Analyzing the Joint Dynamics of Several Networks

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Multiple Networks

Social actors are embedded in multiple networks

friendship, esteem, collaboration, advice, enmity, ...

friendship, bullying, defending, dislike, ...

collaborative projects, client referral, information sharing, ...

When studying network dynamics,
studying between-network dependencies can be illuminating.

A multiple or multivariate social network is a set of n social actors, on which R relations are defined (Wasserman & Faust, 1994; Pattison & Wasserman, 1999).

The study of multiple networks is quite traditional:
e.g., White, Boorman & Breiger (1976);
Boorman & White (1976); Pattison (1993);
later on, authors including Ibarra, Krackhardt, Padgett,
Lazega, Lomi, did empirical research on multiple networks.

... on the variety of how relations can affect relations ...

(cf. also the algebraic approach;
e.g., work by Pattison & Breiger.)

It's a multilevel issue (but not nested):

ties, dyads, actors, triads, subgroups, ...

Different relations can impinge on one another in many different ways.

Example: friendship ⇒ advice asking; ego is ⊗.

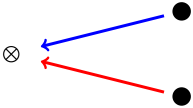
In the first place, within-dyad.

direct association (within tie)
‘friends become advisors’

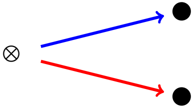
mixed reciprocity
‘friendship reciprocated by asking advice’

A second category operates via actors.

mixed popularity
‘those popular as friends are asked a lot for advice’

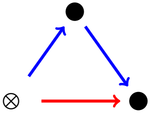


mixed activity
‘those mentioning many friends also mention many advisors’

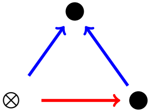


Next category: triads.

mixed transitive closure
'friends of friends
become advisors'

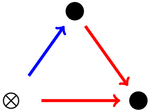


agreement
'those with the same friends
become advisors'



More triads.

other mixed transitive closure
'advisors of friends
become advisors'



Actor orientation: only the bottom tie is the dependent variable.

And there are more mixed triads.

This type of cross-network dependencies is discussed for cross-sectional observations in Wasserman & Pattison (1999), with examples in Lazega & Pattison (1999).

For longitudinal observations the dependencies are multiplied, because we must distinguish between the dependent and the explanatory (antecedent – subsequent) relations.

This can also be applied to *signed graphs* in which case balance theory can be applied.

In addition, the actors in the network can be *affiliated* with various groupings or events:

this can be represented by *two-mode* ('bipartite') networks, where there are
a set $\mathcal{N}$ of actors (the 'actor mode') and
a set $\mathcal{M}$ of groupings (the 'group mode');
and the tie $i \to j$ for $i \in \mathcal{N}, j \in \mathcal{M}$
means that i is a member of grouping j.

For the combination of a one-mode and a two-mode network, other mutual influences between the networks are possible.

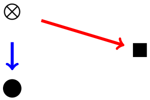
skip bipartite influence models

dependencies in bipartite networks

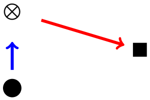
Within-dyad dependencies are undefined.

Actor-level dependencies are meaningful.

mixed activity



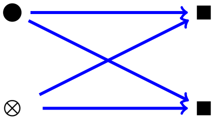
mixed popularity
$\Rightarrow$ activity



Transitivity for bipartite networks: 4-cycles

An interlude:
for bipartite networks, other structures are important
than for one-mode networks.

We meet each other
in various groups.

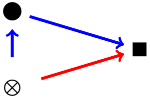


Robins and Alexander (2004):
transitivity in bipartite networks expressed by 4-cycles.

Closed triads are impossible in bipartite networks;
but they are possible as mixed patterns.

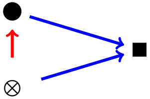
One-with-two-mode triads.

One-mode tie ⇒
two-mode agreement
'I go to places
where my friends are'



Two-mode agreement ⇒
one-mode tie

'Those who go to the same places
become friends'



... outline of further presentation ...

1. specify statistical model:
   actor-based model for multiple networks;
2. sketch procedure for parameter estimation;
3. example.

Actor-based models

Actor-based models are defined here
as extensions of actor-based models
for dynamics of single networks (Snijders 2001, 2017).

1. The actors control their outgoing ties.
2. For panel data: employ a continuous-time model to represent unobserved endogenous network evolution.
3. The ties have inertia: they are *states* rather than *events*.
4. The multiple relations together develop stochastically according to a Markov process.
5. At any single moment in time, only one tie variable may change: no coordination.

6. Changes in each network are modeled as choices by actors in their outgoing ties, with probabilities depending on '*objective functions*' of the network state that would obtain after this change.
7. The process is a **co-evolution** of the multiple networks: the networks develop simultaneously, transitions in each are influenced by all the others.

The objective ('goal') functions, determining the evolution,
are specified separately for each of the R dependent networks.

Notation

Denote tie variable for r^{th} relation from i to j by

$$X_{ij}^{(r)} = \begin{cases} 1 & \text{if} \quad i \stackrel{r}{\to} j \\ 0 & \text{if} \quad \text{not } i \stackrel{r}{\to} j, \end{cases}$$

where this depends on time t.

By X we denote the collection of all R relations:
array $\left(X_{ij}^{(r)}\right)$ for $r = 1, \ldots, R;\ i = 1, \ldots, n;\ j = 1, \ldots, n$

The statistical model is a *process model*:

an agent-based simulation model,
which simulates the development of the multiple networks
from one observation to the next;

statistical modeling consists of fitting such a simulation model
to the observed network data, and testing
which model components are required to give a good fit.

The model is defined by its smallest possible steps,
the 'microsteps', which consist of a change in one tie variable:

extend one new tie / withdraw one existing tie.

offon $\otimes \longrightarrow \bullet$

$\Rightarrow$How rapidly does this happen?
$\Rightarrow$What is the probability of this particular tie change,
compared to other changes?

Decompose model in :

the average *frequency* of changes,
rate functions :
$\lambda_i^{(r)}(x)$ = rate at which i can change r-relations;

and the *probabilities* of particular changes,
objective functions $f_i^{(r)}$:
changes in r-relations have higher probabilities
accordingly as $f_i^{(r)}(x)$ would become higher,
$\sim$ myopic optimization of $f_i^{(r)}(x) +$ error term.

Model for rate of change often can be simple:
rate of change $\lambda_i^{(r)}(x)$ depends only on r,
some relations change faster than others.

Rate of change of relation r is $\lambda_+^{(r)} = \sum_i \lambda_i^{(r)}$;

total rate of change is $\lambda_+^{(+)} = \sum_r \lambda_+^{(r)}$.

Outline of model dynamics / simulation algorithm

Model for microstep (smallest possible change):

1. Next event takes place after time interval
   with exponentially distributed length, average duration $\lambda_+^{(+)}$.
   Step: Increment t by such a random variable.
2. The probability that this is an event where
   actor i may change an r-tie is

$$\frac{\lambda_i^{(r)}}{\lambda_+^{(+)}} \ .$$

   Step: Choose r, i with this probability.

Outline of algorithm – continued

3. For this r and i, actor i may change one outgoing r-tie, or leave all outgoing tie variables $X_{ij}^{(r)}$ unchanged.
   The probability of changing toward any new situation x (x differs only in one tie variable from current situation!) is proportional to

$$\exp\left(f_i^{(r)}(x)\right) .$$

Step: Given that actor i may change a tie in relation r, the event that tie variable $X_{ij}^{(r)}$ is toggled ($X_{ij}^{(r)} \Rightarrow 1 - X_{ij}^{(r)}$) has probability

$$\frac{\exp\left(f_i^{(r)}(x \text{ changed in } x_{ij}^{(r)})\right)}{\sum_h \exp\left(f_i^{(r)}(x \text{ changed in } x_{ih}^{(r)})\right)} .$$

Model specification

The objective function can be conveniently modeled as a weighted sum (cf. generalized linear modeling),

$$f_i^{(r)}(\beta, x) = \sum_{k=1}^{L} \beta_k^{(r)} \, s_{ik}^{(r)}(x) \, ,$$

where $s_{ik}^{(r)}(x)$ are 'effects' and $\beta_k^{(r)}$ their weights,
which jointly drive the dynamics for relation r,
given the current state of this *and all other* relations.

These effects will represent the 'internal' dynamics
of the network, as dependent on its own current state
and on exogenous variables ('covariates');

and, for multiple dependent networks,
also the cross-network dependencies.

Testable hypotheses and 'control mechanisms' are represented
by the choice of the effects $s_{ik}^{(r)}(x)$.

Within-network dependencies and covariate effects have been discussed extensively elsewhere.

A few examples of cross-network dependencies are presented, with formulae for $s_{ik}^{(\mathrm{red})}(x)$.

Since this a component of the objective function for $X^{(\mathrm{red})}$, this network is the dependent relation
– all others have an explanatory role.

direct association

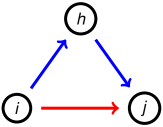
$$\sum_{j=1}^{n} x^{(\mathrm{blue})}(i,j)\, x^{(\mathrm{red})}(i,j)$$

mixed reciprocity

$$\sum_{j=1}^{n} x^{(\mathrm{blue})}(j,i)\, x^{(\mathrm{red})}(i,j)$$

mixed transitive closure

$$\sum_{j,h=1}^{n} x^{(\text{blue})}(i,h)\, x^{(\text{blue})}(h,j)\, x^{(\text{red})}(i,j)$$



Other formulae also are defined by mixed expressions incorporating one network in the 'dependent' and the others in 'explanatory' roles.

Estimation

Assume that $\left(X_{ij}^{(r)}\right)$ is observed for time points $t_1, \ldots, t_M$:
panel data (repeated measures) on multiple networks.

The estimation conditions on $X(t_1)$:
model tendencies of change, not initial state.

Estimation methods:
ML / Bayesian / Method of Moments.

All are computationally intensive MCMC methods.

Method of Moments is computationally faster
and quite efficient.

The Method of Moments operates
by equating observed statistics
to their expected values given the parameter values.
For each parameter there must be a statistic
that is sensitive to this parameter.

Consider the case of $M = 2$ observations;
the estimation conditions on $X(t_1)$.

If, for a given dependent network $X^{(r)}$, with objective function

$$f_i^{(r)}(\beta, x) = \sum_{k=1}^{L} \beta_k^{(r)} \, s_{ik}^{(r)}(x) \, ,$$

we consider an effect $s_{ik}^{(r)}(x)$ that depends only on
this network $x^{(r)}$ itself, then good results
are obtained by using the statistic

$$S_k := \sum_i s_{ik}^{(r)}\big(X(t_2)\big)$$

and requiring, as part of the moment equation, that

$$\mathrm{E}_\beta\big\{S_k \mid X(t_1)\big\} \;=\; s_k^{\mathrm{observed}} \, .$$

This can be implemented by an MCMC approximation
using the Robbins-Monro method of stochastic approximation.

Now consider a statistic that expresses cross-network dependencies.

Which statistic is sensitive for the parameters expressing cross-network dependencies?

The example will be given for the parameter that is the weight of direct association,

$$\sum_{j=1}^{n} x^{(\text{blue})}(i,j)\, x^{(\text{red})}(i,j)$$

for the case of $M = 2$ repeated observations.

Consider direct association:

leading to

The statistic for fitting the corresponding parameter is

$$\sum_{i=1}^{n} \sum_{j=1}^{n} x^{(\text{blue})}(i,j)(t_1)\, x^{(\text{red})}(i,j)(t_2)$$

note the use of t_1 and t_2:
use explanatory network at previous observation, dependent network at the next.

Example

Research with Vanina Torlo and Alessandro Lomi.

International MBA program in Italy;
75 students; 3 waves.

1. *Friendship*
2. *Advice*:
   To whom do you go for help if you missed a class, etc.

Univariate results

Effect	*Friendship* par.	(s.e.)	*Advice* par.	(s.e.)
outdegree (density)	–1.852***	(0.274)	–2.591***	(0.209)
reciprocity	1.134*	(0.530)	1.897***	(0.231)
transitive triplets	0.341***	(0.052)	0.308***	(0.046)
transitive reciprocated triplets	–0.341***	(0.098)	–0.021	(0.086)
indegree - popularity	0.012	(0.007)	0.0359***	(0.0066)
outdegree - popularity	–0.0424***	(0.0052)	–0.052	(0.033)
outdegree - activity	–0.075†	(0.041)	0.017	(0.014)
reciprocal degree - activity	0.123	(0.079)	–0.104*	(0.043)
gender alter	0.062	(0.074)	0.011	(0.091)
gender ego	–0.146†	(0.077)	–0.281**	(0.095)
same gender	0.306***	(0.071)	0.163†	(0.090)
same nationality	0.264**	(0.085)	0.455***	(0.116)
performance alter	–0.021	(0.021)	0.080**	(0.030)
performance squared alter	N.A.	(N.A.)	N.A.	(N.A.)
performance ego	–0.109***	(0.024)	–0.073*	(0.030)
performance squared ego	N.A.	(N.A.)	0.029**	(0.010)
performance difference squared	–0.0224***	(0.0048)	–0.0307***	(0.0070)

† $p < 0.1$; * $p < 0.05$; ** $p < 0.01$; *** $p < 0.001$; overall maximum convergence ratio 0.14.

The "five-parameter" model was used, but pruned;
the `N.A.` indications refer to parameters that were fixed-and-tested using the score-type test.

This is an example of using the score-type test as a confirmation that effects left out of the model are indeed non-significant.

Note about the decimals :

At least a precision of 10% of a standard error should be reported.

This means that it should be avoided to report a decimal number ending, after the leading zeros, by one single non-zero digit.

now the multivariate results ... :

Coevolution results: within-network effects

	Friendship		*Advice*	
Effect	par.	(s.e.)	par.	(s.e.)
outdegree (density)	–2.944***	(0.155)	–3.751***	(0.264)
reciprocity	1.605***	(0.252)	1.133***	(0.245)
transitive triplets	0.178***	(0.024)	0.210***	(0.053)
transitive recipr. triplets	–0.143***	(0.039)	0.027	(0.090)
indegree - popularity	0.0370***	(0.0096)	0.0443***	(0.0075)
outdegree - popularity	–0.0294***	(0.0067)	0.024	(0.027)
outdegree - activity	0.0071	(0.0082)	0.050***	(0.015)
recipr. degree - activity	–0.007	(0.031)	–0.118**	(0.042)
gender alter	0.043	(0.071)	0.027	(0.097)
gender ego	–0.092	(0.073)	–0.202*	(0.094)
same gender	0.194**	(0.070)	0.048	(0.091)
same nationality	0.213**	(0.081)	0.358**	(0.121)
perf. alter	–0.035†	(0.021)	0.139***	(0.033)
perf. ego	–0.103***	(0.021)	–0.014	(0.031)
perf. squared ego	–		0.043***	(0.010)
perf. difference squared	–0.0189***	(0.0045)	–0.0272***	(0.0074)

† $p < 0.1$; * $p < 0.05$; ** $p < 0.01$; *** $p < 0.001$; overall maximum convergence ratio 0.07.

Coevolution results: cross-network effects

Effect	*Friendship* par.	(s.e.)	*Advice* par.	(s.e.)
advice	1.602***	(0.246)	–	
incoming advice	0.810***	(0.193)	–	
friendship	–		1.426***	(0.233)
incoming friendship	–		0.565**	(0.217)
mixed indegree popularity	–0.044**	(0.015)	–0.031*	(0.013)
mixed outdegree popularity	–0.066***	(0.017)	–0.0044	(0.0058)
mixed outdegree activity	–0.046*	(0.023)	–0.046***	(0.011)
WWX closure	0.049	(0.103)	0.035	(0.038)
WXX closure	0.094	(0.087)	0.052	(0.042)
XWX closure	0.062†	(0.036)	–0.034	(0.038)

† $p < 0.1$; * $p < 0.05$; ** $p < 0.01$; *** $p < 0.001$; overall maximum convergence ratio 0.07.

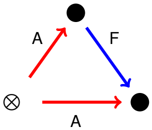
Conclusions (1)

Positive dyad-level effects,
direct effects stronger than reciprocal ('incoming') effects.

Negative actor-level (degree-related) effects friendship ⇔ advice:
Specialization between friendship / advice,
w.r.t. incoming ties as well as outgoing ties.

Conclusions (2)

No strong triadic cross-network effects.
'XWX closure Friendship ⇒ Advice' (`[cl.XWX]`) is weakly significant;
the manual tells us this is "friends of advisors becoming advisors".



dependent Advice
explanatory Friendship

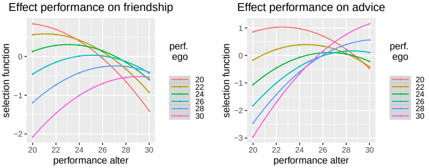
So there is a weak suggestion that
friends of advisors become, or stay, advisors.

Conclusions (3)

Cross-dependencies between friendship and advice do change
the representation of the internal dynamics:
tendency toward reciprocation of advice
is partly mediated by friendship.

Also homophily in advice is partially mediated by friendship.

Effects of performance on friendship and advice



Co-evolution of friendship and advice:
selection functions for performance.

Figures give a much clearer impression than the parameter values.

Figures constructed using `SelectionTables.r`.

Example

Discussion

⇒ See Snijders, Lomi & Torlò in *Social Networks*, 2013.

⇒ See Snijders & Lomi in *Network Science*, 2019.

⇒ Testing cross-network dependencies in dynamics of multiple networks gives interesting new possibilities for hypothesis testing.

⇒ Elaborated along the lines of actor-based modeling.

⇒ Compared to modeling dynamics of single networks, this approach attenuates the Markov assumption by extending the state space to a multiple network.

- ⇒ Other perspectives possible
  by combining one-mode and two-mode networks.
- ⇒ The method is available in RSiena.
  This works for a small number (e.g., 2–6) of networks,
  and a limited number of actors (up to a few hundred).
- ⇒ If there are implication relations between the networks,
  e.g., two networks might be mutually exclusive,
  or one might be a sub-network of the other,
  then this constraint is observed, noted in the `print01Report`,
  and respected in the simulations.
  This gives possibilities for networks with valued ties
  by using different dichotomies.