a5 Statistics TT 2003 **Exercises 2** Dr. Gesine Reinert

1. Let $X_{(n)}$ be the largest value in a sample of size n drawn from the uniform distribution on $[0, \theta]$. Show that $\frac{X_{(n)}}{\theta}$ is a pivot. Using this pivot, find the $100(1-\alpha)\%$ confidence interval for θ which has shortest expected length. Discuss how you would test the hypothesis that θ takes a specific value θ_0 for such a sample.

2. An investigator wants to show that first-born children score higher on IQ tests than second-borns. In one school district, he finds 400 two-child families with both children enrolled in elementary school. He gives the children a vocabulary test, consisting of 40 words which the child has to define; 2 points are given for a correct answer, and 1 point for a partially correct answer. The results are that the 400 first-born average 29, and their sample standard deviation is 10; the 400 second-borns average 28, and their sample standard deviation is 10.

   (a) Find a 95% confidence interval for the difference of the means.

   (b) Which assumptions did you make?

   (c) What can you conclude? ? ?

3. Let $X_1, X_2, \ldots, X_n$ be a random sample from the Gamma distribution with parameter α and λ; having density

$$f(x; (\alpha, \lambda)) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}.$$

   (a) Find a method of moment estimator for α and λ.

   (b) Find the maximum likelihood estimator for λ if $\alpha = 1$.

   (c) Discuss how to find the maximum likelihood estimator for (α, λ) in the general case.

4. Let $X_1, X_2, \ldots, X_n$ be a random sample from $\mathcal{N}(\mu, \sigma^2)$ where $\mu = \sigma^2 = \theta$. Show that the maximum likelihood estimator of θ is

$$\hat{\theta}_n = \frac{1}{2}\left\{\left(1 + \frac{4}{n}\sum_{j=1}^{n} X_j^2\right)^{\frac{1}{2}} - 1\right\}.$$

5. Let $X_1, X_2, \ldots, X_n$ be a random sample from the uniform distribution on $(\theta, \theta + 1)$. Find a maximum-likelihood estimator of θ.

6. A standard probability model used for data on wages is the Pareto distribution, which has probability density function

$$f(x;\theta) = \theta\alpha^{\theta}x^{-(\theta+1)}, \quad x \geq \alpha, \theta > 0,$$

where the constant α represents a statutory minimum wage. Find the maximum-likelihood estimator of θ from a random sample of size n.

There is used to be no statutory minimum wage in the UK (now it is £4.20 per hour for adults of age 22 or older, £3.60 per hour for adults age 18-21). Modify your previous result to account for α also being an unknown parameter. Show that

$$P(\hat{\alpha} > y) = \left(\frac{\alpha}{y}\right)^{n\theta},$$

where $\hat{\alpha}$ is the maximum likelihood estimator of α and hence find the expectation and the variance of $\hat{\alpha}$. Is $\hat{\alpha}$ a consistent estimator of α?

7. The (unknown) probability that a drawing pin will fall upwards when it is dropped on the floor is p. To estimate p, a pin is dropped on the floor until it lands upwards the first time; the number of trials needed until it lands upwards the first time is recorded. Find the maximum likelihood estimator for p using these data, and calculate its mean and its variance.