \pi_{0k} \delta_{\theta_k^*} & \theta_k^* &\sim H \\ \pi_{0k} &= \beta_{0k} \prod_{l=1}^{k-1} (1 - \beta_{0l}) & \beta_{0k} &\sim \text{Beta}(1, \gamma) \\ G_j &= \sum_{k=1}^{\infty} \pi_{jk} \delta_{\theta_k^*} \\ \pi_{jk} &= \beta_{jk} \prod_{l=1}^{k-1} (1 - \beta_{jl}) & \beta_{jk} &\sim \text{Beta}(\alpha \beta_{0k}, \alpha(1 - \sum_{l=1}^k \beta_{0l}))\end{aligned}$$

# Hierarchical Pòlya Urn Scheme

- ▶ Let  $G \sim DP(\alpha, H)$ .
- ▶ We can visualize the Pòlya urn scheme as follows:



where the arrows denote to which  $\theta_k^*$  each  $\theta_i$  was assigned and

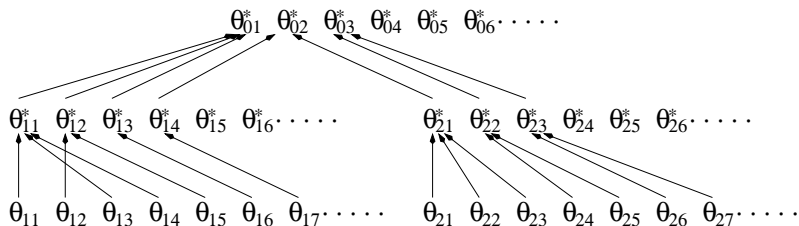
$$\theta_1, \theta_2, \dots \sim G \text{ i.i.d.}$$

$$\theta_1^*, \theta_2^*, \dots \sim H \text{ i.i.d.}$$

(but  $\theta_1, \theta_2, \dots$  are not independent of  $\theta_1^*, \theta_2^*, \dots$ ).

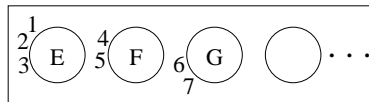
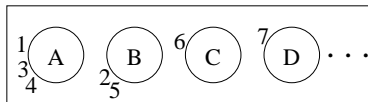
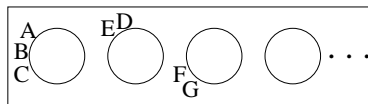
# Hierarchical Pòlya Urn Scheme

- ▶ Let  $G_0 \sim DP(\gamma, H)$  and  $G_1, G_2 | G_0 \sim DP(\alpha, G_0)$ .
- ▶ The hierarchical Pòlya urn scheme to generate draws from  $G_1, G_2$ :



# Chinese Restaurant Franchise

- ▶ Let  $G_0 \sim DP(\gamma, H)$  and  $G_1, G_2 | G_0 \sim DP(\alpha, G_0)$ .
- ▶ The Chinese restaurant franchise describes the clustering of data items in the hierarchy:



# Outline

Relating Different Representations of Dirichlet Processes

Representations of Hierarchical Dirichlet Processes

Extended Bibliography

# Bibliography I

## Dirichlet Processes and Beyond in Machine Learning

**Dirichlet Processes** were first introduced by [Ferguson 1973], while [Antoniak 1974] further developed DPs as well as introduced the mixture of DPs. [Blackwell and MacQueen 1973] showed that the Pólya urn scheme is exchangeable with the DP being its de Finetti measure. Further information on the Chinese restaurant process can be obtained at [Aldous 1985, Pitman 2002]. The DP is also related to Ewens' Sampling Formula [Ewens 1972]. [Sethuraman 1994] gave a constructive definition of the DP via a stick-breaking construction. DPs were rediscovered in the machine learning community by [Neal 1992, Rasmussen 2000].

**Hierarchical Dirichlet Processes** (HDPs) were first developed by [Teh et al. 2006], although an aspect of the model was first discussed in the context of infinite hidden Markov models [Beal et al. 2002]. HDPs and generalizations have been applied across a wide variety of fields.

**Dependent Dirichlet Processes** are sets of coupled distributions over probability measures, each of which is marginally DP [MacEachern et al. 2001]. A variety of dependent DPs have been proposed in the literature since then [Srebro and Roweis 2005, Griffin 2007, Caron et al. 2007]. The infinite mixture of Gaussian processes of [Rasmussen and Ghahramani 2002] can also be interpreted as a dependent DP.

**Indian Buffet Processes** (IBPs) were first proposed in [Griffiths and Ghahramani 2006], and extended to a two-parameter family in [Ghahramani et al. 2007]. [Thibaux and Jordan 2007] showed that the de Finetti measure for the IBP is the beta process of [Hjort 1990], while [Teh et al. 2007] gave a stick-breaking construction and developed efficient slice sampling inference algorithms for the IBP.

**Nonparametric Tree Models** are models that use distributions over trees that are consistent and exchangeable. [Blei et al. 2004] used a nested CRP to define distributions over trees with a finite number of levels. [Neal 2001, Neal 2003] defined Dirichlet diffusion trees, which are binary trees produced by a fragmentation process. [Teh et al. 2008] used Kingman's coalescent [Kingman 1982b, Kingman 1982a] to produce random binary trees using a coalescent process. [Roy et al. 2007] proposed annotated hierarchies, using tree-consistent partitions first defined in [Heller and Ghahramani 2005] to model both relational and featural data.

**Markov chain Monte Carlo Inference** algorithms are the dominant approaches to inference in DP mixtures. [Neal 2000] is a good review of algorithms based on Gibbs sampling in the CRP representation. Algorithm 8 in [Neal 2000] is still one of the best algorithms based on simple local moves. [Ishwaran and James 2001] proposed blocked Gibbs sampling in the stick-breaking representation instead due to the simplicity in implementation. This has been further explored in [Porteous et al. 2006]. Since then there has been proposals for better MCMC samplers based on proposing larger moves in a Metropolis-Hastings framework [Jain and Neal 2004, Liang et al. 2007a], as well as sequential Monte Carlo [Fearhead 2004, Mansinghka et al. 2007].

**Other Approximate Inference Methods** have also been proposed for DP mixture models. [Blei and Jordan 2006] is the first variational Bayesian approximation, and is based on a truncated stick-breaking representation. [Kurihara et al. 2007] proposed an

# Bibliography II

## Dirichlet Processes and Beyond in Machine Learning

improved VB approximation based on a better truncation technique, and using KD-trees for extremely efficient inference in large scale applications. [Kurihara et al. 2007] studied improved VB approximations based on integrating out the stick-breaking weights. [Minka and Ghahramani 2003] derived an expectation propagation based algorithm. [Heller and Ghahramani 2005] derived tree-based approximation which can be seen as a Bayesian hierarchical clustering algorithm. [Daume III 2007] developed admissible search heuristics to find MAP clusterings in a DP mixture model.

**Computer Vision and Image Processing.** HDPs have been used in object tracking

[Fox et al. 2006, Fox et al. 2007b, Fox et al. 2007a]. An extension called the transformed Dirichlet process has been used in scene analysis [Sudderth et al. 2006b, Sudderth et al. 2006a, Sudderth et al. 2008], a related extension has been used in fMRI image analysis [Kim and Smyth 2007, Kim 2007]. An extension of the infinite hidden Markov model called the nonparametric hidden Markov tree has been introduced and applied to image denoising [Kivinen et al. 2007a, Kivinen et al. 2007b].

**Natural Language Processing.** HDPs are essential ingredients in defining nonparametric context free grammars

[Liang et al. 2007b, Finkel et al. 2007]. [Johnson et al. 2007] defined adaptor grammars, which is a framework generalizing both probabilistic context free grammars as well as a variety of nonparametric models including DPs and HDPs. DPs and HDPs have been used in information retrieval [Cowans 2004], word segmentation [Goldwater et al. 2006b], word morphology modelling [Goldwater et al. 2006a], coreference resolution [Haghighi and Klein 2007], topic modelling [Blei et al. 2004, Teh et al. 2006, Li et al. 2007]. An extension of the HDP called the hierarchical Pitman-Yor process has been applied to language modelling [Teh 2006a, Teh 2006b, Goldwater et al. 2006a]. [Savova et al. 2007] used annotated hierarchies to construct syntactic hierarchies. These on nonparametric methods in NLP include [Cowans 2006, Goldwater 2006].

**Other Applications.** Applications of DPs, HDPs and infinite HMMs in bioinformatics include

[Xing et al. 2004, Xing et al. 2007, Xing et al. 2006, Xing and Sohn 2007a, Xing and Sohn 2007b]. DPs have been applied in relational learning [Shafto et al. 2006, Kemp et al. 2006, Xu et al. 2006], spike sorting [Wood et al. 2006a, Görür 2007]. The HDP has been used in a cognitive model of categorization [Griffiths et al. 2007]. IBPs have been applied to infer hidden causes [Wood et al. 2006b], in a choice model [Görür et al. 2006], to modelling dyadic data [Meeds et al. 2007], to overlapping clustering [Heller and Ghahramani 2007], and to matrix factorization [Wood and Griffiths 2006].

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