Multilevel Longitudinal Network Analysis

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Overview

A Introduction: the Stochastic Actor-Oriented Model
and the Siena program;
coevolution.

B Analysis of Multilevel Networks
multiple node sets, multiple networks.

C Multilevel Analysis of Networks
multiple parallel networks.

D Multilevel Network Event Models.

Stochastic Actor-Oriented Model

Kinds of data:

1. **Network panel data**

   = repeated measures of a network on the same node set
   (some exogenous node changes are permitted).
2. **Network and behavior panel data**

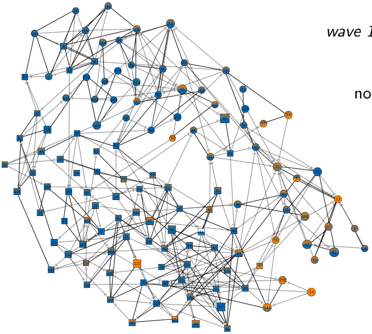
   = the same, with also behavioral dependent variables measured
   ⇒coevolution of networks and behavior.
3. **Multivariate network / behavior panel data**

   = the same, multiple networks and/or multiple behaviors
   ⇒further coevolution.

Networks may also be two-mode networks:
rectangular adjacency matrices.

E.g.: Study of smoking initiation and friendship
(following up on earlier work by P. West, M. Pearson & others).
One school year group from a Scottish secondary school
starting at age 12-13 years, was monitored over 3 years,
3 observations, at appr. 1-year intervals,
160 pupils (with some turnover: 129 always present),
with sociometric & behaviour questionnaires.

Smoking: values 1–3;
drinking: values 1–5;



wave 1 girls: circles

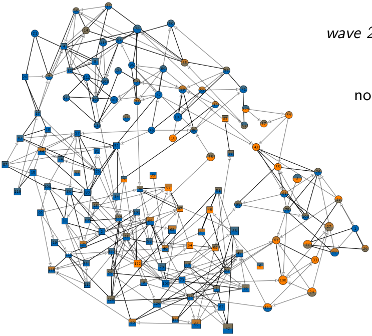
boys: squares

node size: pocket money

color: top = drinking

bottom = smoking

(orange = high)



wave 2 girls: circles

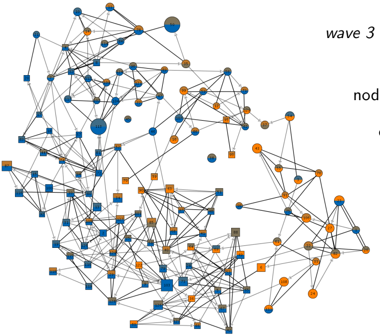
boys: squares

node size: pocket money

color: top = drinking

bottom = smoking

(orange = high)



wave 3

girls: circles
boys: squares

node size: pocket money

color: top = drinking
bottom = smoking
(orange = high)

Questions:

⇒ how to model network dynamics from such data?

⇒ how to model joint dependence between networks and actor attributes such as drinking and smoking?

The Glasgow cohort data set is a panel,
and it is natural to assume *latent change* going on
between the observation moments:
continuous time probability model,
discrete time observations.

Panel data sets are common for networks representing relations between human actors like friendship, advice, esteem, which can be regarded as *states* rather than *events*.

Continuous-time Markov chains: simplicity

Holland & Leinhardt (1977) framework for network dynamics:

1. continuous-time Markov models for panel data
   (changes between observations being unobserved);
   this allows expressing feedback: network builds upon itself;
2. decompose change in smallest constituents,
   i.e., single tie changes: **ministeps**.
3. This means that coordination between actors is not modeled; only feedback, as the actors constitute each others' changing environment.

Note: continuous-time modeling for non-network panel data
developed by Kalbfleisch & Lawless, Bergstrom, Singer, et al.

Simulations – actor orientation

A simulation approach allows to extend this
to include triadic and other complex dependencies.

Actor-oriented perspective (Snijders, 1996, 2001)
('*SAOM* = Stochastic Actor-oriented Model') :

in a directed network,
tie changes are modeled as resulting from
actions by nodes = actors to change their outgoing ties;

An alternative is a *tie-oriented* perspective (Koskinen & Snijders, 2013)
in the ERGM tradition ('*LERGM*'):

tie changes are modeled as dependent on the current network
without a specific process role for the nodes.

Stochastic actor-oriented models: principles

- ⇒ model for network dynamics;
- ⇒ probability model is continuous-time stochastic process, observations are discrete-time;
- ⇒ unobserved changes are ministeps (one variable at the time)
- ⇒ estimation theory elaborated for panel data (i.e., finitely many observation moments, mostly just a few: ≥ 2);
- ⇒ elaborated also for network & behaviour panel data, multivariate networks, two-mode networks;
- ⇒ actor-oriented: in line with social science theories that focus on choices by nodes = actors (can be individuals or organizations) ;
- ⇒ estimation by R package RSiena .

Notation

1. *Actors* $i = 1, \ldots, n$ (network nodes).
2. Array X of *ties* between them : one binary network X;
   $X_{ij} = 0$ (or 1) if there is no tie (or there is a tie), from i to j.
   Matrix X is *adjacency matrix* of digraph.
   Can be extended to multiple networks or discrete ordered values.
   X_{ij} is a *tie indicator* or *tie variable*.
3. Exogenously determined independent variables:
   actor-dependent covariates v, dyadic covariates w.
   These can be constant or changing over time.
4. Continuous time parameter t,
   observation moments $t_1, \ldots, t_M$.

Model assumptions

1. $X(t)$ is a Markov process.
   Strong assumption;
   covariates and state space extensions may enhance plausibility.
2. Condition on the first observation $X(t_1)$, do not model it:
   no assumption of a stationary marginal distribution.
3. At any time moment, only one tie variable X_{ij} can change.
   This precludes swapping partners or coordinated group formation.
   Such a change is called a *ministep* (also 'micro-step').
4. Heuristic: Each actor "controls" her outgoing ties
   collected in the row vector $\big(X_{i1}(t), ..., X_{in}(t)\big)$.
   Actors have full information on all variables (can be weakened).

Timing model: rate functions

'how quick is change?'

At randomly determined moments t,
actors i get opportunity to change a tie variable X_{ij} : **ministep**.
(Actors are also permitted to leave things unchanged.)

Each actor i has a **rate function** $\lambda_i(\alpha, x)$, with $\lambda_+(\alpha, x) = \sum_i \lambda_i(\alpha, x)$:

1. Waiting time until next ministep for current state x
   $\sim$ Exponential$\big(\lambda_+(\alpha, x)\big)$;
2. $\text{P}\big\{\text{Next ministep is for actor } i\big\} = \dfrac{\lambda_i(\alpha, x)}{\lambda_+(\alpha, x)}$.

Rate functions may be constant between waves ($\sim$ homogeneous Poisson processes) or depend on actor characteristics or positions.

Choice model: objective functions

'what is the direction of change?'

The **objective function** $f_i(\beta, x^{\text{old}}, x^{\text{new}})$ for actor i
models change probabilities to go from x^{old} to x^{new}
(cf. potential function for x^{new}).
x^{old} and x^{new} are two consecutive network states differing by only one tie.

Ministep: When actor i gets an opportunity for change,
s/he has the possibility to change *one* outgoing tie variable X_{ij},
or leave everything unchanged.

By $x^{(\pm ij)}$ is denoted the network obtained from x
when x_{ij} is changed ('toggled') into $1 - x_{ij}$.
Formally, $x^{(\pm ii)}$ is defined to be equal to x.

Probabilities in ministep

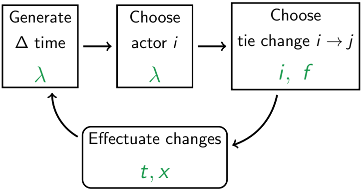
Conditional on actor i being allowed to make a change,
i.e., i taking a ministep,
the probability that X_{ij} changes into $1 - X_{ij}$ is

$$p_{ij}(\beta, x) = \frac{\exp\left(f_i(\beta, x, x^{(\pm ij)})\right)}{\sum_{h=1}^{n} \exp\left(f_i(\beta, x, x^{(\pm ih)})\right)} ,$$

and p_{ii} is the probability of not changing anything.

Higher values of the objective function indicate
the preferred direction of changes.

Simulation algorithm network dynamics



Model specification :

Objective function f_i reflects network effects
(endogenous) and covariate effects (exogenous).
Convenient specification of objective function is a linear combination.

In basic model specifications,
objective function does not depend on the 'old' network:

$$f_i(\beta, x^{\text{old}}, x^{\text{new}} = x) = \sum_{k=1}^{L} \beta_k \, s_{ik}(x) \,,$$

where the weights β_k are statistical parameters
indicating strength of '*effect*' $s_{ik}(x)$.

Dependence on actor-dependent covariates (v_i)
or dyad-dependent (w_{ij}) is left out of the notation.

Examples of effects (1)

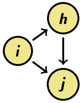
Some possible network effects for actor i, e.g.:

1. *out-degree effect*, controlling the density / average degree,
   $s_{i1}(x) = x_{i+} = \sum_j x_{ij}$
2. *reciprocity effect*, number of reciprocated ties
   $s_{i2}(x) = \sum_j x_{ij} x_{ji}$

Examples of effects (2)

Various effects related to network closure:

3. *transitive triplets effect*,
   number of transitive patterns in i's ties
   $(i \to j,\ i \to h,\ h \to j)$
   $s_{i3}(x) = \sum_{j,h} x_{ij} x_{ih} x_{hj}$



transitive triplet

Examples of effects (3)

4. GWESP effect (cf. ERG models)
   (*geometrically weighted edgewise shared partners*)
   which gives a more moderate contribution of transitivity

$$\text{GWESP}(i,\alpha) \;=\; \sum_j x_{ij}\, e^{\alpha} \left\{ 1 \;-\; \left(1 - e^{-\alpha}\right)^{\sum_h x_{ih} x_{hj}} \right\} .$$

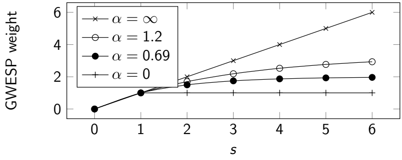


Figure: Weight of tie $i \to j$ for $s = \sum_h x_{ih} x_{hj}$ two-paths.

Examples of effects (4)

Various objective function effects associated with actor covariate v.

Those to whom '*ego*' i is tied are called i's '*alters*'.

1. *covariate-related popularity*, 'alter'
   sum of covariate over all of i 's alters
   $s_{i1}(x) = \sum_j x_{ij}\, z_j$;
2. *covariate-related activity*, 'ego'
   i's out-degree weighted by covariate
   $s_{i2}(x) = z_i\, x_{i+}$;
3. *covariate-related interaction*, 'ego × alter'
   $s_{i3}(x) = z_i \sum_j x_{ij}\, z_j$;

Rate function: basic specification

The rate function $\lambda_i(x)$ defines how often actor i gets opportunities for change / makes microsteps.

The observations are taken at times ('waves') $t_1, t_2, \ldots, t_M$ $(M \geq 2)$ and the time interval $[t_m, t_{m+1}]$ is called a *period*.

A basic feature is to include period-dependent multiplicative parameters

$$\lambda_i(\rho, \alpha, x) \, = \, \rho_m \, \lambda_i^0(\alpha, x)$$

where $\lambda_i(\rho, \alpha, x)$ applies in period $[t_m, t_{m+1}]$.
$\lambda_i^0(\alpha, x)$ may be constant, or depend on i and x and parameters α;
the free parameters ρ_m reflect that observations may be taken any time, without interfering with the network process,
and that we have no prior knowledge about the 'speed of social time'.

Rate function: extended specification

A non-constant rate function $\lambda_i(\alpha, x)$
represents that some actors change their ties more quickly than others, depending on covariates or network position.

Dependence on covariates:

$$\lambda_i(\rho, \alpha, x) = \rho_m \, \exp(\sum_h \alpha_h \, v_{hi}) \, .$$

Dependence on network position, e.g., on an actor variable V:

$$\lambda_i(\rho, \alpha, x) = \rho_m \, \exp(\alpha_1 \, v_i) \; .$$

ρ_m is a period-dependent base rate.

Estimation

For estimating the parameters, if there are complete continuous-time data (all ministeps known), we could use maximum likelihood.

For panel data, estimation is less straightforward.

Estimation methods have been developed using
Method of Moments, *Generalized Method of Moments*,
Bayes, and *Maximum Likelihood* methods.

Method of Moments is used the most:
statistical efficiency quite good, time efficiency good.

Estimation: Method of moments

Method of moments ('estimating equations') 'MoM' :
Choose a suitable statistic $Z = (Z_1, \ldots, Z_K)$,
the statistic Z must be *sensitive* to the parameter θ in the sense that

$$\frac{\partial \mathrm{E}_\theta(Z_k)}{\partial \theta} > 0 \ ;$$

determine value $\hat{\theta}$ of θ for which
observed and expected values of Z are equal:

$$E_{\hat{\theta}}\{Z\} = z \ .$$

Statistics for MoM

Assume that there are $M = 2$ observation moments,
and rates are constant: $\lambda_i(x) = \rho$.

ρ determines the expected "amount of change".

A sensitive statistic for ρ is the Hamming distance,

$$C = \sum_{\substack{i,j=1 \\ i \neq j}}^{g} \mid X_{ij}(t_2) - X_{ij}(t_1) \mid \ ,$$

the "observed total amount of change".

For the weights β_k in the objective function

$$f_i(\beta, x) = \sum_{k=1}^{L} \beta_k \, s_{ik}(x) \, ,$$

a higher value of β_k means that all actors
'strive more strongly after' a high value of $s_{ik}(x)$,
so $s_{ik}(x)$ will tend to be higher for all i, k.

This leads to the statistic

$$S_k = \sum_{i=1}^{n} s_{ik}\big(X(t_2)\big) \ .$$

This statistic will be sensitive to β_k :
a higher β_k will tend to lead to higher values of S_k.

(Recall: the model is conditional on $x(t_1)$.)

How to solve the moment equation?

Moment equation $E_{\hat{\theta}}\{Z\} = z$ is difficult to solve, as

$$\mathrm{E}_{\theta}\{Z\}$$

cannot be calculated explicitly.

However, the solution can be approximated, e.g., by the Robbins-Monro (1951) method for stochastic approximation.

Iteration step (cf. Newton-Raphson) :

$$\hat{\theta}_{N+1} = \hat{\theta}_N - a_N \, D^{-1}(z_N - z) \; , \qquad (1)$$

where z_N is a simulation of Z with parameter $\hat{\theta}_N$,
D is a suitable matrix, and $a_N \to 0$.

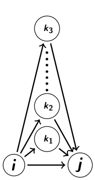
This yields (with tuning) a surprisingly stable algorithm.

Example: Glasgow data

The following page presents estimation results for the Glasgow data: friendship network between 160 pupils, observed at 3 yearly waves.

The model was the result of an extensive goodness of fit exercise, considering distributions of outdegrees, indegrees, and triad motifs.

Transitive closure is represented by the geometrically weighted shared partners ('gwesp') effect:



Effect	par.	(s.e.)
rate (period 1)	11.404	(1.289)
rate (period 2)	9.155	(0.812)
outdegree (density)	–3.345***	(0.229)
reciprocity: creation	4.355***	(0.485)
reciprocity: maintenance	2.660***	(0.418)
GWESP: creation	3.530***	(0.306)
GWESP: maintenance	0.315	(0.414)
indegree – popularity	–0.068*	(0.028)
outdegree – popularity	–0.012	(0.055)
outdegree – activity	0.109**	(0.036)
reciprocated degree – activity	–0.263***	(0.066)
sex (F) alter	–0.130†	(0.076)
sex (F) ego	0.056	(0.086)
same sex	0.442***	(0.078)

Some conclusions:

Evidence for reciprocity; transitivity;
reciprocity stronger for creating than for maintaining ties;
transitivity only for creating ties;
gender homophily;
those with many reciprocated ties are less active
in establishing new ties or maintaining existing ties.

Note: definition of reciprocated degree – activity:

$$s_{ik}(x) = \sum_j x_{ij}\, x_{i+}^{\text{rec}}$$

where

$$x_{i+}^{\text{rec}} = \sum_j x_{ij}\, x_{ji}.$$

A2. Non-directed networks

The actor-driven modeling is less straightforward
for non-directed relations,
because two actors are involved in deciding about a tie.

Various modeling options are possible,
representing different ways of *coordination*
between the two actors at both sides of the tie.

1. Forcing (dictatorial) model:
   one actor takes the initiative and unilaterally imposes
   that a tie is created or dissolved.
2. Unilateral initiative with reciprocal confirmation:
   one actor takes the initiative and proposes a new tie
   or dissolves an existing tie;
   if the actor proposes a new tie, the other has to confirm,
   otherwise the tie is not created.
   (Cf. Jackson & Wolinsky, 1996)

3. Pairwise conjunctive model:
   a pair of actors is chosen and reconsider whether a tie will exist between them; a new tie is formed if both agree.
4. Pairwise disjunctive (forcing) model:
   a pair of actors is chosen and reconsider whether a tie will exist between them;
   a new tie is formed if at least one wishes this.
5. Pairwise compensatory (additive) model:
   a pair of actors is chosen and reconsider whether a tie will exist between them; this is based
   on the sum of their utilities for the existence of this tie.

Option 1 is close to the actor-driven model for directed relations.

In options 3–5, the pair of actors (i, j) is chosen
depending on the product of the rate functions $\lambda_i \lambda_j$
(under the constraint that $i \neq j$).

The numerical interpretation of the ratio function
differs between options 1–2 compared to 3–5.

In options 2–5, the decision about the tie is taken on the basis of the objective functions f_i , f_j of both actors.

A3. Co-evolution

In the SAOM for a single network,
the actors change their network neighbourhoods :
these *co-evolve* as the common changing environment = system state.

This can be extended to a system with multiple variables:
other networks, discrete actor-level variables, two-mode networks.

The basic ideas remain the same:
continuous-time Markov chain, now with larger state space;
heuristic: actors can change outgoing ties and their own variables;
at times of change, only one variable can change;
behavior is **discrete**, changes –1 / 0 / +1 / (ministeps!).

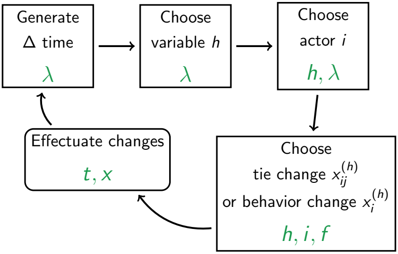
Rate functions, objective functions, specified
separately **for each** dependent variable.

Dependence of rate and objective functions of variable Y
on another dependent variable Z expresses directed dependence:
effect $Z \Rightarrow Y$.

Given the longitudinal panel data, this allows
to estimate separately dependence $Z \Rightarrow Y$ and $Y \Rightarrow Z$;
under the assumption that the model holds...

Computer simulation algorithm

The co-evolution Markov chain is a succession of ministeps;
variables can be networks or actor-level variables.



Networks and Behaviour Studies

Co-evolution of a network and one or more actor variables
representing behavioural tendencies of actors
are *Networks and Behaviour Studies* that can be used to
study mechanisms of social influence and social selection.

E.g.: network of adolescents,
co-evolution friendship network ⇔ smoking behaviour.

Actor-oriented models for networks and behavior

Each actor "controls" his outgoing ties
collected in the row vector $\big(X_{i1}(t), ..., X_{in}(t)\big)$,
and also her behavior $Z_i(t) = \big(\ Z_{i1}(t), ..., Z_{iH}(t)\ \big)$,
where it is assumed there are $H \geq 1$ behavior variables.

Network change process and behavior change process
run concurrently, with a common state space,
with transition probabilities depending on this joint state.

A3. Co-evolution Networks and behaviour

Outline of model definition

Microsteps now can be either a change to the network,
or a change to the behavior.

Rate functions are defined separately
for changes in network: λ^X
for changes in behavior h: λ^{Z_h} .

Objective functions likewise are defined separately
for the network f^X and the behaviors f^{Z_h} .

X and Z are interdependent because the objective functions
depend on both variables; they are applied as

$$f_i^X\big(X(t), Z(t)\big) \text{ and } f_i^{Z_h}\big(X(t), Z(t)\big)$$

Behavior ministep

Whenever actor i may make a change in variable h of Z,
she changes only one behavior, say z_{ih}, to the new value v
(recall: changes can be –1, 0, +1).
The new vector is denoted by $z(i, h \rightsquigarrow v)$.
Change probabilities are given by

$$p_{ihv}(\beta, z, x) = \frac{\exp(f(i, h, v))}{\sum_u \exp(f(i, h, u))}$$

where

$$f(i, h, v) = f_i^{Z_h}(\beta, z(i, h \rightsquigarrow v), x) \ .$$

Specification of behavior model

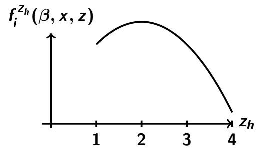
Objective function for behavior $f_i^{Z_h}$ also is linear predictor:

$$f_i^{Z_h}(\beta, x, z) = \sum_{k=1}^{L} \beta_k^{Z_h} \, s_{ik}^{Z_h}(x, z) \, ,$$

Some basic effects:

1. *Linear shape* ,
   $s_{i1}^{Z_h}(x, z) = z_{ih}$
2. *quadratic shape*, 'effect behavior on itself',
   $s_{i2}^{Z_h}(x, z) = z_{ih}^2$
   Quadratic shape effect important for model fit.

For a negative quadratic shape parameter,
the model for behavior is a unimodal preference model.



For positive quadratic shape parameters ,
the behavior objective function can be bimodal
('positive feedback').

3. *behavior-related similarity*,
   sum of behavior similarities
   between i and his friends
   $s_{i3}^{Z_h}(x, z) = \sum_j x_{ij} \left(1 - |\, z_{ih} - z_{jh} \,|\right)$,
   if Z_h assumes values between 0 and 1;
   may be divided by $x_{i+} = \sum_j x_{ij}$;
4. *average behavior alter* — an alternative to similarity:
   $s_{i4}^{Z_h}(x, z) = z_{ih} \frac{1}{x_{i+}} \sum_j x_{ij} z_{jh}$
5. *popularity-related tendency*, (in-degree)
   $s_{i5}^{Z_h}(x, z) = z_{ih}\, x_{+i}$
6. *activity-related tendency*, (out-degree)
   $s_{i6}^{Z_h}(x, z) = z_{ih}\, x_{i+}$

Effects (3) and (4) both express the idea of influence,
but in mathematically different ways.
Theory plus data will have to differentiate between them.

7. *dependence on other behaviors* $(h \neq \ell)$,
   $s_{i7}^{Z_h}(x, z) = z_{ih}\, z_{i\ell}$

For both the network and the behavior dynamics,
extensions are possible depending on the network position.

Multivariate Networks

Co-evolution of several networks allows studying how these networks influence each other. The same co-evolution principles apply.

E.g.: networks in organizations,
relevant networks are advice – collaboration – friendship;
other example: bullying in schools,
some relevant networks are friendship – bullying – defending.

A multitude of mixed structural effects are interesting.
E.g.: **direct entrainment**: advisors become friends;
mixed transitive closure patterns:
advisors of friends become advisors, etc.;
actor-level dependencies:
those who have many friendships give less advice.

Two-mode Networks

Networks may also be two-mode networks,
where the first node set is the set of actors
and the second node set can be, e.g., activities, meeting places
representing further contextual aspects; or cognitions.

For co-evolution of friendship, associations, and behavior this opens the possibility to study whether actors are influenced by their friends or by their co-members of associations.

Two-mode networks have less structural possibilities
because the second mode is unrelated to the first mode.

The procedures are implemented in the R package

R
S imulation
I nvestigation for
E mpirical
N etwork
A nalysis

which is available from CRAN and (up-2-date) R-Forge
http://www.stats.ox.ac.uk/siena/

Material, papers, can be found on SIENA website.

Some references (time-ordered)

- Tom A.B. Snijders (2001). The Statistical Evaluation of Social Network Dynamics. *Sociological Methodology*, 31, 361–395.
- Johan H. Koskinen and Tom A.B. Snijders (2007). Bayesian inference for dynamic social network data. *Journal of Statistical Planning and Inference*, 13, 3930–3938.
- Tom Snijders, Christian Steglich, and Michael Schweinberger (2007), Modeling the co-evolution of networks and behaviour. Pp. 41–71 in *Longitudinal models in the behavioral and related sciences*, eds. Kees van Montfort, Han Oud and Albert Satorra; Lawrence Erlbaum.
- Steglich, C.E.G., Snijders, T.A.B. and Pearson, M. (2010). Dynamic Networks and Behavior: Separating Selection from Influence. *Sociological Methodology*, 40, 329–392.
- Tom A.B. Snijders, Johan Koskinen, and Michael Schweinberger (2010). Maximum Likelihood Estimation for Social Network Dynamics. *Annals of Applied Statistics*, 4, 567–588.
- Johan H. Koskinen and Tom A.B. Snijders (2013). Longitudinal models. Pp. 130–140 in *Exponential Random Graph Models*, edited by Dean Lusher, Johan Koskinen, and Garry Robins. Cambridge University Press.

Some references (continued)

- Tom A.B. Snijders, Alessandro Lomi, and Vanina Torlò (2013). A model for the multiplex dynamics of two-mode and one-mode networks, with an application to employment preference, friendship, and advice. *Social Networks*, 35, 265–276.
- Viviana Amati, Felix Schönenberger, Tom A.B. Snijders (2015). Estimation of stochastic actor-oriented models for the evolution of networks by generalized method of moments. *Journal de la Société Française de Statistique*, 156, 140–165.
- Tom A.B. Snijders (2016). The Multiple Flavours of Multilevel Issues for Networks, pp. 15-46 in Emmanuel Lazega and Tom A.B. Snijders (eds), *Multilevel Network Analysis for the Social Sciences; Theory, Methods and Applications*. Springer.
- Tom A.B. Snijders and Mark Pickup (2016). Stochastic actor-oriented models for network dynamics. In Victor, J. N., Lubell, M., and Montgomery, A. H., editors, *Oxford Handbook of Political Networks*. Oxford University Press, still in press.
- Tom A.B. Snijders (2017). Stochastic Actor-Oriented Models for Network Dynamics. *Annual Review of Statistics and its Application*, 4, 343-363.
- See SIENA manual and homepage.

Extra 1. Convergence criterion

In the implementation in RSiena, the algorithm runs for a preset number of iterations, after which convergence is assessed.

Determine how close we are to $\mathbf{E}_\theta\{Z\} = z$

One way to measure this uses the 't-ratios for convergence',

$$\text{tconv}_k = \frac{\overline{Z}_{Nk} - z_k}{\text{s.d.}(Z_{1k}, \ldots, Z_{Nk})} ,$$

where $Z = (Z_{n1}, \ldots, Z_{nK})$, and requires, e.g.,

$$\max_k |\text{tconv}_k| \leq 0.1 .$$

Overall maximum convergence ratio

A better criterion turns out to be the maximum t-ratio for convergence for any linear combination of the parameters,

$$\text{tconv.max} = \max_b \left\{ \frac{b'(\overline{Z}_N - z)}{\sqrt{b'\,\widehat{\text{Cov}}(Z)\,b}} \right\} .$$

This is equal to $\Big($use Cauchy-Schwarz, $\Sigma = \widehat{\text{Cov}}(Z)\Big)$

$$\max_c \left\{ \frac{c'\,\Sigma^{-1/2}(\overline{Z} - z)}{\sqrt{c'c}} \right\} = (\overline{Z} - z)'\,\Sigma^{-1}(\overline{Z} - z) .$$

The definition implies that

$$\text{tconv.max} \geq \max_k |\text{tconv}_k| .$$

The current rule for practical use is that

$$\text{tconv.max} \leq 0.25 \ \ \underline{\text{and}} \ \ \max_k |\text{tconv}_k| \leq 0.1 \ .$$