Time Series Solution Sketches Sheet 1 HT 2010

1. Let $\{X_t\}$ be the ARMA(1, 1) process,

$$X_t - \phi X_{t-1} = \epsilon_t + \theta\epsilon_{t-1}, \qquad \{\epsilon_t\} \sim \text{WN}(0, \sigma^2),$$

where $|\phi| < 1$ and $|\theta| < 1$. Show that the acf of $\{X_t\}$ is given by

$$\rho(1) = \frac{(1+\phi\theta)(\phi+\theta)}{1+\theta^2+2\phi\theta}, \qquad \rho(h) = \phi^{h-1}\rho(1) \quad \text{for } h \geqslant 1.$$

Solution: From $E(X_t) = \phi E(X_{t-1})$, and using $\phi < 1$ and stationarity we get $E(X_t) = E(X_{t-1}) = 0$.

For $k \geqslant 2$: multiplying $X_t = \phi X_{t-1} + \epsilon_t + \theta\epsilon_{t-1}$ by X_{t-k} and taking expectations we get $\gamma_k = \phi\gamma_{k-1}$, and hence $\gamma_k = \phi^{k-1}\gamma_1$ for $k \geqslant 2$.

Multiplying the same equation by X_t and taking expectations we get

$$\gamma_0 = \phi\gamma_1 + E[X_t(\epsilon_t + \theta\epsilon_{t-1})]$$

and

$$\begin{aligned} X_t &= \phi X_{t-1} + \epsilon_t + \theta\epsilon_{t-1} = \phi[\phi X_{t-2} + \epsilon_{t-1} + \theta\epsilon_{t-2}] + \epsilon_t + \theta\epsilon_{t-1} \\ &= \phi^2 X_{t-2} + \phi\epsilon_{t-1} + \phi\theta\epsilon_{t-2} + \epsilon_t + \theta\epsilon_{t-1} \end{aligned}$$

so

$$\begin{aligned} \gamma_0 &= \phi\gamma_1 + E[(\phi^2 X_{t-2} + \phi\epsilon_{t-1} + \phi\theta\epsilon_{t-2} + \epsilon_t + \theta\epsilon_{t-1})(\epsilon_t + \theta\epsilon_{t-1})] \\ &= \phi\gamma_1 + \sigma^2[\phi\theta + 1 + \theta^2]. \end{aligned}$$

Also

$$\begin{aligned} \gamma_1 &= E(X_t X_{t+1}) = E[X_t(\phi X_t + \epsilon_{t+1} + \theta\epsilon_t)] \\ &= \phi\gamma_0 + E[(\phi X_{t-1} + \epsilon_t + \theta\epsilon_{t-1})(\epsilon_{t+1} + \theta\epsilon_t)] = \phi\gamma_0 + \theta\sigma^2. \end{aligned}$$

We can now solve the two equations involving γ_0, γ_1, and then find γ_k, and hence ρ_k, as required.

2. Consider a process consisting of a linear trend plus an additive noise term,

$$X_t = \beta_0 + \beta_1 t + \epsilon_t$$

where β_0 and β_1 are fixed constants, and where the ϵ_t are independent random variables with zero means and variances σ^2. Show that X_t is non-stationary, but that the first difference series $\nabla X_t = X_t - X_{t-1}$ is second-order stationary, and find the acf of ∇X_t.

Solution: $E(X_t) = E(\beta_0 + \beta_1 t + \epsilon_t) = \beta_0 + \beta_1 t$ which depends on t, hence X_t is non-stationary.

Let $Y_t = \nabla X_t = X_t - X_{t-1}$. Then

$$Y_t = \beta_0 + \beta_1 t + \epsilon_t - \{\beta_0 + \beta_1(t-1) + \epsilon_{t-1}\} = \beta_1 + \epsilon_t - \epsilon_{t-1}.$$

So

$$\begin{aligned} \text{cov}(Y_t, Y_{t+k}) &= \text{cov}(\epsilon_t - \epsilon_{t-1}, \epsilon_{t+k} - \epsilon_{t+k-1}) \\ &= E(\epsilon_t\epsilon_{t+k} - \epsilon_{t-1}\epsilon_{t+k} - \epsilon_t\epsilon_{t+k-1} + \epsilon_{t-1}\epsilon_{t+k-1}) \\ &= \begin{cases} 2\sigma^2 & k = 0 \\ -\sigma^2 & k = 1 \\ 0 & k \geqslant 2. \end{cases} \end{aligned}$$

Hence Y_t is stationary and its acf is

$$\rho_k == \begin{cases} 1 & k = 0 \\ -\frac{1}{2} & k = 1 \\ 0 & k \geqslant 2. \end{cases}$$

3. Let $\{S_t, t = 0, 1, 2, \dots\}$ be the random walk with constant drift μ, defined by $S_0 = 0$ and

$$S_t = \mu + S_{t-1} + \epsilon_t, \qquad t = 1, 2, \dots,$$

where $\epsilon_1, \epsilon_2, \dots$ are independent and identically distributed random variables with mean 0 and variance σ^2. Compute the mean of S_t and the autocovariance of the process $\{S_t\}$. Show that $\{\nabla S_t\}$ is stationary and compute its mean and autocovariance function.

Solution:

$$\begin{aligned} S_t &= \epsilon_t + \mu + S_{t-1} = \epsilon_t + \mu + \epsilon_{t-1} + \mu + S_{t-2} \\ &= \epsilon_t + \epsilon_{t-1} + 2\mu + S_{t-2} = \dots = \sum_{j=0}^{t-1} \epsilon_{t-j} + t\mu + S_0 \end{aligned}$$

So $E(S_t) = 0 + t\mu + 0 = t\mu$. Thus the mean depends on t, the process is not stationary.

For the autocovariance of S_t, the autocovariance at lag k is

$$E[\{S_t - t\mu\}\{S_{t+k} - (t+k)\mu\}] = E(\sum_{j=0}^{t-1} \epsilon_{t-j} \sum_{i=0}^{t+k-1} \epsilon_{t+k-i}) = \sum_{j=0}^{t-1} E(\epsilon_{t-j}\epsilon_{t-j}) = t\sigma^2$$

since, when moving from the first line to the second line of the above display, $E(\epsilon_{t-j}\epsilon_{t+k-i}) = 0$ unless $i = j + k$.

Note: for $k = 0, \dots, t-1$,

$$\begin{aligned} Cov(S_t, S_{t-h}) &= Cov\left(\sum_{j=0}^{t-1} \epsilon_{t-j}, \sum_{i=0}^{t-h-1} \epsilon_{t-i}\right) = \sum_{j=0}^{t-1} \sum_{j=0}^{t-h-1} Cov(\epsilon_{t-j}, \epsilon_{t-i}) \\ &= (t-h)\sigma^2. \end{aligned}$$

$Y_t = \nabla S_t = S_t - S_{t-1} = \mu + \epsilon_t$, which is clearly stationary as the observations are independent; we have $E(Y_t) = \mu$ and $Var(Y_t) = \sigma^2$.

For the autocovariance of Y_t, note $Y_t - \mu = \epsilon_t$, and similarly $Y_{t'} - \mu = \epsilon_{t'}$, and so for $t \neq t'$ each Y_t depends on a different ϵ_t, and therefore cov$(Y_t, Y_{t'}) = 0$ for all $t \neq t'$. So the autocovariance function is σ^2 at lag 0, and is zero at all other lags.

4. If

$$X_t = a\cos(\lambda t) + \epsilon_t$$

where $\epsilon_t \sim \text{WN}(0, \sigma^2)$, and where a and λ are constants, show that $\{X_t\}$ is not stationary.

Solution: $E(X_t) = E(a\cos(\lambda t) + \epsilon_t) = a\cos(\lambda t)$, which depends on t, so X_t is not stationary.