Statistical Methods for Network Dynamics (⋆)

Metodi Statistici Per L'Analisi Dinamica Delle Reti

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Riassunto:
Nel contributo è presentata una rassegna dei piú recenti modelli statistici e metodi di stima per l'analisi longitudinale di reti sociali. Per rappresentare i processi sottostanti le dinamiche di rete, è utile pensare ai dati di panel come ad osservazioni provenienti da un processo a tempo continuo definito sullo spazio dei grafi orientati. Vengono discussi e illustrati modelli stocastici tie-oriented e actor-oriented in grado di riflettere sia dinamiche endogene che effetti di variabili esogene. Tali modelli non consentono il calcolo esplicito ma possono essere sviluppati specifici schemi di simulazione. Sono inoltre proposti metodi di approssimazione stocastica per la stima dei parametri. Un esempio di applicazione di questi modelli è condotto sui dati reticolari provenienti da uno studio sul precursore della comunicazione via e-mail.

Keywords: continuous-time Markov process, dyadic data, method of moments, panel data, random utility, Robbins-Monro procedure, social networks, simulation models

1. Introduction

Social networks provide a natural approach for the study of social and economic interaction structures. A network consists of a set of points (or nodes) and the ties between them. The points and the ties can have different meanings depending on the context. For instance, the points may be the pupils in a classroom while the set of ties refers to the friendship relations between them; or the set of points may be firms, where the ties represent their collaborative links. Alternatively, the set of points may be countries and the ties represent bilateral trade agreements. In general, the points in the social network represent a relevant set of social or economic actors. Networks of relations between social actors are increasingly recognized as crucial social opportunities and constraints for the behavior and performance of the actors. The well-being of individuals in their social context is conditioned not only by their individual characteristics and behavior but also by their social ties; the economic production of goods and services is conditioned by the networks between firms as well as by networks between individuals inside the firms. Within the social sciences (first in sociology and social anthropology, more recently also in economics, social psychology, and education) this has led to the establishment of "social network studies" as a productive research field. Overviews and recent developments can be found in, e.g., Wasserman and Faust (1994), Doreian and Stokman (1997), Leenders and Gabbay (1999), Lin, Cook, and Burt (2001), Monge and Contractor (2003), Brass *et al.* (2004), and Carrington, Scott, and Wasserman (2005).

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A network can be denoted by the finite set $\mathcal{N} = \{1, \ldots, n\}$ of actors on which a relation is defined which can be represented by a nonreflexive directed graph (digraph) or, alternatively, an adjacency matrix with a structurally zero diagonal. The $n \times n$ adjacency matrix $\boldsymbol{x} = \big(x_{ij}\big)$ indicates by $x_{ij} = 1$ or $x_{ij} = 0$, respectively, that there is a tie, or there is no tie, from actor i to actor j. The nonreflexivity means that self-ties are not considered, so that $x_{ii} = 0$ for all i. The variables x_{ij} are referred to as *tie variables*.

When considering the structure of a network it is evident that there will usually be strong dependencies between tie variables. All variables in row i of the adjacency matrix refer to ties issuing from the same 'sending' actor i; similarly, the elements of column j refer to ties directed to the same 'receiving' actor j. Social processes of reciprocity, e.g., reciprocation of friendship or mutual collaboration, will lead to a dependence between the tie variable x_{ij} and the reciprocally placed variable x_{ji}.

More complicated types of dependency involve more than two actors. The most well-known of these is transitivity of choices: "friends of my friends are my friends". This implies that when there are ties from i to j and from j to h ($x_{ij} = 1$ and $x_{jh} = 1$), there will be a tendency toward the existence also of the tie from i to h ($x_{ih} = 1$).

When making stochastic models of network data, such dependencies will be translated into stochastic dependencies between the tie variables which then are represented by capital letters X_{ij} to expressing their stochastic nature. The types of dependency mentioned imply that it is impossible to separate the set of variables in the adjacency matrix $\boldsymbol{X}$ into subsets which are mutually independent. Statistical inference concerning social networks is directed both at modeling the dependence structure within the network and the dependence on exogenous explanatory variables. Such explanatory variables can be attributes of the individual actors, such as the gender or age of persons, or the turnover or profit of companies; but they can also be dyadic variables, i.e., attributes of pairs or ordered pairs of actors, such as the distance between the dwellings of persons, or the existence of a board overlap between companies.

Increasing attention is being given in social network analysis to longitudinal data. Controlling for earlier states of the network simplifies the studies of dependence structures, both from a substantive and from a statistical point of view. This presentation focuses on methods of inferential statistics for the analysis of longitudinal network data, continuing the work presented in Snijders (2001, 2005). It may be interesting to note that interesting developments are taking place also in the construction of stochastic models for network dynamics using techniques of statistical mechanics, see, e.g., Newman *et al.* (2002) and Albert and Barabási (2002). The latter models give good insights in how simple rules can give rise to interesting and nontrivial network topologies, but they are too restricted to give empirically credible models of observed network dynamics and to estimate and test a wide array of possible elements of such dynamics. Other interesting models have been proposed, based on sociological theories, e.g., Carley (1991), Mark (1998), Macy *et al.* (2003), and Bearman, Moody and Stovel (2004) with a very interesting empirical analysis. All of these articles, although they are very interesting, do not consider issues of statistical inference and therefore are not further considered here.

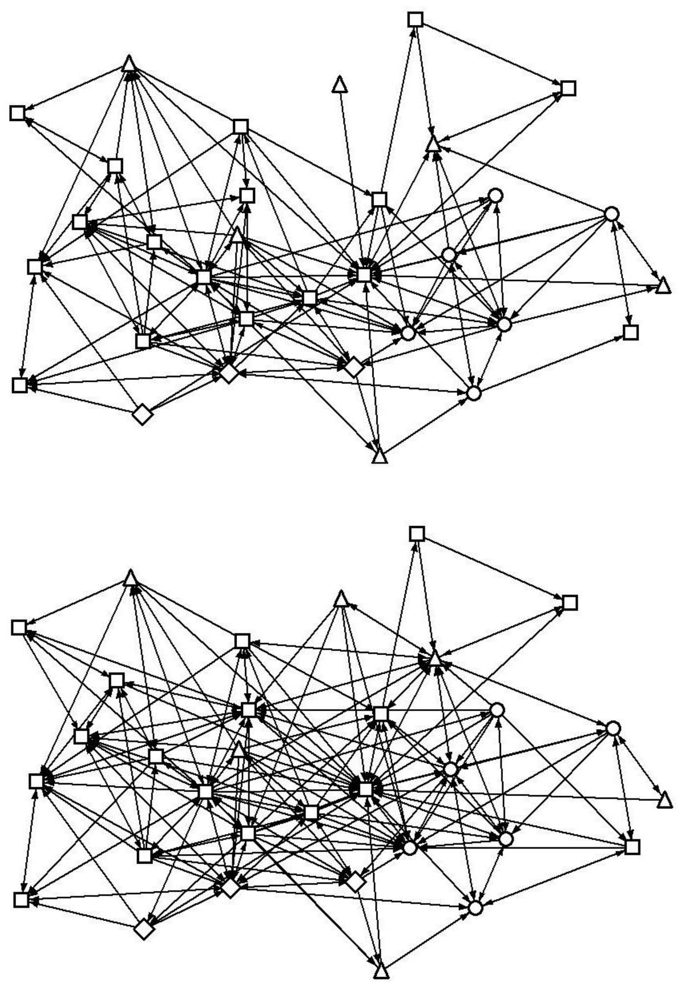
The most usual type of longitudinal network data is panel data, where for $M \geq 2$ time points an observation $\boldsymbol{x}(t_m)$ is available of the network on the same set $\mathcal{N}$ of actors. Individual covariates will be denoted by $\boldsymbol{v} = \big(v_i\big)$ and dyadic covariates by $\boldsymbol{w} = \big(w_{ij}\big)$. These can also be changing over time; this will not be made explicit in the notation.

2. Example: The EIES data

As an example, we use the Electronic Information Exchange System (EIES) data collected by Freeman and Freeman (1980), discussed also in Wasserman and Faust (1994). This is a network of 32 researchers who participated in an early study on the effects of electronic information exchange, a precursor of email communication. Two measures of acquaintanceship are used, collected before and after the study (8 months apart). The data as reproduced by Wasserman and Faust were dichotomized: 1 ("positive tie") for being a friend or close friend of the other, 0 ("no tie") otherwise. In addition, the discipline of the researcher is used as a categorical individual-bound covariate. It is coded 1 for sociologists (of whom there were 17), 2 for anthropologists (6), 3 for mathematicians and statisticians (3), and 4 for psychologists (6).

At the first measurement, there were 152 ties, which given that there are $n(n-1) = 992$ possible ties leads to a density of 152/992 = 0.15. Of these friendship ties, 10 had disappeared at the second measurement, while 62 new ties were created. The density increased to 204/992 = 0.21. The figure indicates the two observed networks.

Figure 1: *EIES friendship network, observed at two time points.*
$\square$ *sociologists;* $\circ$ *anthropologists;* $\diamond$ *mathematicians and statisticians;* $\triangle$ *psychologists.*



The questions asked concerning this data set is whether there is a tendency in the network dynamics of a preference for friendship to others of the same discipline; and whether there is evidence for reciprocity of choices, transitivity of choices, and for preferring others who are already popular in the sense of receiving many friendship choices.

3. Stochastic Models for Network Dynamics

A flexible class of models for panel data on networks can be obtained by assuming that the data are momentary observations of a continuous-time Markov process, in which each tie variable $X_{ij}(t)$ develops in stochastic dependence on the entire network $X(t)$. The elements of the intensity matrix of this Markov process will be denoted $q(\mathbf{x}, \tilde{\mathbf{x}})$. Thus, $\mathrm{P}\{\mathbf{X}(t+\epsilon) = \tilde{\mathbf{x}} \mid \mathbf{X}(t) = \mathbf{x}\} \approx \epsilon q(\mathbf{x}, \tilde{\mathbf{x}})$ if $\mathbf{x} \neq \tilde{\mathbf{x}}$.

Utilizing Markov process models for network dynamics was proposed already by Holland and Leinhardt (1977). It is quite natural to assume the existence of an underlying continuous-time process that is observed only at a few moments. The assumption that this is a Markov process, however, is very strong. On one hand, this assumption is induced by the available data: there is not much than one could do except assume Markov process. On the other hand, by including a richer set of covariates it may be possible to make this assumption more and more realistic as the scientific insights in the modeled processes increases.

It is also natural to assume that the tie variables $X_{ij}(t)$ develop conditionally independently of each other, given the current network $\mathbf{X}(t)$. This implies that at each single moment, no more than one tie variable $X_{ij}(t)$ can change its value. The intensity matrix $\big(q(\mathbf{x}, \tilde{\mathbf{x}})\big)$ can then be represented by specifying only the non-zero non-diagonal elements, which can be denoted

$$q_{ij}(\mathbf{x}) \;=\; q(\mathbf{x}, \tilde{\mathbf{x}}) \tag{1}$$

where the matrix $\tilde{\mathbf{x}}$ is defined by

$$\tilde{x}_{hk} \;=\; \begin{cases} x_{hk} & \text{if } (h,k) \neq (i,j) \\ 1 - x_{ij} & \text{if } (h,k) = (i,j). \end{cases}$$

All other elements $q(\mathbf{x}, \tilde{\mathbf{x}})$ are assumed to be 0. The value $q_{ij}(\mathbf{x})$ can be interpreted as the propensity for the arc variable X_{ij} to change into its opposite $(1 - X_{ij})$, given that the current state of the network is $\mathbf{X} = \mathbf{x}$. The matrix $\mathbf{x}$ where x_{ij} is replaced by a 0 or 1, respectively, will be denoted by $\mathbf{x}(i,j,0)$ and $\mathbf{x}(i,j,1)$. Thus, $\tilde{\mathbf{x}} = \mathbf{x}(i,j,1-x_{ij})$.

3.1. Tie-oriented dynamics

One way to model the network dynamics, proposed by P.E. Pattison and G.L. Robins (personal communication), is to assume the existence of a *potential function* $f(\mathbf{x})$ that governs the stochastic process in the sense that the process can be regarded as a stochastic optimizer of $f(\mathbf{x})$ like is used, e.g., in simulated annealing. This is expressed by the intensity matrix

$$q_{ij}(\mathbf{x}) \;=\; \rho \, \frac{\exp\big(f(\mathbf{x}(i,j,1-x_{ij}))\big)}{\exp\big(f(\mathbf{x}(i,j,0))\big) + \exp\big(f(\mathbf{x}(i,j,1))\big)} \;. \tag{2}$$

This can be interpreted as the result of two sub-processes. In the first place, for each tie variable $X_{ij}(t)$ there is an independent Poisson processes going on, with intensity parameter ρ; in the second place, when an event occurs in this Poisson process, the value of $X_{ij}(t)$ is newly determined with log odds $f(\mathbf{x}(i,j,1)) - f(\mathbf{x}(i,j,0))$. The Poisson processes can be regarded as defining the moments where ties *can* change; whether they actually do change depends on whether the new random choice for X_{ij} yields a value different from the preceding value.

This is the Gibbs sampling process for the probability distribution

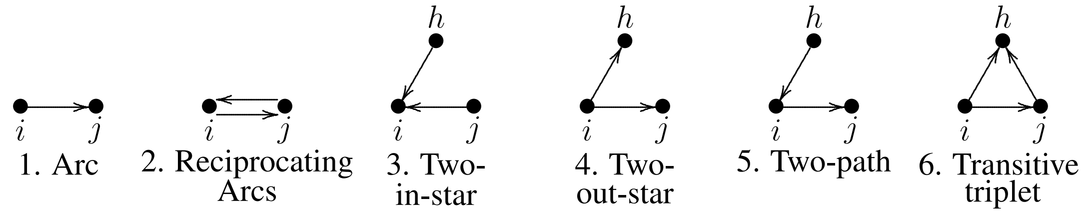
$$\frac{\exp\big(f(\mathbf{x})\big)}{\kappa}\ , \tag{3}$$

where κ is the normalizing constant. Such processes are reviewed for a wide array of physical systems by Newman and Barkema (1999); for social networks they are treated, e.g., by Snijders (2002), Pattison and Robins (2005), and Snijders *et al.* (2006). It was proposed by Frank and Strauss (1986), Frank (1991), and Wasserman and Pattison (1996) to model non-longitudinal network data by probability distributions defined by (3) where the function $f(\mathbf{x})$ is a linear combination of network statistics,

$$f(\mathbf{x}) \ = \ \sum_k \beta_k s_k(\mathbf{x})\ . \tag{4}$$

The resulting families of probability distributions are called *exponential random graph models*. Such functions can as well be used as potential functions for modeling longitudinal network data. The choice of the functions $s_k(\mathbf{x})$ defines the statistical model and has to express the dependence between the tie variables – both in the non-longitudinal and the longitudinal case. Important examples for $s_k(\mathbf{x})$ are counts of network configurations such as the following.

Figure 2: *Some local network configurations.*



The statistics for these six configurations are defined by

$$\begin{array}{rclrcl} s_1(\mathbf{x}) & = & \sum_{i,j\in\mathcal{N}} x_{ij}\ , & s_2(\mathbf{x}) & = & \sum_{i,j\in\mathcal{N}} x_{ij}\, x_{ji}\ , \\ s_3(\mathbf{x}) & = & \sum_{i,j,h\in\mathcal{N}} x_{hi}\, x_{ji}\ , & s_4(\mathbf{x}) & = & \sum_{i,j\in\mathcal{N}} x_{ih}\, x_{ij}\ , \\ s_5(\mathbf{x}) & = & \sum_{i,j\in\mathcal{N}} x_{hi}\, x_{ij}\ , & s_6(\mathbf{x}) & = & \sum_{i,j,h\in\mathcal{N}} x_{ij}\, x_{jh}\, x_{ih}\ . \end{array} \tag{5}$$

Holland and Leinhardt (1975) already remarked that such local configuration counts can be used to represent much of the network structure. This is further discussed in the mentioned literature, and was taken up for biological applications more recently by Milo *et al.* (2002). Snijders *et al.* (2006) explain that for modeling non-longitudinal data of social networks it is not adequate to use only counts of the configurations represented in

Figure 2, because models with only these configurations can lead to degeneracy; this is briefly discussed below in Section 4.

In addition to these and other statistics that depend only on the network and thus may be said to be purely structural, the potential function can also have terms depending on individual or dyadic covariates, i.e., attributes of the actors or of pairs of actors; and terms reflecting interactions between network structure and attributes.

When using a function of the type (4), the log odds in the determination of the new value of X_{ij} is given by

$$f(\mathbf{x}(i,j,1)) - f(\mathbf{x}(i,j,0)) \;=\; \sum_k \beta_k \big(s_k(\mathbf{x}(i,j,1)) - s_k(\mathbf{x}(i,j,0))\big)\;.$$

Wasserman and Pattison (1996) call $s_k(\mathbf{x}(i,j,1)) - s_k(\mathbf{x}(i,j,0))$ the *change statistics*.

The tie-based network dynamic expressed by (2) and (4) corresponds well with the exponential random graph model defined by (3) and (4) because it has the latter as its limiting distribution, and because it is the Gibbs sampling procedure when cycling randomly through the constituent variables x_{ij}, as discussed, e.g., in Pattison and Robins (2005); of course there are many other Markov processes with this limiting distribution, cf. Snijders (2002).

3.2. Actor-oriented dynamics

Social science theory as well as intuition also suggest that it can be meaningful to think of the network dynamics as being driven by the social actors who make up the node set $\mathcal{N}$. This is in line with the methodological approach of structural individualism (Wippler, 1978; Udehn, 2002). Such an approach leads to stochastic actor-oriented models, proposed in Snijders (1996), where the actors $i \in \mathcal{N}$ are represented as actors stochastically optimizing an objective function $f_i(\mathbf{x})$ which represents the resultant of their goals and restrictions. The basic elements of the actor-oriented models presented here are the following.

1. The actors control their outgoing ties.
2. The ties have inertia:
   at any single moment in time, only one variable $X_{ij}(t)$ may change.
3. Changes are made by the actors to optimize their situation,
   as it will obtain immediately after this change.
4. The assessment by actors of their situation comprises a random element,
   expressing aspects not modeled explicitly.

Elements 3 and 4 can be summarized by saying that the actors perform a myopic stochastic optimization of their objective function $f_i(\mathbf{x})$, which therefore must be interpreted as the short-term objectives of the actors.

Such an objective function $f_i(\mathbf{x})$ should be formulated from the viewpoint of the actor, whereas in the tie-oriented model it was formulated globally, as a function of the whole network. The actor's viewpoint is naturally reflected by characteristics of the pattern of ties in which actor i is involved. Still referring to Figure 2, this is satisfied by the objective function

$$f_i(\mathbf{x}) \;=\; \sum_k \beta_k s_{ik}(\mathbf{x})\;. \tag{6}$$

where s_k is defined, e.g., as the number of configurations of a particular type in which actor i is involved as the focal actor, such as

$$s_1(\mathbf{x}) \;=\; \sum_{j\in\mathcal{N}} x_{ij} \;\text{ and }\; s_6(\mathbf{x}) \;=\; \sum_{j,h\in\mathcal{N}} x_{ij}\, x_{jh}\, x_{ih}\;. \tag{7}$$

To specify the model, it is necessary also to define the constraints under which the actors perform their myopic optimization, and it is natural again to construct this as a two-step process: first a stochastic choice is made as to the time point when the next change can be made and as to the actor who can make this change; then the actor makes the change as a stochastic optimization of his/her objective function.

Two examples of the first step in the process will be proposed here: a tie-based process and an actor-based process.

3.2.1. Tie-based opportunities, actor-oriented choice

This section presents a model, suggested by C.E.G. Steglich (personal communication), that combines a tie-based process for generating opportunities for changing the network with an actor-oriented choice model. In the tie-based opportunity process, a random pair (i, j) $(i \neq j)$ is chosen, and actor i gets the opportunity to change the tie variable X_{ij}. These events happen in a Poisson process at a rate ρ. It is assumed that the actor considers the situation immediately after this contemplated change. If the current network is $\mathbf{x}$, the possible new results are $\mathbf{x}(i, j, 0)$ and $\mathbf{x}(i, j, 1)$, and it is assumed that the value attached to this result is the sum of the objective function and a random residual,

$$f_i\big(\mathbf{x}(i, j, h)\big) \;+\; U(h)$$

where $h = 0$ or 1, and $U(0), U(1)$ are independent random variables with a Gumbel distribution. The actor chooses the outcome with the highest value. The well-known correspondence between stochastic optimization with Gumbel residuals and logistic regression (Maddala, 1983) implies that here also the probabilities of the new values $X_{ij} = 1, 0$ are determined by the log odds $\mathbf{x}(i, j, 1) - \mathbf{x}(i, j, 0)$. The intensity matrix is

$$q_{ij}(\mathbf{x}) \;=\; \rho\,\frac{\exp\big(f_i(\mathbf{x}(i, j, 1 - x_{ij}))\big)}{\exp\big(f_i(\mathbf{x}(i, j, 0))\big) + \exp\big(f_i(\mathbf{x}(i, j, 1))\big)}\;. \tag{8}$$

This can be regarded as a hybrid model with tie-based assignment of potential changes (e.g., random meetings of pairs of persons) and actor-oriented determination of directed ties.

Whether this model differs from the pure tie-oriented model depends on whether the change statistics corresponding to the potential function $f(\mathbf{x})$ differ from those for the objective function $f_i(\mathbf{x})$. E.g., referring to (5) and (7), for s_1 the change statistics is 1 in either case, whereas for s_6 it is

$$\sum_{j,h\in\mathcal{N}} \big\{x_{ij}x_{jh} \;+\; x_{ij}x_{ih} \;+\; x_{jh}x_{ih}\big\}$$

for the pure tie-oriented model and

$$\sum_{j,h\in\mathcal{N}} \big\{x_{ij}x_{jh} \;+\; x_{jh}x_{ih}\big\}$$

for the hybrid model.

3.2.2. A fully actor-oriented model

The actor-based opportunity process is obtained by giving each actor an opportunity for change according to a Poisson process at a rate ρ; when actor i has such an opportunity for change, the actor-oriented approach is implemented as follows. The actor reconsiders the collection of his outgoing ties, and is given the opportunity to select one of the tie variables X_{ij} $(j \neq i)$ and change it. The value attached to the current situation and to the possible new situations is again represented by the objective function of the new situation plus a random residual which is assumed to have a Gumbel distribution. The set of obtainable values is

$$f_i(\mathbf{x}) + U(0) \quad \text{and} \quad f_i\big(\mathbf{x}(i,j,1-x_{ij})\big) + U(j) \quad (j = 1, \ldots, n; j \neq i) .$$

It is convenient to represent the new situation $\mathbf{x}(i,j,1-x_{ij})$ by $\mathbf{x}(i \rightsquigarrow j)$ and the current situation formally by $\mathbf{x} = \mathbf{x}(i \rightsquigarrow i)$. The probabilities for the new situation then are given by the multinomial logit expressions (cf. Maddala, 1983)

$$p_{ij}(\mathbf{x}) = \frac{\exp\big(f_i(\mathbf{x}(i \rightsquigarrow j)) - f_i(\mathbf{x})\big)}{\sum_{h=1}^{n} \exp\big(f_i(\mathbf{x}(i \rightsquigarrow h)) - f_i(\mathbf{x})\big)} , \tag{9}$$

where $j = i$ formally refers to keeping the existing situation unchanged. The intensity matrix (1) can be written as

$$q_{ij}(\mathbf{x}) = \rho \, p_{ij}(\mathbf{x}) . \tag{10}$$

3.3. Model extensions

The basic model specifications defined above can be extended in various ways. One possible extension is to let the rates of change depend on covariates or on current network structure. Another possibility is to introduce an asymmetry between the values of ties when they are formed and their values when they are lost. E.g., for friendship dynamics, there is theoretical and empirical evidence that the additional value of a tie added by its being reciprocated is higher when considering a potential loss of the tie than when considering the potential new formation of the tie. Such extensions are discussed in Snijders (2001, 2005).

For the model specification it should be noted that the "social time" which determines the speed of change of the network is not necessarily the same as the physical time elapsing between consecutive observation moments. Given the absence of the extraneous definition of this "social time", it is not a restriction to set to 1 the total time elapsed between each pair of consecutive observations. If there are $M \geq 3$ observation moments, it is advisable to specify distinct rate parameters ρ_m governing the frequency of opportunities for change between t_m and t_{m+1}. Accordingly, the symbol ρ will denote the vector $(\rho_1, \ldots, \rho_{M-1})$. Then ρ_m denotes the expected number of opportunities for change between t_m and t_{m+1}; per ordered pair (i,j) in the case of tie-based opportunities, and per actor i in the case of actor-based opportunities.

4. Degeneracy

A fundamental difficulty with the Exponential Random Graph Models (ERGMs) that are so closely related to the tie-based models is the degeneracy which is basically the same as the phenomenon of phase changes in physical models (e.g., Newman and Barkema, 1999) and discussed for the ERGM case in Snijders (2002), Handcock (2003), Snijders *et al.* (2006), and other references cited there. A simple example of this degeneracy arises as follows. The empirical phenomenon of (imperfect) transitivity of relations can be reflected by incorporating the number of transitive triplets (s_6 in (5) or (7)) as a term in the potential or objective function. This term will receive a positive weight β_6. If this weight is not too small, the tie-based dynamic model, which is one of the usual algorithms for obtaining random samples from the ERGM distribution, will have a rather high probability to produce a complete graph (i.e., a digraph with $X_{ij} = 1$ for all $i \neq j$) within a limited amount of time. The complete graph is a quasi-absorbing state in the sense that the probability to loose more than a few ties within a very long time period is negligible. This means that the probability distribution is concentrated on a very small and not practically meaningful set of outcomes, and it renders such specifications of the ERGM meaningless as a statistical model for non-trivial network data. This is the motivation for the proposal of other specifications in Snijders *et al.* (2006).

The same phenomenon can be observed for the actor-oriented models. It is not as detrimental for modeling longitudinal network data, however, as it is for modeling cross-sectional network data. The longitudinal model does not make the assumption of stationarity of the Markov chain. The probability of the sample path leading to a complete or nearly complete graph within the time frame of the observations will usually be negligible for reasonable parameter estimates. Even if the parameter estimates yield a limiting distribution that is nearly degenerate, this is of no practical concern for longitudinal modeling because it refers to an extrapolation usually to the far future.

5. Estimation

These models can be simulated on computers in rather straightforward ways (cf. Snijders, 2005). Parameter estimation, however, is more complicated, because the likelihood function or explicit probabilities can be computed only for uninteresting models. This section presents the Method of Moments estimates proposed in Snijders (2001). Work is under way on development of Maximum Likelihood estimators. In the following, the parameter vector (ρ, β) is denoted by θ.

It is undesirable in practice to make the assumption that the distribution of the process is stationary. Instead, for each observation moment t_m ($m \leq M-1$) the observed network $\mathbf{x}(t_m)$ can be used as a conditioning event for the distribution of $\mathbf{X}(t_{m+1})$. The Method of Moments requires that a vector of statistics $U_{m+1} = U(\mathbf{X}(t_m), \mathbf{X}(t_{m+1}))$ is utilized, such that the expected value $\mathrm{E}_\theta U(\mathbf{X}(t_m), \mathbf{X}(t_{m+1}))$ is sensitive to the parameter θ. Given the conditioning, the moment equations, or estimating equations, can then be written as

$$\sum_{m=1}^{M-1} \mathrm{E}_\theta\{U(\mathbf{X}(t_m), \mathbf{X}(t_{m+1})) \mid \mathbf{X}(t_m) = \mathbf{x}(t_m)\} \;=\; \sum_{m=1}^{M-1} U(\mathbf{x}(t_m), \mathbf{x}(t_{m+1})) \;. \tag{11}$$

It turns out that suitable statistics are the following. The number of changed ties between consecutive observations,

$$\sum_{i,j} |X_{ij}(t_{m+1}) \;-\; X_{ij}(t_m)| \;,$$

is especially sensitive to the rate of change ρ_m. Statistics sensitive especially to β are for the tie-oriented model the potential function

$$f(\mathbf{X}(t_{m+1}))$$

and for the actor-oriented models the sum of the individual objective functions

$$\sum_i f_i(\mathbf{X}(t_{m+1})) \ .$$

To solve the estimating equation (11), in the absence of ways to calculate analytically the expected values, stochastic approximation methods can be used. Variants of the Robbins-Monro (1951) algorithm have been used with good success. This is a stochastic iteration method which produces a sequence of estimates $\theta^{(N)}$ which is intended to converge to the solution of (11). Denote the observed networks by $\mathbf{x}(t_m)$ for $1 \leq m \leq M$. For a given provisional estimate $\theta^{(N)}$, the model is simulated so that for each $m = 1, \ldots, M-1$, a simulated random draw is obtained from the conditional distribution of $\mathbf{X}(t_{m+1})$ conditional on $\mathbf{X}(t_m) = \mathbf{x}(t_m)$. This simulated network is denoted $\mathbf{X}^{(N)}(t_{m+1})$. Denote $U_m^{(N)} = U(\mathbf{x}(t_m), \mathbf{X}^{(N)}(t_{m+1}))$, and $U^{(N)} = \sum_{m=1}^{M-1} U_m^{(N)}$, and let u^{obs} be the right-hand side of (11). Then the iteration step in the Robbins-Monro algorithm for obtaining the Method of Moments estimate is given by

$$\theta^{(N+1)} = \theta^{(N)} - a_N \, D^{-1} \left(U^{(N)} - u^{\text{obs}} \right) \ , \tag{12}$$

where D is a suitable matrix and a_N a sequence of positive constants tending to 0. Tuning details of the algorithm are given in Snijders (2001). The experience with the convergence of this algorithm is quite good. The standard errors can be computed using the standard formulae of standard errors for the Method of Moments, based on the delta method, and applying simulation methods; also see Schweinberger and Snijders (2006).

6. Example

The Electronic Information Exchange System (EIES) data introduced above were analyzed using the three models introduced above: (A) purely tie-oriented; (B) tie-based opportunities with actor-oriented dynamics; (C) purely actor-oriented. For each model, estimates according to three specifications were obtained : a pure similarity specification, where only the similarity of the discipline plays a role; a purely structural specification – i.e., a model driven only by network structure; and a specification that combines network structure and disciplinary similarity. The disciplinary similarity is expressed by dyadic covariates defined as $w_{hij} = 1$ if actors i and j both have the discipline sociology (for $h = 1$), anthropology (for $h = 2$), mathematics/statistics (for $h = 2$), or psychology (for $h = 4$); and $w_{hij} = 0$ otherwise.

In all models a term representing the value of the number of ties is included. This term must be present in any model to make it meaningful, and can be compared functionally to an intercept term in a regression model. The structural network effects represent the value of reciprocated ties, of transitive ties, and of ties to popular others, popularity being measured by the square root of the actor's indegree. The square is taken because it is plausible that the value of a friendship tie to a popular person has decreasing marginal returns when popularity is measured by the indegree. Preliminary analyses showed that counts of the other configurations in Figure 2 did not need to be included.

For the tie-oriented model, the potential function is

$$f(\mathbf{x}) = \beta_1 \sum_{i,j} x_{ij} + \sum_{h=1}^{4} \beta_{h+1} \sum_{i,j} w_{hij} + \beta_6 \sum_{i,j} x_{ij}\, x_{ji} + \beta_7 \sum_{i,j,h} x_{ij}\, x_{jh}\, x_{ih} + \beta_8 \sum_{i,j} x_{ij} \sqrt{\sum_h x_{hj}}\, . \tag{13}$$

The parameters β_2 to β_5 measure the discipline-specific value of having a tie between actors with the same discipline. Parameter β_6 measures the value of tie reciprocation, β_7 of transitivity, and β_8 of popularity. The three specifications are defined by setting parameters β_6 to β_8 to 0 for the first specification, and β_2 to β_5 for the second. The effects can be tested by approximate t-tests, which may be called Wald-type tests, the test statistic being the t-ratio for an estimated parameter. A parameter will be interpreted as significantly different from 0 if in absolute value it is as least twice as large as its standard error.

Table 1 presents the results for the purely tie-oriented model. It may be noted that for Model 1, the processes for the dyads $(X_{ij}(t), X_{ji}(t))$ are independent; this is a reparametrisation of the reciprocity model of Wasserman (1980) and Leenders (1995); also see Snijders (2005). The results for the model with tie-based opportunities for change and actor-oriented choice are hardly different, and are not reported here.

Table 1: *Parameter estimates: purely tie-oriented model.*

	Model 1		Model 2		Model 3	
effect	$\hat{\beta}_k$	s.e.	$\hat{\beta}_k$	s.e.	$\hat{\beta}_k$	s.e.
rate	0.16	0.03	0.32	0.05	0.32	0.05
outdegree	0.07	0.37	-6.22	1.82	-6.45	2.00
both sociology	0.08	0.54			-0.08	0.69
both anthropology	0.60	0.55			0.61	0.78
both maths/stats	1.58	0.73			2.82	1.78
both psychology	0.90	0.55			0.38	0.77
reciprocity			4.01	1.85	4.06	2.24
transitivity			0.35	0.20	0.42	0.25
popularity			1.00	0.38	0.97	0.39

The outdegree effect is much lower in Models 2 and 3 than in Model 1 because it is compensated by the four similarity effects. Of the four similarity effects, we see that the preference for a friend from the same discipline seems strongest for the mathematicians/statisticians, and the t-ratio leads to a significant result for this effect in Model 1 but not in Model 3, which includes the control for structural network effects. The tendency toward reciprocity is significant in Model 2 but not in Model 3. The tendency toward transitivity is not significant; the preference for popular friends is significant.

The objective function for the actor-oriented model mirrors the potential function (13), but is defined from the actor's viewpoint. In network terminology, it is a function of the personal network of actor i, defined here as the induced digraph where the set of actors is the set of all those whose geodesic distance from i is at most 2. The objective function is

$$\begin{aligned} f_i(\mathbf{x}) \; &= \; \beta_1 \sum_j x_{ij} \, + \, \sum_{h=1}^{4} \beta_{h+1} \sum_j w_{hij} \, + \\ & \beta_6 \sum_j x_{ij} \, x_{ji} \, + \, \beta_7 \sum_{j,h} x_{ij} \, x_{jh} \, x_{ih} \, + \, \beta_8 \sum_j x_{ij} \sqrt{\sum_h x_{hj}} \, . \end{aligned} \tag{14}$$

Table 2 presents results for the actor-oriented model. Since the process of opportunities for change is different and the choice situation is different, the estimates are not comparable to those of Table 1. However, conceptually the tests of whether the parameters differ from 0 do have similar interpretations as for the tie-oriented models. Here again the only similarity effect that at least comes near significance is the similarity preference among mathematicians/statisticians. Contrasting with the tie-oriented results, this is significant for both Models 4 and 6. Also in contrast to Table 1, the effects of reciprocity, transitivity, and popularity all are significant both in Model 5 and in Model 6.

Table 2: *Parameter estimates: actor-oriented model.*

	Model 4		Model 5		Model 6	
effect	$\hat{\beta}_k$	s.e.	$\hat{\beta}_k$	s.e.	$\hat{\beta}_k$	s.e.
rate	2.46	0.31	2.63	0.35	2.58	0.34
outdegree	0.08	0.21	-2.63	0.47	-2.71	0.52
both sociology	-0.03	0.27			-0.05	0.29
both anthropology	0.36	0.33			0.15	0.33
both maths/stats	0.97	0.42			1.14	0.43
both psychology	0.51	0.35			0.12	0.37
reciprocity			1.50	0.34	1.47	0.35
transitivity			0.15	0.06	0.17	0.07
popularity			0.42	0.12	0.43	0.13

Whether the tie-oriented or the actor-oriented model provides a better representation of the data is hard to determine at this moment. These models are not nested in one another, and currently no methods are available for estimating likelihoods or otherwise assessing the overall fit of the model. A robust conclusion that arises from the application of both models is that there is no evidence for friendship preference for others of the same discipline for sociologists, anthropologists, or psychologists; and there is evidence for a preference for popular others. It remains an open question whether the significant results under the actor-oriented model for reciprocity, transitivity, and preference among mathematicians/statisticians for those of the same discipline, are plausible evidence, and this will have to be decided by further study of the relative fit of these two – or other – models.

7. Discussion

Although the statistical modeling of network dynamics started already with Holland and Leinhardt (1977) and Wasserman (1980), this area has been in rapid development only since recent years. The availability of methods for analysis has been a stimulus also for the collection of longitudinal network data. Plausible models and good methods for parameter estimation and testing have been developed, and are available in the *SIENA* program (Snijders *et al.*, 2005). However, the currently available array of procedures still needs to be extended. Work is now being done to develop Maximum Likelihood and Bayesian estimators (Koskinen and Snijders, 2006), which will be a useful supplement to the Method of Moments estimators described above. This will also allow the study of the efficiency of the latter estimators. Some limited simulation studies have supported the Wald-type tests used here; score-type tests associated to the Method of Moments estimators were developed by Schweinberger (2006). The example in this paper underlines the need for methods to assess fit of models, and compare non-nested models. This could be done formally based on estimated likelihoods, or informally based on the comparison of observed and expected values of relevant statistics that are not used for parameter estimation.

The open question of assessing fit also invites speculation about the robustness of the results against the use of models of which the fit is not beyond doubt. There is an inherent tension between the complexity of processes of network dynamics, and the limited amount of data that can in practice be observed concerning these processes. One issue is that the models proposed here are Markov processes. For a two-wave data set such as the EIES data there are no clear alternatives to making such an assumption, but the assumption is certainly debatable. Including more information in the state space (by covariates, by considering valued rather than dichotomous ties, etc.) can relax the doubts concerning such an assumption. Another issue is the difference between the tie-oriented and actor-oriented models. Which type of model is to be preferred is a matter both of social science theory and of empirical fit. It will be important to know, supported by simulation studies and/or mathematical results, the extent to which results based on particular models for network dynamics are robust to deviations from the precise assumptions made. In addition, it will be useful to develop still other models, e.g., models accounting for actor heterogeneity (like were developed for non-longitudinal network data, e.g., by Nowicki and Snijders, 2001; Hoff, Raftery, and Handcock, 2002; or van Duijn, Snijders, and Zijlstra, 2004) or measurement error. Models where not only the network but also actor characteristics are the dynamic dependent variables are also in development (Steglich, Snijders, and Pearson, 2006).

Other open questions are about mathematical properties of the estimators and tests proposed. Simulation studies support the conjecture that the Method of Moments estimators have asymptotically normal distributions, but this has not been proven. It is unknown if the solution to the moment equation (11), under certain conditions, is unique. Similar questions can be asked about the Maximum Likelihood estimators. All this indicates that there is ample scope for future work on methods of statistical inference for network dynamics.

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