Linear Growth for Greedy Lattice Animals

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Abstract

Let $d \geq 2$, and let $\{X_{\mathbf{v}}, \mathbf{v} \in \mathbb{Z}^d\}$ be an i.i.d. family of non-negative random variables with common distribution F. Let $N(n)$ be the maximum value of $\sum_{\mathbf{v}\in\xi} X_{\mathbf{v}}$ over all connected subsets ξ of $\mathbb{Z}^d$ of size n which contain the origin. This model of "greedy lattice animals" was introduced by Cox et al. (1993) and Gandolfi and Kesten (1994), who showed that if $\mathbb{E}X_{\mathbf{0}}^d(\log^+ X_{\mathbf{0}})^{d+\epsilon} < \infty$ for some $\epsilon > 0$, then $N(n)/n \to N$ a.s. and in $\mathcal{L}_1$ for some $N < \infty$. Using related but partly simpler methods, we derive the same conclusion under the slightly weaker condition that $\int_0^\infty \left(1 - F(x)\right)^{1/d} dx < \infty$, and show that $N \leq c\int_0^\infty \left(1 - F(x)\right)^{1/d} dx$ for some constant c. We also give analogous results for the related "greedy lattice paths" model.

Key words: lattice animals, self-avoiding paths, superadditivity, concentration inequality
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1 Introduction

Let $d \geq 2$, and let $\{X_{\mathbf{v}}, \mathbf{v} \in \mathbb{Z}^d\}$ be an i.i.d. family of non-negative random variables, with common distribution F. For a finite subset ξ of $\mathbb{Z}^d$, the *weight* $S(\xi)$ of ξ is defined by

$$S(\xi) = \sum_{\mathbf{v}\in\xi} X_{\mathbf{v}}. \tag{1.1}$$

A *greedy lattice animal of size* n is a connected subset of $\mathbb{Z}^d$ of size n containing the origin, whose weight is maximal among all such sets. Let $N(n)$ be this maximum weight.

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This model is presented by Cox, Gandolfi, Griffin and Kesten (1993), and a variety of applications in statistical physics, queueing theory and percolation are described. Under the condition that

$$\mathbb{E} X_{\mathbf{0}}^d (\log^+ X_{\mathbf{0}})^{d+\epsilon} < \infty \text{ for some } \epsilon > 0, \tag{1.2}$$

they show that there exists an $N < \infty$ such that

$$\limsup_{n\to\infty} \frac{N(n)}{n} \le N \text{ almost surely.} \tag{1.3}$$

Gandolfi and Kesten (1994) proceed to show that, under the same condition, one in fact has more strongly that there exists an $N < \infty$ such that

$$\frac{N(n)}{n} \to N \text{ almost surely and in } \mathcal{L}_1. \tag{1.4}$$

In this paper, we reproduce the conclusions of Cox et al. (1993) and of Gandolfi and Kesten (1994) under a slightly weaker condition, and in addition obtain an explicit bound for the limit N in terms of the distribution F. The methods used are related to those of the above papers, but are simpler in parts; in particular the need for the rather intricate probability estimates used there is avoided.

Our ultimate result is the following:

Theorem 1.1 *There exists a constant $c < \infty$ such that if F satisfies*

$$\int_0^\infty \Big(1 - F(x)\Big)^{1/d} dx < \infty, \tag{1.5}$$

then there exists an N with

$$\frac{N(n)}{n} \to N \textit{ almost surely and in } \mathcal{L}_1 \tag{1.6}$$

as $n \to \infty$, and

$$N \le c \int_0^\infty \Big(1 - F(x)\Big)^{1/d} dx. \tag{1.7}$$

Condition (1.5) is a touch weaker than (1.2); for example, it is implied by the condition

$$\mathbb{E} X^d (\log^+ X)^{d-1+\epsilon} < \infty \text{ for some } \epsilon > 0. \tag{1.8}$$

See Section 8 for details.

In the first part of our proof (corresponding to the the results of Cox et al., 1993), we derive a bound on $\mathbb{E}N(n)/n$ in the case where F is a Bernoulli distribution (the result is related to the "power law" proved by Lee, 1997b). This allows us to control the effect on $N(n)$ of the tail of F; an exchange of a maximum and an integral yields the boundedness in n of $\mathbb{E}N(n)/n$ for general F satisfying (1.5). Comparing $N(n)$ to a related process with a superadditive property then yields the almost sure boundedness of $N(n)/n$ as in (1.3), and gives a bound of the form of (1.7) for $\limsup N(n)/n$.

From that point on, it would be possible to complete the proof using a modification of the arguments in Gandolfi and Kesten (1994), based on martingale inequalities and the "method of bounded differences". Instead we deduce from the bound of the form (1.7) a rather stronger truncation result than was used there, and base the rest of the proof on a powerful "concentration of measure" result due to Talagrand (1995). We do however follow Gandolfi and Kesten by partitioning the set of feasible lattice animals of a given size n into sets each containing lattice animals of a given width m, $1 \leq m \leq n$, in order to apply superadditivity arguments. (Here the *width* of ξ is one greater than the difference between the maximum and minimum values of the first coordinate $v(1)$ among the members $\mathbf{v}$ of ξ).

In Section 7 we consider the related *greedy lattice path* model which was also treated by Gandolfi and Kesten. Let $M(n)$ be the maximal weight of a self-avoiding path of length n starting at the origin. Theorem 7.1 gives a result analogous to Theorem 1.1, showing convergence of $M(n)/n$ under condition (1.5). Much of the proof carries over directly from that of Theorem 1.1; certain parts are harder because superadditivity arguments are not so easily applicable. We simplify the path decomposition argument which was used to prove the convergence under condition (1.2) in Gandolfi and Kesten (1994).

To our knowledge, the strongest known necessary condition for (1.3) or (1.4) is that $\mathbb{E}X^d < \infty$ (see Proposition 3.4 and the remark which follows). The gap between this and condition (1.5) is discussed in Section 9, along with various models and results related to those mentioned above.

1.1 Notation

We write $\mathbf{0}$ for the origin of $\mathbb{Z}^d$, and $\mathbf{1}$ for the point of $\mathbb{Z}^d$ all of whose coordinates are equal to 1. For $\mathbf{v} \in \mathbb{Z}^d$, we write $v(i)$ for the ith coordinate of $\mathbf{v}$, $1 \leq i \leq d$, and $\|\mathbf{v}\| = \max_{1 \leq i \leq d} |v(i)|$; for $l \in \mathbb{Z}$, we write $l\mathbf{v}$ for the point of $\mathbb{Z}^d$ whose ith coordinate is $lv(i)$ for $1 \leq i \leq d$. For $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^d$, we write $\|\mathbf{u} - \mathbf{v}\|$ for $\max_{1 \leq i \leq d} |u(i) - v(i)|$. For $m \leq n \in \mathbb{Z}$, we write $[m, n]^d$ for the cube $\{\mathbf{v} : m \leq v(i) \leq n, 1 \leq i \leq d\}$ of size $(n - m + 1)^d$, and write $B(\mathbf{v}, m)$

for the cube $\{\mathbf{z} : \|\mathbf{z}-\mathbf{v}\| \le m\}$ of size $(2m+1)^d$.

We regard $\mathbb{Z}^d$ as a graph in the normal way; two points are adjacent iff they are (Euclidean) distance exactly 1 apart; thus any point has exactly $2d$ neighbours.

We assume throughout that $\{X_{\mathbf{v}}, \mathbf{v} \in \mathbb{Z}^d\}$ are i.i.d. and non-negative. We write $\mathbb{P}$ for the probability measure governing $\{X_{\mathbf{v}}\}$, and $\mathbb{E}$ for the expectation with respect to $\mathbb{P}$; sometimes we write $\mathbb{P}_F$ and $\mathbb{E}_F$ when we wish to stress the dependence on the common distribution F of the variables $\{X_{\mathbf{v}}\}$; here $F(x) = \mathbb{P}(X_{\mathbf{0}} \le x)$, $x \ge 0$. We will write $\mathrm{Ber}(p)$ for the Bernoulli distribution with parameter p under which $\mathbb{P}(X=1) = 1-\mathbb{P}(X=0) = p$.

A *lattice animal* is a finite connected subset of $\mathbb{Z}^d$. Let $A(n)$ be the set of lattice animals of size n which contain the point $\mathbf{0}$. Defining the weight $S(\xi)$ of a lattice animal ξ as at (1.1), we have

$$N(n) = \max_{\xi \in A(n)} S(\xi). \tag{1.9}$$

2 Boundedness in Expectation via the Bernoulli Case

The following result is Lemma 1 of Cox et al. (1993), and describes how a lattice animal may be covered by a configuration on a lattice of larger scale:

Lemma 2.1 *Let $1 \le l \le n$ and let $\xi \in A(n)$. Then there exists a sequence $\{\mathbf{u}_0, \dots, \mathbf{u}_r\}$ in $\mathbb{Z}^d$, where $r = \lfloor 2n/l \rfloor$, such that $\mathbf{u}_0 = \mathbf{0}$, such that $\|\mathbf{u}_i - \mathbf{u}_{i-1}\| \le 1$ for all $1 \le i \le r$, and such that*

$$\xi \subset \bigcup_{i=0}^{r} B(l\mathbf{u}_i, 2l).$$

Consider the case where the $X_{\mathbf{v}}$ have Bernoulli(p) distribution. The next result provides a "power law" for the behaviour of $N(n)$ as p becomes small. It will allow us to bound the effect of the tail of F on $\mathbb{E}_F N(n)/n$ when F is a distribution satisfying (1.5).

Lee (1997b, Theorem 2) shows that there exists a constant c such that

$$p^{-1/d} \limsup_{n\to\infty} \frac{N(n)}{n} \le c$$

$\mathbb{P}_{\mathrm{Ber}(p)}$—a.s., for all p. The basis of our argument is similar to that of Lee, but we extend it to provide control over $\mathbb{E}\,N(n)$ which is uniform in n as well as in p:

Proposition 2.2 *There is a constant $c < \infty$ such that, for all $p \in (0,1]$ and all $n \in \mathbb{N}$,*

$$p^{-1/d}\mathbb{E}_{Ber(p)}\frac{N(n)}{n} \leq c.$$

Proof:

If $np^{1/d} \leq 1$, then

$$\begin{aligned} p^{-1/d}\mathbb{E}_{\mathrm{Ber}(p)}\frac{N(n)}{n} &\leq \frac{1}{np^{1/d}}\mathbb{E}_{\mathrm{Ber}(p)}\sum_{\mathbf{v}\in\mathbb{Z}^d:\|\mathbf{v}\|<n} X_{\mathbf{v}} \\ &\leq \frac{(2n)^d p}{np^{1/d}} \\ &= 2^d(np^{1/d})^{d-1} \\ &\leq 2^d. \end{aligned} \tag{2.1}$$

So suppose that $np^{1/d} > 1$. We will apply Lemma 2.1 with $l = \lceil p^{-1/d}\rceil$. Note that the number of sequences $\mathbf{u}_0, \dots, \mathbf{u}_r$ (with $r = \lfloor 2n/l \rfloor \leq 2np^{1/d}$) which satisfy the the properties given in Lemma 2.1 is $3^{dr} \leq 9^{dnp^{1/d}}$, and that for any such sequence, the number of points contained in $\bigcup_{i=0}^r B(l\mathbf{u}_i, 2l)$ is no greater than $(r+1)(4l+1)^d \leq 3np^{1/d}(9p^{-1/d})^d$.

For $s > 0$, we then have

$$\begin{aligned} \mathbb{P}_{\mathrm{Ber}(p)}\left(\frac{N(n)}{np^{1/d}} \geq s\right) &= \mathbb{P}_{\mathrm{Ber}(p)}\left(\max_{\xi\in A(n)}\sum_{\mathbf{v}\in\xi} X_{\mathbf{v}} \geq np^{1/d}s\right) \\ &\leq \mathbb{P}_{\mathrm{Ber}(p)}\left(\max_{\mathbf{u}_0,\dots,\mathbf{u}_r}\sum_{\mathbf{v}\in\bigcup_0^r B(l\mathbf{u}_i,2l)} \geq np^{1/d}s\right) \\ &\leq \sum_{\mathbf{u}_0,\dots,\mathbf{u}_r}\mathbb{P}_{\mathrm{Ber}(p)}\left(\sum_{\mathbf{v}\in\bigcup_0^r B(l\mathbf{u}_i,2l)} \geq np^{1/d}s\right) \\ &\leq \sum_{\mathbf{u}_0,\dots,\mathbf{u}_r} e^{-np^{1/d}s}\mathbb{E}_{\mathrm{Ber}(p)}\left[\exp\left(\sum_{\mathbf{v}\in\bigcup_0^r B(l\mathbf{u}_i,2l)} X_{\mathbf{v}}\right)\right] \\ &= \sum_{\mathbf{u}_0,\dots,\mathbf{u}_r} e^{-np^{1/d}s}\left[\mathbb{E}_{\mathrm{Ber}(p)}e^{X_{\mathbf{0}}}\right]^{\left|\bigcup_0^r B(l\mathbf{u}_i,2l)\right|} \\ &\leq \sum_{\mathbf{u}_0,\dots,\mathbf{u}_r} e^{-np^{1/d}s}\left[\mathbb{E}_{\mathrm{Ber}(p)}e^{X_{\mathbf{0}}}\right]^{3np^{1/d}(9p^{-1/d})^d} \\ &= \sum_{\mathbf{u}_0,\dots,\mathbf{u}_r} e^{-np^{1/d}s}(1-p+pe)^{3np^{1/d}(9p^{-1/d})^d} \\ &\leq 9^{dnp^{1/d}}e^{-np^{1/d}s}\left((1-p+pe)^{1/p}\right)^{3np^{1/d}9^d} \end{aligned}$$

$$
\begin{aligned}
&\leq 9^{dnp^{1/d}} e^{-np^{1/d}s}\left(e^{(e-1)}\right)^{3np^{1/d}9^d} \\
&= e^{-np^{1/d}s}\left[9^d e^{3\times 9^d(e-1)}\right]^{np^{1/d}}.
\end{aligned}
$$

Now take y large enough that $e^{-y}9^d e^{3\times 9^d(e-1)} \leq 1$. (This condition does not depend on n or on p). Then

$$
\begin{aligned}
p^{-1/d}\mathbb{E}_{\mathrm{Ber}(p)}\left(\frac{N(n)}{n}\right) &\leq y + \mathbb{E}_{\mathrm{Ber}(p)}\left[\frac{N(n)}{np^{1/d}} - y\right]_+ \\
&= y + \int_y^\infty \mathbb{P}_{\mathrm{Ber}(p)}\left(\frac{N(n)}{np^{1/d}} \geq s\right) ds \\
&\leq y + \frac{1}{np^{1/d}} e^{-np^{1/d}y}\left[9^d e^{3\times 9^d(e-1)}\right]^{np^{1/d}} \\
&= y + \frac{1}{np^{1/d}}\left[e^{-y}9^d e^{3\times 9^d(e-1)}\right]^{np^{1/d}} \\
&\leq y + 1. \qquad (2.2)
\end{aligned}
$$

The right hand sides of (2.1) and (2.2) are independent of p and n, so the desired result follows. □

A straightforward exchange of a maximum and an integral now yields the boundedness of $\mathbb{E}_F N(n)/n$ for all distributions F satisfying (1.5):

Theorem 2.3 *There is a constant $c < \infty$ such that, for all distributions F satisfying (1.5),*

$$
\sup_n \mathbb{E}_F \frac{N(n)}{n} \leq c\int_0^\infty (1 - F(x))^{1/d} dx. \qquad (2.3)
$$

Proof: For any lattice animal ξ, we have

$$
\begin{aligned}
S(\xi) &= \sum_{\mathbf{v}\in\xi} X_{\mathbf{v}} \\
&= \int_0^\infty \#\{\mathbf{v} \in \xi : X_{\mathbf{v}} > x\} dx.
\end{aligned}
$$

Then

$$
\begin{aligned}
N(n) &= \max_{\xi\in\mathcal{A}(n)} S(\xi) \\
&= \max_{\xi\in\mathcal{A}(n)} \int_0^\infty \#\{\mathbf{v} \in \xi : X_{\mathbf{v}} > x\} dx \\
&\leq \int_0^\infty \Big[\max_{\xi\in\mathcal{A}(n)} \#\{\mathbf{v} \in \xi : X_{\mathbf{v}} > x\}\Big] dx. \qquad (2.4)
\end{aligned}
$$

Since the integrand is always non-negative, and then since the random variables $\{I(X_{\mathbf{v}} > x), \mathbf{v} \in \mathbb{Z}^d\}$ are i.i.d. with common distribution $\mathrm{Ber}(1 - F(x))$, we have

$$\begin{aligned}
\mathbb{E}_F N(n) &\leq \int_0^\infty \Big[\mathbb{E}_F \max_{\xi \in \mathcal{A}(n)} \sharp\{\mathbf{v} \in \xi : X_{\mathbf{v}} > x\}\Big] dx \\
&= \int_0^\infty \Big[\mathbb{E}_{\mathrm{Ber}(1-F(x))} N(n)\Big] dx \\
&\leq \int_0^\infty [cn(1 - F(x))^{1/d}] dx,
\end{aligned}$$

where c is the constant established in Proposition 2.2, giving (2.3) as required.

□

3 Almost Sure Boundedness via Superadditivity

For $m, n \in \mathbb{Z}$, $m < n$, let $Q(m, n)$ be the maximum weight of a lattice animal of size not more than $(d + 1)(n - m)$, contained in the cube $[m, n]^d$ of size $(n - m + 1)^d$, and including the point $m\mathbf{1}$ and a point adjacent to $n\mathbf{1}$, but not including the point $n\mathbf{1}$ itself. Certainly the set of lattice animals described in this definition is non-empty, since there are paths of length $d(n - m) + 1$ from $m\mathbf{1}$ to $n\mathbf{1}$ contained in $[m, n]^d$.

The following properties are immediate from the definition of $Q(m, n)$:

Non-negativity:

$$Q(m, n) \geq 0 \text{ for all } m < n; \tag{3.1}$$

Stationarity:

The collections $\{Q(m, n), m < n\}$ and $\{Q(m + 1, n + 1), m < n\}$ have the same joint distributions; (3.2)

Superadditivity:

$$Q(l, m) + Q(m, n) \leq Q(l, n) \text{ for all } l < m < n. \tag{3.3}$$

We will use the collection Q as both an upper bound and a lower bound for the process N:

Lemma 3.1

(i) $Q(0,n) \le N((d+1)n)$ *for all* n.
(ii) $N(n) \le Q(-n,n)$ *for all* n.

Proof: Part (i) follows immediately from the definition of $Q(0,n)$. For part (ii), note that there are paths of length $2dn$ in $[-n,n]^d \setminus \{n\mathbf{1}\}$ going from the point $-n\mathbf{1}$ to a point adjacent to $n\mathbf{1}$, and passing through the point $\mathbf{0}$. If ξ is any lattice animal in the set $A(n)$ (i.e. a lattice animal of size n which contains $\mathbf{0}$), then the union of ξ and such a path is a lattice animal of size no more than $(2d+1)n \le (d+1)2n$, contained in $[-n,n]^d \setminus \{n\mathbf{1}\}$ and including the point $-n\mathbf{1}$ and a point adjacent to $n\mathbf{1}$. The result follows. □.

Lemma 3.2 *There exists* $q \in [0,\infty]$ *such that:*

$$\text{(i)}\ \lim_{n\to\infty} \frac{\mathbb{E}Q(0,n)}{n} = q;$$
$$\text{(ii)}\ \lim_{n\to\infty} \frac{Q(0,n)}{n} = q \text{ a.s. and in } \mathcal{L}_1;$$
$$\text{(iii)}\ \lim_{n\to\infty} \frac{Q(-n,n)}{2n} = q \text{ a.s. and in } \mathcal{L}_1.$$

If (1.5) holds, then

$$q \le (d+1)c\int_0^\infty \Big(1-F(x)\Big)^{1/d} dx, \tag{3.4}$$

where c *is the constant given by Theorem 2.3.*

Proof: Parts (i) and (ii) follow from properties (3.1), (3.2) and (3.3) and the fact that $\{X_{\mathbf{v}}, \mathbf{v} \in \mathbb{Z}^d\}$ are independent, using Kingman's subadditive ergodic theorem, (or rather a superadditive version of it). Part (iii) follows similarly from a "two-sided" version of the same theorem – which is, for example, a special case of Theorem 2.7 of Akcoglu and Krengel (1981). The bound (3.4) is implied by part (i) of Lemma 3.1 and by Theorem 2.3. □

We can now obtain immediately the almost sure boundedness of $N(n)/n$:

Theorem 3.3 *If (1.5) holds, then*

$$\limsup_{n\to\infty} \frac{N(n)}{n} \le 2(d+1)c\int_0^\infty \Big(1-F(x)\Big)^{1/d} dx,$$

almost surely, where c *is the constant given by Theorem 2.3.*

Proof: The result follows from part (iii) of Lemma 3.2 and part (ii) of Lemma 3.1. □

In passing, we note the following:

Proposition 3.4 *The following are equivalent:*

$$\text{(i)} \limsup_{n\to\infty} \frac{N(n)}{n} = \infty \ a.s.$$
$$\text{(ii)} \lim_{n\to\infty} \frac{N(n)}{n} = \infty \ a.s.$$
$$\text{(iii)} \limsup_{n\to\infty} \mathbb{E}\,\frac{N(n)}{n} = \infty.$$
$$\text{(iv)} \lim_{n\to\infty} \mathbb{E}\,\frac{N(n)}{n} = \infty.$$

The equivalence follows again from Lemmas 3.1 and 3.2. In particular, it was shown by Cox et al. (1993) that $\mathbb{E}\,X^d = \infty$ is a sufficient condition for (i) to hold – in fact even for the stronger conclusion

$$\limsup_{n\to\infty} \max_{\mathbf{v}\in[-n,n]^d} \frac{X_\mathbf{v}}{n} = \infty \text{ a.s.}$$

to hold – thus the conclusion (ii) (which was proved in the same paper under a stronger condition) is also true whenever $\mathbb{E}\,X^d = \infty$.

4 Truncations

For a lattice animal ξ and for $y > 0$, we define the "y-truncated" weight of ξ by

$$S^{(y)}(\xi) = \sum_{\mathbf{v}\in\xi} \min[X_\mathbf{v}, y], \tag{4.1}$$

and then define

$$N^{(y)}(n) = \max_{\xi\in A(n)} S^{(y)}(\xi). \tag{4.2}$$

From this definition, we have immediately that, for all $n \in \mathbb{N}$ and $y > 0$,

$$N^{(y)}(n) \le N(n) \le N^{(y)}(n) + \max_{\xi\in A(n)} \sum_{\mathbf{v}\in\xi} [X_\mathbf{v} - y]_+. \tag{4.3}$$

The following result gives a bound on the growth rate of the last term on the RHS of (4.3). Under condition (1.5), the RHS of (4.4) will tend to 0 as $y \to \infty$; this will allow us to approximate the quantity $\limsup_{n\to\infty} N(n)/n$ arbitrarily closely by $\limsup_{n\to\infty} N^{(y)}(n)/n$ for appropriate y, and so to work for most of Section 6 with the quantities $X_\mathbf{v}$ replaced by the truncated versions $\min[X_\mathbf{v}, y]$.

Lemma 4.1 *For any $y > 0$,*

$$\limsup_{n\to\infty} \frac{1}{n} \max_{\xi\in A(n)} \sum_{\mathbf{v}\in\xi} [X_{\mathbf{v}} - y]_+ \leq 2(d+1)c \int_y^\infty \left(1 - F(x)\right)^{1/d} dx \ a.s. \tag{4.4}$$

Proof: The expression

$$\max_{\xi\in A(n)} \sum_{\mathbf{v}\in\xi} [X_{\mathbf{v}} - y]_+$$

corresponds to an expression for $N(n)$ in which $X_{\mathbf{v}}$ has been replaced by $[X_{\mathbf{v}} - y]_+$.

Now the random variables $[X_{\mathbf{v}} - y]_+$, $\mathbf{v} \in \mathbb{Z}^d$, are i.i.d. and non-negative with distribution $F^{(>y)}$, where

$$F^{(>y)}(x) = F(x + y),\ x \geq 0.$$

We have

$$\int_0^\infty \left(1 - F^{(>y)}(x)\right)^{1/d} dx = \int_y^\infty \left(1 - F(x)\right)^{1/d} dx,$$

so (4.4) follows directly from Theorem 3.3, applied to the situation where the distibution F is replaced by the distribution $F^{(>y)}$. $\square$

Remark 4.2 Let $f : \mathbb{Z}^d \to \mathbb{R}^+$ be any function such that $f(\mathbf{v}) \to \infty$ as $\|\mathbf{v}\| \to \infty$. Let $\hat{X}_{\mathbf{v}} = \min\{X_{\mathbf{v}}, f(\mathbf{v})\}$. One can deduce straightforwardly from Lemma 4.1 that, under condition (1.5),

$$\frac{1}{n} \max_{\xi\in\mathcal{A}(n)} \sum_{\mathbf{v}\in\xi} (X_{\mathbf{v}} - \hat{X}_{\mathbf{v}}) \to 0$$

a.s. as $n \to \infty$. This was proved by Gandolfi and Kesten (1994) under condition (1.2) for the case $f(\mathbf{v}) = \log(\|\mathbf{v}\| + 1)$.

5 A Concentration Inequality

The following concentration inequality is based on a result of Talagrand (1995). When we use it, $\mathcal{C}$ will correspond to a set of lattice animals each of a given size R, and the variables Y_i will correspond to truncated weights $\min[X_{\mathbf{v}}, y]$. The particular usefulness of the result for our purposes is that the bound provided depends only on R, and is independent of K.

Lemma 5.1 *Let $Y_i, 1 \le i \le K$ be independent random variables, such that*

$$\mathbb{P}(0 \le Y_i \le y) = 1$$

for each i. Let $\mathcal{C}$ be a set of subsets of $\{1, 2, \ldots, K\}$, such that

$$\max_{C \in \mathcal{C}} |C| \le R,$$

and let

$$Z = \max_{C \in \mathcal{C}} \sum_{i \in C} Y_i.$$

Then

$$\mathbb{P}\Big(|Z - \mathbb{E}\, Z| \ge u\Big) \le \exp\left(-\frac{u^2}{16Ry^2} + 64\right). \tag{5.1}$$

Proof: Let M be a median of the random variable Z, and let $s > 0$. Theorem 8.1.1 of Talagrand (1995) then implies that

$$\mathbb{P}\Big(|Z - M| \ge s\Big) \le 4 \exp\left(-\frac{s^2}{4Ry^2}\right). \tag{5.2}$$

We then have

$$\begin{aligned}
|\mathbb{E}\, Z - M| &\le \mathbb{E}\, |Z - M| \\
&= \int_0^\infty \mathbb{P}\Big(|Z - M| \ge s\Big) ds \\
&\le \int_0^\infty 4 \exp\left(-\frac{s^2}{4Ry^2}\right) ds \\
&= 8y\sqrt{R} \int_0^\infty \exp(-x^2) dx \\
&\le 16y\sqrt{R}\, .
\end{aligned}$$

If $u \ge 32y\sqrt{R}$, we can combine this again with (5.2) to give

$$\begin{aligned}
\mathbb{P}\Big(|Z - \mathbb{E}\, Z| \ge u\Big) &\le \mathbb{P}\Big(|Z - M| + |\mathbb{E}\, Z - M| \ge u\Big) \\
&\le \mathbb{P}\Big(|Z - M| \ge u - 16y\sqrt{R}\Big) \\
&\le \mathbb{P}\Big(|Z - M| \ge u/2\Big) \\
&\le 4 \exp\left(-\frac{(u/2)^2}{4Ry^2}\right) \\
&= 4 \exp\left(-\frac{u^2}{16Ry^2}\right)
\end{aligned}$$

$$< \exp\left(-\frac{u^2}{16Ry^2} + 64\right).$$

If $u < 32y\sqrt{R}$, then (5.1) holds trivially, since the RHS is at least 1. $\square$

6 Convergence Almost Surely and in $\mathcal{L}_1$

For $\mathbf{z} \in \mathbb{Z}^d$ and $l, n \in \mathbb{N}$, $l \leq n$, define $\hat{A}(\mathbf{z}, n, l)$ to be the set of lattice animals ξ of size n such that $\mathbf{z} \in \xi$, and such that $\min_{\mathbf{v}\in\xi} v(1) = z(1)$ and $\max_{\mathbf{v}\in\xi} v(1) = z(1) + l - 1$. (Here $v(1)$ and $z(1)$ represent the first coordinates of $\mathbf{v}$ and of $\mathbf{z}$). One could say that $\hat{A}(\mathbf{z}, n, l)$ is the set of lattice animals of size n, of *width* l, and including $\mathbf{z}$ as a *leftmost point.* Classifying lattice animals by their width in this way will enable us to apply arguments based on superadditivity.

For $\alpha \in \mathbb{R}$, $\alpha \geq 1$, $y > 0$, $z \in \mathbb{Z}^d$ and $m \in \mathbb{N}$, define

$$W^{(y)}(\mathbf{z}, m, \alpha) = \max_{\xi\in\hat{A}(\mathbf{z},\lfloor\alpha m\rfloor,m)} S^{(y)}(\xi). \tag{6.1}$$

Here $S^{(y)}(\xi)$ is the y-truncated weight of ξ as defined at (4.1). Thus $W^{(y)}(\mathbf{z}, m, \alpha)$ is the maximum (y-truncated) weight of a lattice animal of size $\lfloor\alpha m\rfloor$ and width m which includes $\mathbf{z}$ as a leftmost point.

For each $\alpha \geq 1$, $y > 0$, we then define

$$W_\alpha^{(y)} = \sup_m \frac{\mathbb{E}W^{(y)}(\mathbf{0}, m, \alpha)}{m}. \tag{6.2}$$

Finally we set

$$N = \sup_{\alpha\geq 1}\sup_{y\geq 0} \frac{W_\alpha^{(y)}}{\alpha}. \tag{6.3}$$

(This supremum will not be finite for all F, but for F satisfying (1.5), we will show that the bound (1.7) holds).

We note the following properties of the quantities $W^{(y)}(\mathbf{z}, m, \alpha)$. Part (iii) corresponds, essentially, to the observation that any lattice animal of size n containing the origin must have a leftmost point somewhere in $[-n, n]^d$ and have width m for some $1 \leq m \leq n$.

Lemma 6.1 *For all $\alpha \geq 1$, $y > 0$:*

(i) For all $m \in \mathbb{N}$, $W^{(y)}(\mathbf{z}, m, \alpha)$ has the same distribution as $W^{(y)}(\mathbf{0}, m, \alpha)$ for all $\mathbf{z} \in \mathbb{Z}^d$.

(ii) For all $m \in \mathbb{N}$,

$$W^{(y)}(\mathbf{0}, m, \alpha) \leq N^{(y)}(\lfloor \alpha m \rfloor). \tag{6.4}$$

(iii) For all $n \in \mathbb{N}$,

$$N^{(y)}(n) \leq \max_{\mathbf{z} \in [-n,n]^d} \max_{1 \leq m \leq n} W^{(y)}(\mathbf{z}, m, n/m). \tag{6.5}$$

Proof: Part (i) follows from the fact that $\{X_{\mathbf{v}}, \mathbf{v} \in \mathbb{Z}^d\}$ are i.i.d. and so certainly stationary – taking $\mathbf{z}$ rather than $\mathbf{0}$ amounts merely to a translation.

Parts (ii) and (iii) follow from the definitions (4.2) and (6.1) and the observations that

$$\hat{A}(\mathbf{0}, \lfloor \alpha m \rfloor, m) \subseteq A(\lfloor \alpha m \rfloor)$$

for all $\alpha \geq 1$, $m \in \mathbb{N}$, and that

$$A(n) \subseteq \bigcup_{\mathbf{z} \in [-n,n]^d} \bigcup_{1 \leq m \leq n} \hat{A}(\mathbf{z}, n, m)$$

for all $n \in \mathbb{N}$, respectively. □

Next we apply the concentration inequality of Section 5 to control the deviation of the quantities $W^{(y)}(\mathbf{z}, m, \alpha)$ from their expectations:

Lemma 6.2 *Let $y > 0$, $\alpha \geq 1$, $m \in \mathbb{N}$, $\mathbf{z} \in \mathbb{Z}^d$. Then*

$$\mathbb{P}\left(\left| W^{(y)}(\mathbf{z}, m, \alpha) - \mathbb{E} W^{(y)}(\mathbf{z}, m, \alpha) \right| \geq u \right) \leq \exp\left(-\frac{u^2}{16 \alpha m y^2} + 64 \right).$$

Proof: We have

$$W^{(y)}(\mathbf{z}, m, \alpha) = \max_{\xi \in \hat{A}(\mathbf{z}, \lfloor \alpha m \rfloor, m)} \sum_{\mathbf{v} \in \xi} \min[X_{\mathbf{v}}, y].$$

Since $\hat{A}(\mathbf{z}, \lfloor \alpha m \rfloor, m)$ is a finite set, and all of the lattice animals contained in it have size no greater than αm, and since $0 \leq \min[X_{\mathbf{v}}, y] \leq y$ for all $\mathbf{v}$, we can apply Lemma 5.1 with $R = \alpha m$ to give the result. □

We now note a superadditivity property of the sequences $\{\mathbb{E} W^{(y)}(\mathbf{0}, m, \alpha), m \in \mathbb{N}\}$:

Lemma 6.3 *For any $y > 0$ and $\alpha \geq 1$, and for all $m_1, m_2 \in \mathbb{N}$,*

$$\mathbb{E}W^{(y)}(\mathbf{0}, m_1 + m_2, \alpha) \geq \mathbb{E}W^{(y)}(\mathbf{0}, m_1, \alpha) + \mathbb{E}W^{(y)}(\mathbf{0}, m_2, \alpha).$$

Proof: Let ξ_1 be a lattice animal in $\hat{A}(\mathbf{0}, \lfloor \alpha m_1 \rfloor, m_1)$ for which $W^{(y)}(\mathbf{0}, m_1, \alpha)$ attains its maximum – (see the definition (6.1)) – and let $\mathbf{z}'$ be a *rightmost point* of ξ_1; i.e. a point in ξ_1 such that $z'(1) = \max_{\mathbf{v} \in \xi_1} v(1) = m_1 - 1$. If there is non-uniqueness in the choice of ξ_1 or of $\mathbf{z}'$, then make the choice by any method which is independent of $\{X_{\mathbf{v}}, \mathbf{v} \in \mathbb{Z}^d, v_1 \geq m_1\}$.

Let $\mathbf{z}''$ be the point obtained by increasing the first coordinate of $\mathbf{z}'$ by 1; then $z''(1) = m_1$, and $\mathbf{z}''$ is adjacent to $\mathbf{z}'$.

Let ξ_2 be a lattice animal in $\hat{A}(\mathbf{z}'', \lfloor \alpha m_2 \rfloor, m_2)$ for which $W^{(y)}(\mathbf{z}'', m_2, \alpha)$ attains its maximum.

Then ξ_1 and ξ_2 are disjoint (since $\xi_1 \subset \{\mathbf{v} : 0 \leq v(1) < m_1\}$ and $\xi_2 \subset \{\mathbf{v} : m_1 \leq v(1) < m_1 + m_2\}$), and $\xi_1 \cup \xi_2$ is a subset of a lattice animal in $\hat{A}(\mathbf{0}, \lfloor \alpha(m_1 + m_2) \rfloor, m_1 + m_2)$, (since $\mathbf{z}' \in \xi_1$ is adjacent to $\mathbf{z}'' \in \xi_2$ and since $\lfloor \alpha(m_1 + m_2) \rfloor \geq \lfloor \alpha m_1 \rfloor + \lfloor \alpha m_2 \rfloor$). Thus we have

$$\begin{aligned} W^{(y)}(\mathbf{0}, m_1 + m_2, \alpha) &\geq S^{(y)}(\xi_1 \cup \xi_2) \\ &= S^{(y)}(\xi_1) + S^{(y)}(\xi_2) \\ &= W^{(y)}(\mathbf{0}, m_1, \alpha) + W^{(y)}(\mathbf{z}'', m_2, \alpha). \end{aligned} \tag{6.6}$$

But by the independence of $\{X_{\mathbf{v}}, v(1) < m_1\}$ and $\{X_{\mathbf{v}}, m_1 \leq v(1) < m_1 + m_2\}$, the random variable $W^{(y)}(\mathbf{z}'', m_2, \alpha)$ has the same distribution as $W^{(y)}(\mathbf{0}, m_2, \alpha)$. Hence taking expecations in (6.6) gives the desired result. $\square$

We combine the previous two lemmas to show that the supremum $W_\alpha^{(y)}$ defined at (6.2) in fact represents the linear growth rate of $W^{(y)}(\mathbf{0}, m, \alpha)$ as m becomes large:

Lemma 6.4 *Let $\alpha \geq 1$, $y > 0$.*

$$\frac{\mathbb{E}W^{(y)}(\mathbf{0}, m, \alpha)}{m} \to W_\alpha^{(y)} \text{ as } m \to \infty. \tag{6.7}$$

and

$$\frac{W^{(y)}(\mathbf{0}, m, \alpha)}{m} \to W_\alpha^{(y)} \text{ almost surely, as } m \to \infty. \tag{6.8}$$

Proof: Property (6.7) follows from the definition (6.2) and from the superadditivity property for the sequence $\{\mathbb{E}W^{(y)}(\mathbf{0}, m, \alpha), m \geq 1\}$ established in

Lemma (6.3), (see for example Walters, 1982, Theorem 4.9).

Now, given any $\epsilon > 0$, let m_0 be large enough that

$$\left|\frac{\mathbb{E}W^{(y)}(\mathbf{0},m,\alpha)}{m} - W_\alpha^{(y)}\right| \le \frac{\epsilon}{2} \text{ for all } m \ge m_0. \tag{6.9}$$

(Such an m_0 exists by (6.7)). Then, for $m \ge m_0$, we can apply Lemma 6.2 and (6.9) to give

$$\begin{aligned}
\mathbb{P}\left(\left|\frac{W^{(y)}(\mathbf{0},m,\alpha)}{m} - W_\alpha^{(y)}\right| \ge \epsilon\right) \\
&\le \mathbb{P}\left(\left|\frac{W^{(y)}(\mathbf{0},m,\alpha)}{m} - \mathbb{E}\,\frac{W^{(y)}(\mathbf{0},m,\alpha)}{m}\right| \ge \frac{\epsilon}{2}\right) \\
&= \mathbb{P}\left(\left|W^{(y)}(\mathbf{0},m,\alpha) - \mathbb{E}\,W^{(y)}(\mathbf{0},m,\alpha)\right| \ge \frac{\epsilon m}{2}\right) \\
&\le \exp\left(-\frac{(\epsilon m/2)^2}{16\alpha m y^2} + 64\right) \\
&= \exp\left(-\frac{\epsilon^2 m}{64\alpha y^2} + 64\right). \qquad (6.10)
\end{aligned}$$

Since the sum of the RHS of (6.10) over all m is finite, we have by Borel-Cantelli that

$$\limsup_{m\to\infty}\left|\frac{W^{(y)}(\mathbf{0},m,\alpha)}{m} - W_\alpha^{(y)}\right| \le \epsilon$$

almost surely. This holds for all $\epsilon > 0$, giving the required result. $\square$

The previous lemma will give us the lower bound that we need. To get a corresponding upper bound, we argue similarly, using the upper bound for $N^{(y)}(n)$ given by Lemma 6.1(iii):

Lemma 6.5

$$\limsup_{n\to\infty}\frac{N^{(y)}}{n} \le N \textit{ a.s., for all } y > 0.$$

Proof: Let $\epsilon > 0$. From Lemma 6.1(iii), we have, for any n,

$$\begin{aligned}
\mathbb{P}\left(\frac{1}{n}N^{(y)}(n) \ge N + \epsilon\right) &\le \mathbb{P}\left(\frac{1}{n}\max_{\mathbf{z}\in[-n,n]^d}\max_{1\le m\le n} W^{(y)}(\mathbf{z},m,n/m) \ge N+\epsilon\right) \\
&\le \sum_{\mathbf{z}\in[-n,n]^d}\sum_{1\le m\le n}\mathbb{P}\left(\frac{1}{n}W^{(y)}(\mathbf{z},m,n/m) \ge N+\epsilon\right).
\end{aligned} \tag{6.11}$$

Now, for any α, y, m,

$$\begin{aligned}\mathbb{E}W^{(y)}(\mathbf{z},m,\alpha) &= \mathbb{E}W^{(y)}(\mathbf{0},m,\alpha)\\ &\leq mW_\alpha^{(y)}\\ &\leq m\alpha N,\end{aligned}$$

from the definitions (6.2) and (6.3) of $W_\alpha^{(y)}$ and N. Thus, for all y, $\mathbf{z}$, m, n,

$$\mathbb{E}W^{(y)}(\mathbf{z},m,n/m) \leq nN. \tag{6.12}$$

Then for all $m \leq n$, $z \in \mathbb{Z}^d$, we can apply Lemma 6.2 with $\alpha = n/m$ to give

$$\begin{aligned}\mathbb{P}\left(\frac{1}{n}W^{(y)}(\mathbf{z},m,n/m) \geq N+\epsilon\right)\\ &\leq \mathbb{P}\left(\frac{1}{n}\left(W^{(y)}(\mathbf{z},m,n/m) - \mathbb{E}W^{(y)}(\mathbf{z},m,n/m)\right) \geq \epsilon\right)\\ &\leq \mathbb{P}\left(\left|W^{(y)}(\mathbf{z},m,n/m) - \mathbb{E}W^{(y)}(\mathbf{z},m,n/m)\right| \geq n\epsilon\right)\\ &\leq \exp\left(-\frac{(n\epsilon)^2}{16ny^2}+64\right)\\ &= \exp\left(-\frac{n\epsilon^2}{16y^2}+64\right).\end{aligned} \tag{6.13}$$

Finally, from (6.11) and (6.13), we have that

$$\begin{aligned}\mathbb{P}\left(\frac{1}{n}N^{(y)}(n) \geq N+\epsilon\right) &\leq \sum_{\mathbf{z}\in[-n,n]^d}\sum_{1\leq m\leq n}\exp\left(-\frac{n\epsilon^2}{16y^2}+64\right)\\ &= n(2n+1)^d\exp\left(-\frac{n\epsilon^2}{16y^2}+64\right).\end{aligned} \tag{6.14}$$

Since the sum of the RHS of (6.14) over all n is finite, we have from Borel-Cantelli that

$$\limsup_{n\to\infty}\frac{1}{n}N^{(y)}(n) \leq N+\epsilon \text{ a.s.}$$

Since this holds for all $\epsilon > 0$, the result follows. $\square$

The bounds we have established from above and below allow us to complete the proof of the main result:

Proof of Theorem 1.1:
We define N as at (6.3). Then, under condition (1.5), the bound (1.7) follows from the domination of the quantities $W^{(y)}$ by the quantities $N^{(y)}$ given

by Lemma 6.1(ii), and the bound on $\mathbb{E}N(n)/n$ (and hence, by (4.3), on $\mathbb{E}N^{(y)}(n)/n$) given by Theorem 2.3. The value of c can be taken as that established in Proposition 2.2.

Now, for all $\alpha \geq 1$, $y \geq 0$,

$$\begin{aligned}
\liminf_{n\to\infty} \frac{N(n)}{n} &\geq \liminf_{n\to\infty} \frac{N\big(\lfloor \alpha \lfloor n/\alpha \rfloor \rfloor\big)}{n} \\
&= \liminf_{n\to\infty} \frac{N\big(\lfloor \alpha \lfloor n/\alpha \rfloor \rfloor\big)}{\alpha \lfloor n/\alpha \rfloor} \\
&\geq \liminf_{m\to\infty} \frac{N\big(\lfloor \alpha m \rfloor\big)}{\alpha m} \\
&\geq \frac{1}{\alpha} \liminf_{m\to\infty} \frac{W^{(y)}(\mathbf{0}, m, \alpha)}{m} \quad \text{(from Lemma 6.1(ii))} \\
&\geq \frac{1}{\alpha} W_\alpha^{(y)} \text{ a.s.} \qquad\qquad \text{(from (6.8))}.
\end{aligned}$$

Thus

$$\liminf_{n\to\infty} \frac{N(n)}{n} \geq \sup_{y,\alpha} \frac{W_\alpha^{(y)}}{\alpha} = N \tag{6.15}$$

as desired.

On the other hand, if (1.5) holds, then, by Lemma 4.1, for any $\epsilon > 0$ we can fix a y such that

$$\limsup_{n\to\infty} \frac{1}{n} \max_{\xi \in A(n)} \sum_{\mathbf{v}\in\xi} [X_{\mathbf{v}} - y]_+ \leq \epsilon$$

almost surely. Then, from (4.3),

$$\begin{aligned}
\limsup_{n\to\infty} \frac{1}{n} N(n) &\leq \limsup_{n\to\infty} \frac{1}{n} N^{(y)}(n) + \epsilon \\
&\leq N + \epsilon,
\end{aligned} \tag{6.16}$$

almost surely, by Lemma 6.5. This holds for all $\epsilon > 0$; combining this with (6.15) gives the a.s. convergence in (1.6).

For the $\mathcal{L}_1$ convergence, note that, by Lemmas 3.1 and 3.2,

$$\frac{1}{n} N(n) \leq \frac{1}{n} Q(-n, n),$$

and that

$$\frac{1}{n} Q(-n, n) \to 2q \text{ in } \mathcal{L}_1, \tag{6.17}$$

where $q < \infty$ under condition (1.5). Thus the dominated convergence theorem and the a.s. convergence already established give the $\mathcal{L}_1$ convergence desired.

□

7 Greedy Self-Avoiding Lattice Paths

A sequence $\pi = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ of n vertices of $\mathbb{Z}^d$ is a *self-avoiding path of length* n if $\mathbf{v}_i \neq \mathbf{v}_j$ for all $i \neq j$ and $\mathbf{v}_i$ is adjacent to $\mathbf{v}_{i+1}$ for $1 \leq i \leq n-1$.

Let $\Pi(n)$ be the set of self-avoiding paths of length n starting at $\mathbf{0}$. For $\pi = (\mathbf{v}_1, \ldots, \mathbf{v}_n) \in \Pi(n)$, we write also π for the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ of size n consisting of the points on the path π. The weight $S(\pi)$ of a path π is then defined by (1.1).

Let

$$M(n) = \max_{\pi \in \Pi(n)} S(\pi). \tag{7.1}$$

$M(n)$ is the weight of a "greedy lattice path" of length n. In this section we prove the following result, which corresponds to Theorem 1.1 for greedy lattice animals.

Theorem 7.1 *If F satisfies (1.5), then there exists M such that*

$$\frac{M(n)}{n} \to M \text{ almost surely and in } \mathcal{L}_1. \tag{7.2}$$

Since $\Pi(n) \subset A(n)$, it's immediate that $M(n) \leq N(n)$, and so $M \leq N$, where $N = \lim_{n\to\infty} N(n)/n$ is established by Theorem 1.1. Thus M will also obey the bound (1.7). Lee (1993) shows that, in fact, the strict inequality $M < N$ holds, except in the special case where the $X_{\mathbf{v}}$ have bounded support and attain their maximum value with probability at least p_c, where p_c is the critical probability for site percolation on $\mathbb{Z}^d$. Lee's results are stated under condition (1.2), but in fact the argument covers any case in which the limits M and N exist almost surely.

We introduce a truncated version of the quantities $M(n)$, as we did for $N(n)$ in Section 4. For $y > 0$, let

$$M^{(y)}(n) = \max_{\pi \in \Pi(n)} S^{(y)}(\pi). \tag{7.3}$$

Note then that as at (4.3), we have

$$\begin{aligned} M^{(y)}(n) \le M(n) &\le M^{(y)}(n) + \max_{\pi \in \Pi(n)} \sum_{\mathbf{v} \in \Pi} [X_{\mathbf{v}} - y]_+ \\ &\le M^{(y)}(n) + \max_{\xi \in A(n)} \sum_{\mathbf{v} \in \xi} [X_{\mathbf{v}} - y]_+, \end{aligned}$$

so that Lemma 4.1 can serve the same purpose as in the previous section.

We follow Gandolfi and Kesten (1994) by considering in particular a subset of $\Pi(n)$ consisting of *cylinder paths*. We call a self-avoiding path a cylinder path if its first point is a leftmost point and its last point is a rightmost point. That is, a self-avoiding path $(\mathbf{v}_1, \dots, \mathbf{v}_n)$ is a cylinder path if $v_1(1) \le v_j(1) \le v_n(1)$ for all $1 \le j \le n$.

Let $C(\mathbf{z}, n, l)$ be the set of self-avoiding cylinder paths of length n and width l which start at $\mathbf{z}$.

Let $\alpha \ge 1$ and $y > 0$. Analogously to the definitions of $W^{(y)}$, $W_\alpha^{(y)}$ and N at (6.1)-(6.3), define

$$R^{(y)}(\mathbf{z}, m, \alpha) = \max_{\pi \in C(\mathbf{z}, \lfloor \alpha m \rfloor, m)} S^{(y)}(\pi). \tag{7.4}$$

and

$$R_\alpha^{(y)} = \sup_m \frac{\mathbb{E}\, R^{(y)}(\mathbf{0}, m, \alpha)}{m}; \tag{7.5}$$

then define

$$M = \sup_{\alpha \ge 1} \sup_{y > 0} \frac{R_\alpha^{(y)}}{\alpha}. \tag{7.6}$$

The quantities $R^{(y)}$ will behave in a conveniently superadditive way, just as the quantites $W^{(y)}$ in the previous section, and the $\liminf$ part of Theorem 7.1 can be established in exactly the same way as that of Theorem 1.1; we will give very brief details. For the $\limsup$ part, we will have to work a little harder than in the previous section, since not every self-avoiding path is a cylinder path, and so in particular there is no upper bound on $M^{(y)}(n)$ in terms of the $R^{(y)}$ which corresponds to the inequality (6.5) between $N^{(y)}(n)$ and the $W^{(y)}$. To complete the argument we give a method for decomposing any lattice path in $\Pi(n)$ into a suitable union of cylinder paths; the method is a simplified version of that used in Gandolfi and Kesten (1994).

7.1 Lower Bound

Lemma 7.2 $R^{(y)}(\mathbf{0}, m, \alpha) \leq M^{(y)}(\lfloor \alpha m \rfloor)$ *for all* $m \in \mathbb{N}$, $\alpha \geq 1$, $y > 0$.

Proof: As in Lemma 6.1(ii), this follows from the fact that $C(\mathbf{0}, \lfloor \alpha m \rfloor, m) \subseteq \Pi(\lfloor \alpha m \rfloor)$. □

Lemma 7.3 *Let* $y > 0$, $\alpha \geq 1$, $m \in \mathbb{N}$, $\mathbf{z} \in \mathbb{Z}^d$. *Then*

$$\mathbb{P}\left(\left|R^{(y)}(\mathbf{z}, m, \alpha) - \mathbb{E}\, R^{(y)}(\mathbf{z}, m, \alpha)\right| \geq u\right) \leq 4 \exp\left(-\frac{u^2}{16 \alpha m y^2} + 64\right). \quad (7.7)$$

Lemma 7.4 *For any* $y > 0$ *and* $\alpha \geq 1$, *and for all* $m_1, m_2 \in \mathbb{N}$,

$$\mathbb{E}\, R^{(y)}(\mathbf{0}, m_1 + m_2, \alpha) \geq \mathbb{E}\, R^{(y)}(\mathbf{0}, m_1, \alpha) + \mathbb{E}\, R^{(y)}(\mathbf{0}, m_2, \alpha).$$

Lemma 7.5 *Let* $\alpha \geq 1$, $y > 0$.

$$\frac{\mathbb{E}\, R^{(y)}(\mathbf{0}, m, \alpha)}{m} \to R_\alpha^{(y)} \text{ as } m \to \infty. \quad (7.8)$$

and

$$\frac{R^{(y)}(\mathbf{0}, m, \alpha)}{m} \to R_\alpha^{(y)} \text{ almost surely, as } m \to \infty. \quad (7.9)$$

Proofs: The proofs of Lemmas 7.3-7.5 are essentially identical to those of Lemmas 6.2-6.4. □

Arguing as at (6.15), we can then derive that

$$\liminf_{n \to \infty} \frac{M(n)}{n} \geq M \text{ a.s.} \quad (7.10)$$

7.2 Upper Bound

The first lemma corresponds to Lemma 9 of Gandolfi and Kesten (1994), and shows that, for large n, all paths of "unusually large" truncated weight contained in $[-n, n]^d$ are fairly short compared to n (of length less than n^δ). This will be used together with the subsequent lemma, which shows that any self-avoiding path of length n can be decomposed into cylinder paths most of which have length at least n^δ.

Lemma 7.6 *Let $0 < \delta < 1$, let $y > 0$, and let $\epsilon > 0$.*

Let $\mathcal{B}(n)$ be the event that there exists a cylinder path π contained in $[-n, n]^d$, with $n^\delta \le |\pi| \le n$, and with $S^{(y)}(\pi) \ge (M + \epsilon)|\pi|$.

Then with probability 1, the event $\mathcal{B}(n)$ occurs for only finitely many n.

Proof: A cylinder path of the type concerned in the event $\mathcal{B}(n)$ has length l for some $n^\delta \le l \le n$, width w for some $1 \le w \le l$, and starting point $\mathbf{z} \in [-n, n]^d$.

Thus we have

$$\mathcal{B}(n) \subseteq \bigcup_{n^\delta \le l \le n} \bigcup_{1 \le w \le l} \bigcup_{\mathbf{z} \in [-n,n]^d} \{R^{(y)}(\mathbf{z}, w, l/w) \ge (M + \epsilon)l\}.$$

Using Lemma 7.3 and the definitions (7.4) and (7.6), we can argue as at (6.13) to get

$$\mathbb{P}\Big(R^{(y)}(\mathbf{z}, w, l/w) \ge (M + \epsilon)l\Big) \le \exp\left(-\frac{\epsilon^2 l}{16y^2} + 64\right)$$

Hence

$$\begin{aligned}\mathbb{P}\big(\mathcal{B}(n)\big) &\le \sum_{n^\delta \le l \le n} \sum_{1 \le w \le l} \sum_{\mathbf{z} \in [-n,n]^d} \exp\left(-\frac{\epsilon^2 l}{16y^2} + 64\right) \\ &\le (2n+1)^{d+2} \exp\left(-\frac{\epsilon^2 n^\delta}{16y^2} + 64\right).\end{aligned}$$

Since the sum of the RHS over all n is finite, the Borel-Cantelli Lemma gives the result. $\square$

Lemma 7.7 *Any self-avoiding path of length n can be represented as the disjoint union of a set of cylinder paths, such that at most $2n^\delta$ paths in the set have length less than n^δ.*

Proof: Let $\pi = (\mathbf{v}_1, \dots, \mathbf{v}_n)$. For the sake of argument, assume that the first-occurring leftmost point of π, say $\mathbf{v}_l$, occurs earlier than the last-occurring rightmost point of π, say $\mathbf{v}_r$. (If not, then reverse the order of the path).

The path π is then the union of an *initial segment* $I = (\mathbf{v}_1, \dots, \mathbf{v}_{l-1})$, a *central segment* $C = (\mathbf{v}_l, \dots, \mathbf{v}_r)$ and a *final segment* $F = (\mathbf{v}_{r+1}, \dots, \mathbf{v}_n)$. The central segment is a cylinder path whose width is the width of π. We take this as the first path in our set.

The first point $\mathbf{v}_{r+1}$ of F is a rightmost point of F. Take the portion of F from $\mathbf{v}_{r+1}$ up to the last-occurring leftmost point of F, say $\mathbf{v}_{l'}$. This portion is itself a cylinder path (viewed in reverse), whose width is the width of F. We add this path to our set. Let F' be the remainder of F, which is $(\mathbf{v}_{l'+1}, \ldots, \mathbf{v}_n)$. The path F' has smaller width than F, and has its first point $\mathbf{v}_{l'+1}$ as a leftmost point. Take the portion of F' from $\mathbf{v}_{l'+1}$ up to the last-occurring rightmost point of F', say $\mathbf{v}_{r'}$. This portion is a cylinder path whose width is the width of F'. Add this path to our set, and let F'' be what remains, which is $(\mathbf{v}_{r'+1}, \ldots, \mathbf{v}_n)$. The path F'' has smaller width than F', and has its first point $\mathbf{v}_{r'+1}$ as a rightmost point. Continue by taking the portion of F'' from $\mathbf{v}_{r'+1}$ up to the last-occurring leftmost point of F'', and so on. Continue this process until the remaining portion is itself a cylinder path.

In this way, F is decomposed into a sequence of cylinder paths with strictly decreasing widths.

Similarly, the initial segment I may be decomposed into a sequence of cylinder paths with strictly decreasing widths.

The central segment C is a cylinder path, and has greater width than any of the cylinder paths comprising F and I.

Hence we have decomposed π into a set of cylinder paths, such that for any w, there are at most two paths in the set with width w. Thus there are fewer than $2n^\delta$ paths in the set with width less than n^δ. Since the length of a path is at least as large as its width, there are fewer than $2n^\delta$ paths in the set with length less than n^δ, as desired. An example of the decomposition is given in Figure 7.1. □

We combine the previous two lemmas to complete the proof:

Proof of Theorem 7.1: Let $0 < \delta < 1/2$, $\epsilon > 0$, and $y > 0$, and define the event $\mathcal{B}(n)$ as in Lemma 7.6.

From Lemma 7.7, any path $\pi \in \Pi(n)$ is the disjoint union of a set of cylinder paths each of which has length at least n^δ, and of a set of single points of size at most $2n^{2\delta}$.

Suppose the event $\mathcal{B}(n)$ does not occur. Then each of these cylinder paths of length at least n^δ has (y-truncated) weight at most $(M + \epsilon)$ times its length. But the y-truncated weight of any of the single points is at most y. Thus

$$\begin{aligned} S^{(y)}(\pi) &\leq (M + \epsilon)|\pi| + 2n^{2\delta} y \\ &= (M + \epsilon)n + 2n^{2\delta} y. \end{aligned}$$

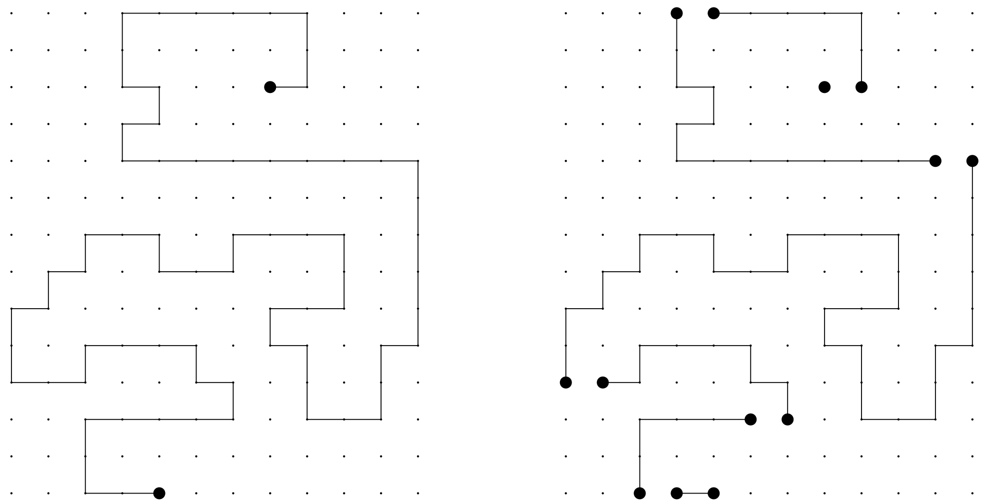

Fig. 7.1. A lattice path and its decomposition into cylinder paths according to the method in the proof of Lemma 7.7. The endpoints of each path are marked. Working from bottom to top, the cylinder paths have widths 2, 4, 6, 12 (the central segment), 8, 5, 1.

Since, by Lemma 7.6, $\mathcal{B}(n)$ almost surely happens only finitely many times, we have

$$\begin{aligned}\limsup_{n\to\infty} \frac{M^{(y)}(n)}{n} &\leq \limsup_{n\to\infty} \frac{1}{n}\Big[(M+\epsilon)n + 2n^{2\delta}y\Big] \\ &= M+\epsilon\end{aligned}$$

almost surely. This holds for all $y > 0$. As at (6.16), it therefore follows under condition (1.5) that

$$\limsup_{n\to\infty} \frac{M(n)}{n} \leq M+\epsilon$$

almost surely, for all ϵ. Combining this with (7.10) gives the a.s. convergence in Theorem 7.1. The $\mathcal{L}_1$ convergence follows by the dominated convergence theorem as at (6.17). □

8 Moment Conditions

In this section we derive sufficient conditions for (1.5) to hold, in terms of the expectation of functions of $X_{\mathbf{0}}$ under F.

Proposition 8.1 *Suppose* $g : (0,\infty) \mapsto (0,\infty)$ *is a function with a strictly* *positive derivative, and with*

$$\int_0^\infty \left(\frac{dg(x)}{dx}\right)^{\left(\frac{-1}{d-1}\right)} dx < \infty.$$

If X is a non-negative random variable with distribution F, then

$$\mathbb{E}g(X) < \infty \Rightarrow \int_0^\infty \Big(1 - F(x)\Big)^{1/d} dx < \infty. \tag{8.1}$$

Proof: Using Hölder's inequality,

$$\begin{aligned}
\int_0^\infty \Big(1 - F(x)\Big)^{1/d} dx &= \int_0^\infty \left[\Big(1 - F(x)\Big)\left(\frac{dg(x)}{dx}\right)\right]^{1/d} \left(\frac{dg(x)}{dx}\right)^{-1/d} dx \\
&\le \left[\int_0^\infty \Big(1 - F(x)\Big)\frac{dg(x)}{dx} dx\right]^{1/d} \left[\int_0^\infty \left(\frac{dg(x)}{dx}\right)^{\frac{-1}{d-1}} dx\right]^{\frac{d-1}{d}} \\
&= \Big[\mathbb{E}g(X) - g(0)\Big]^{1/d} \left[\int_0^\infty \left(\frac{dg(x)}{dx}\right)^{\frac{-1}{d-1}} dx\right]^{\frac{d-1}{d}}.
\end{aligned}$$

□

We use Proposition 8.1 to give a sequence of moment conditions each of which imply (1.5). These conditions become weaker as k increases. For $k = 0$, the condition is simply that $\mathbb{E}X_{\mathbf{0}}^{d+\epsilon}$ is finite; for $k = 1$ we get (1.8), while for $k = 2$ we get the condition

$$\mathbb{E}X_{\mathbf{0}}^d (\log^+ X_{\mathbf{0}})^{d-1} (\log^+ \log^+ X_{\mathbf{0}})^{d-1+\epsilon} < \infty.$$

Proposition 8.2 *Define $l_0(x) = x$, and, for $r \geq 1$, let*

$$l_r(x) = \log^+ l_{r-1}(x).$$

Then for $k \in \mathbb{Z}_+$ and any $\epsilon > 0$, define

$$g_k(x) = x^d \left(\prod_{r=1}^k l_r(x)^{d-1}\right) \Big(l_k(x)\Big)^\epsilon.$$

Then if X is a non-negative random variable with distribution F,

$$\mathbb{E}g(X) < \infty \Rightarrow \int_0^\infty \Big(1 - F(x)\Big)^{1/d} dx < \infty. \tag{8.2}$$

Proof: Fix k and ϵ. $g_k(x)$ is strictly increasing except on some initial interval where it is 0; so there exists $u > 0$ such that $g_k(u) = 1$, such that $0 \leq g_k(x) \leq 1$ for $0 \leq x \leq u$, and such that g_k is strictly increasing on $[u, \infty)$.

We define $\tilde{g}_k(x) = g_k(x)$ for $x > u$, and $\tilde{g}_k(x) = x/u$ for $x \leq u$. Then $\tilde{g}_k(x)$ is continuous and strictly increasing on $[0, \infty)$; also

$$|g_k(x) - \tilde{g}_k(x)| \leq 1 \text{ for all } x,$$

so that

$$\mathbb{E}\, g_k(X) < \infty \Rightarrow \mathbb{E}\, \tilde{g}_k(X) < \infty.$$

Hence, using Proposition 8.1, it will suffice to show that

$$\int_0^\infty \left(\frac{d\tilde{g}_k(x)}{dx}\right)^{\left(\frac{-1}{d-1}\right)} dx < \infty. \tag{8.3}$$

Now

$$\int_0^\infty \left(\frac{d\tilde{g}_k(x)}{dx}\right)^{\left(\frac{-1}{d-1}\right)} dx = \int_0^u \left(\frac{1}{u}\right)^{\left(\frac{-1}{d-1}\right)} dx + \int_u^\infty \left(\frac{dg_k(x)}{dx}\right)^{\left(\frac{-1}{d-1}\right)}.$$

The first term on the RHS is certainly finite, and for the second we have, for $x \geq u$,

$$\begin{aligned}
\frac{dg_k(x)}{dx} &= dx^{d-1}\left(\prod_{r=1}^k l_r(x)^{d-1}\right) l_k(x)^\epsilon + x^d \frac{d}{dx}\left[\left(\prod_{r=1}^k l_r(x)^{d-1}\right) l_k(x)^\epsilon\right] \\
&\geq dx^{d-1}\left(\prod_{r=1}^k l_r(x)^{d-1}\right) l_k(x)^\epsilon \\
&= \left[d^{\left(\frac{-1}{d-1}\right)}\left(\prod_{r=0}^{k-1} l_r(x)^{-1}\right) l_k(x)^{\left(\frac{-\epsilon}{d-1}-1\right)}\right]^{-(d-1)};
\end{aligned}$$

thus

$$\begin{aligned}
\int_u^\infty \left(\frac{dg_k(x)}{dx}\right)^{\left(\frac{-1}{d-1}\right)} dx &\leq d^{\left(\frac{-1}{d-1}\right)} \int_u^\infty \left(\prod_{r=0}^{k-1} l_r(x)^{-1}\right) l_k(x)^{\left(\frac{-\epsilon}{d-1}-1\right)} dx \\
&= d^{\left(\frac{-1}{d-1}\right)} \int_u^\infty \left(-\frac{d-1}{\epsilon}\right) \frac{d}{dx}\left(l_k(x)^{\left(\frac{-\epsilon}{d-1}\right)}\right) dx \qquad (8.4) \\
&= d^{\left(\frac{-1}{d-1}\right)} \left(\frac{d-1}{\epsilon}\right) l_k(u)^{\left(\frac{-\epsilon}{d-1}\right)} \\
&< \infty,
\end{aligned}$$

as required for (8.3). Here to obtain (8.4) we have used the chain rule for differentiation repeatedly, and the fact that for $x \geq u$ and $1 \leq r \leq k$, $l_r(x) = \log l_{r-1}(x)$, so that

$$\frac{d}{dx} l_k(x) = l_{k-1}(x)^{-1} \frac{d}{dx} l_{k-1}(x)$$

$$= \cdots$$
$$= \prod_{r=0}^{k-1} l_r(x)^{-1}.$$

□

9 Further Questions and Comments

(a) What happens when (1.5) does not hold? We have proved that (1.5) is sufficient for the a.s. convergence of $N(n)/n$ in (1.6), and have also seen (Proposition 3.4 and the comment after) that the condition $\mathbb{E}X_{\mathbf{0}}^d < \infty$ is necessary even for the a.s. boundedness as in (1.3). The limiting behaviour of $N(n)/n$ does not seem to be known for any F such that $\mathbb{E}X_{\mathbf{0}}^d < \infty$ but for which (1.5) fails; that is, we do not know of any proof either that (1.5) is not necessary for (1.3) or that $\mathbb{E}X_{\mathbf{0}}^d < \infty$ is not sufficient for (1.6).

If there are in fact F for which (1.5) does not hold, but for which a.s. boundedness or a.s. convergence of $N(n)/n$ does hold, then the loss of power in our argument occurs at (2.4), when the exchange of integral and maximum is performed.

We saw in Proposition 3.4 that $\limsup_{n\to\infty} \mathbb{E}N(n)/n < \infty$ iff $\limsup_{n\to\infty} N(n)/n < \infty$ a.s. (These are also equivalent to the property that $\limsup_{n\to\infty} \mathbb{E}M(n)/n < \infty$, or that $\limsup_{n\to\infty} M(n)/n < \infty$ a.s., as can be seen by comparing $M(n)$ to $Q(0,n)$ and $Q(-n,n)$ as in Lemma 3.1). Could there be distributions F for which these properties hold, but for which a.s. convergence of $N(n)/n$ as in (1.6) or of $M(n)/n$ as in (7.2) fails? Essentially, a result such as Lemma 4.1 is enough to give the a.s. convergence in either case – to apply the methods of Sections 6 and 7 it suffices to have a bound for the LHS of (4.4) which tends to 0 as $y \to \infty$. So if a.s. boundedness holds, but a.s. convergence for $M(n)$ or for $N(n)$ fails, this implies that the LHS of (4.4) is finite for all y, but bounded away from 0 as $y \to \infty$. This seems implausible, but we do not currently have an argument which excludes it. More weakly, is it the case that the LHS of (4.4) tends to 0 whenever $N(n)/n$ (respectively $M(n)/n$) converges a.s.? This would show for example that the a.s. convergence of $N(n)/n$ implies (respectively is implied by) that of $M(n)/n$.

(b) Oriented Lattice Paths. Following on from (a), one can consider models in which the set of feasible configurations is considerably more restricted. For example, let $Y(n)$ be the maximal weight of a path from $\mathbf{0}$ to $n\mathbf{1}$, in which each step must consist of increasing a single coordinate by 1. For $d = 2$, such models are used, for example, in the analysis of systems of queues in tandem (e.g.

see Glynn and Whitt (1991) and Baccelli, Borovkov and Mairesse (2000)). By superadditivity arguments, $Y(n)/n$ converges a.s. to a finite constant whenever $\mathbb{E}Y(n)/n$ is bounded in n. Could there be an F for which this occurs, but for which $\mathbb{E}N(n)/n$ and $\mathbb{E}M(n)/n$ are not bounded?

(c) Continuity of M and N under weak convergence of F. We write $N(F)$ and $M(F)$ for the values of N and M in Theorem 1.1 and Theorem 7.1 which correspond to a given distribution F. Let $\{F_n, n \in \mathbb{N}\}$ be a sequence of distributions which converge weakly to a limiting distribution F as $n \to \infty$. Lee (1997a) shows that $M(F_n) \to M(F)$ and $N(F_n) \to N(F)$ as $n \to \infty$ under the condition that there is a distribution G which stochastically dominates F and all the F_n, and such that (1.2) holds under G.

The distribution G is used as the majorant for an application of the dominated convergence theorem. Using our Theorem 1.1, Lee's argument applies almost identically to give the same conclusion whenever G obeys (1.5).

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