The Inflation Technique for Causal Inference

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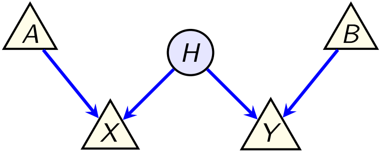
Algebraic Aspects of Causality Reading Group
4th April 2022

Literature

DAG models are often used as causal models, and we may not wish to assume **causal sufficiency** (i.e. all important variables are measured).

The problem of finding constraints in marginalized DAG models has a rich literature. In addition to the work of Robins, Pearl, Geiger, Tian and others on finding **equality** contraints:

Bell (1964) was the first to propose inequalities on a DAG model, which he showed could be violated by quantum models.

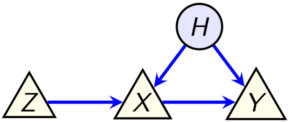


This was followed by Clauser et al. (1969) who developed the **CHSH inequality**.

Literature II

Pearl (1995) introduced the **instrumental inequality**, which gave a constraint on binary instrumental variable models.

Bonet (2001) expanded Pearl's work using computational algebra.

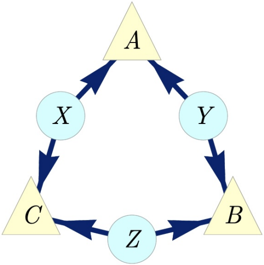


Later, Kang and Tian (2006), Evans (2012), Chaves et al. (2014), Kédagni and Mourifié (2020) and others proposed **graphical approaches** to deriving inequalities.

Manski, Robins and Balke contributed bounds for causal effects, but not for compatibility (though needing consistency does lead to inequalities).

Constraints

Does this graph induce constraints over the observed variables? (A, B and C.)



There are no m-separations. There are also no nested independences.

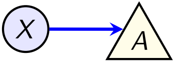
As it turns out, there are *inequality* constraints.

Outline

Notation

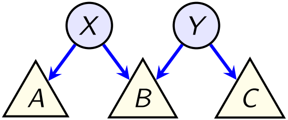
We have a DAG $\mathcal{G}$ with observed and hidden variables ($\boldsymbol{V}$ and $\boldsymbol{L}$).

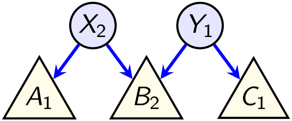
These are denoted by triangles and circles respectively.



Given two DAGs, say $\mathcal{G}$ and $\mathcal{H}$, we say that they are **isomorphic** (and write $\mathcal{G} \sim \mathcal{H}$ if there is a mapping from the nodes of one to the nodes of another that preserves the edge relations.

Example.



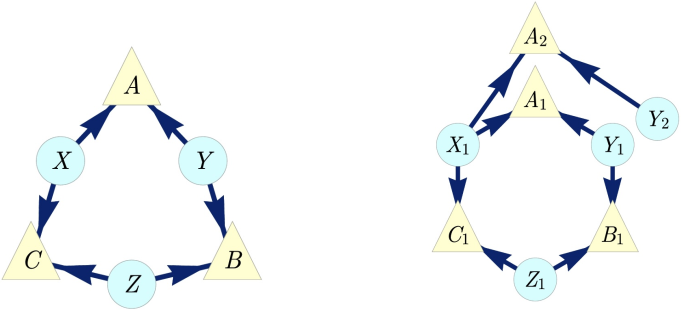


As in this example, this is usually just by dropping indices.

Inflation

An **inflation** (say $\mathcal{H}$) of a DAG $\mathcal{G}$ is one with vertices V_j for which $\mathcal{H}_{\mathrm{an}_{\mathcal{H}}(V_j)} \sim \mathcal{G}_{\mathrm{an}_{\mathcal{G}}(V)}$ for each V_j.

Example.

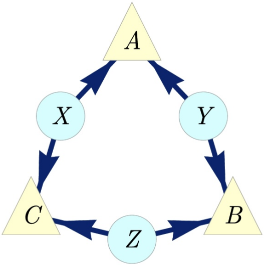


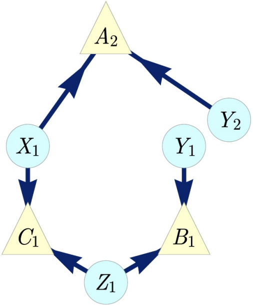
The triangle graph and its ‘capped’ inflation.

Incompatibility

Logically, there is no reason that we cannot impose a distribution on an inflated graph taken from the original graph.

Indeed the inflated graph may impose **additional independences** not shown in the original.





For example, the ‘cut’ inflation above exhibits $A_2 \perp\!\!\!\perp B_1$, but there is no similar independence in $\mathcal{G}$.

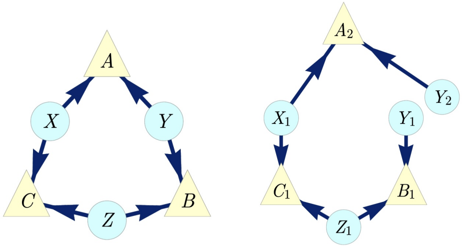
Witnessing Incompatibility

Let $[abc]$ represent $P(A = a, B = b, C = c) = 1$ for each a, b, c.

Is the distribution $([000] + [111])/2$ compatible with the triangle graph?

No! Suppose for contradiction it were. Then this would imply that each of the two way margins were $([00] + [11])/2$, so in the cut inflation we would have

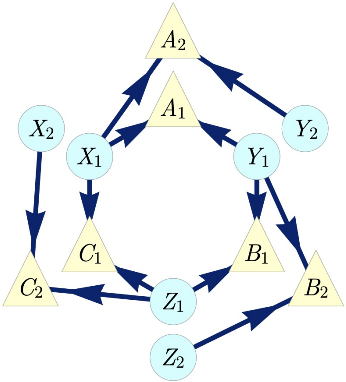
$$B_1 = x \iff C_1 = x \iff A_2 = x.$$



But this implies that $B_1 = A_2$ a.s., which contradicts the graph that says $A_2 \perp\!\!\!\perp B_1$.

Injectable Sets

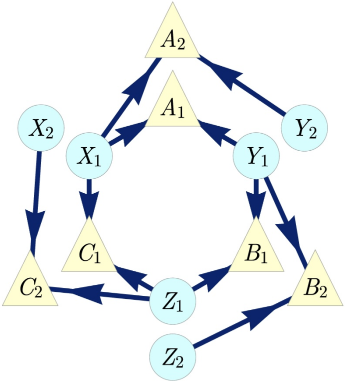
We say that a subset of vertices (say C') in an inflation graph $\mathcal{H}$ is **injectable** if the image of the equivalent vertices $C \sim C'$ in $\mathcal{G}$ is such that $\mathcal{G}_{\mathsf{an}_{\mathcal{G}}(C)} \sim \mathcal{H}_{\mathsf{an}_{\mathcal{H}}(C')}$.



For example, in the graph above $\{B_1, C_1\}$ and $\{B_1, C_2\}$ are both injectable, but $\{A_2, C_2\}$ is not.

Witnessing Incompatibility (again)

What about $([100] + [010] + [001])/3$? Again this distribution is **not** compatible. To see this, consider the 'spiral inflation'.



'Injectable sets' are $\{A_1, B_1, C_1\}$, $\{A_1, B_2\}$, $\{B_1, C_2\}$, $\{C_1, A_2\}$.

Under distribution, have

$$A_2 = 1 \implies C_1 = 0$$
$$B_2 = 1 \implies A_1 = 0$$
$$C_2 = 1 \implies B_1 = 0$$

and A_2, B_2, C_2 all independent.

This means that we must **sometimes** observe $A_2 = B_2 = C_2 = 1$, and this implies that sometimes $A_1 = B_1 = C_1 = 0$, which is a contradiction.

Deriving Inequalities

We can derive non-trivial inequalities for causal structures by starting with trivial ones on an inflated graph, and then putting in conditional independence constraints.

For example, it holds for all variables A, B, C taking values in $\{-1, +1\}$ that:

$$\mathbb{E}AC + \mathbb{E}BC \leq 1 + \mathbb{E}AB.$$

Then note that, in the cut inflation for the triangle graph, we have $A_2 \perp\!\!\!\perp B_1$, so

$$\mathbb{E}AC + \mathbb{E}BC \leq 1 + \mathbb{E}A \cdot \mathbb{E}B$$

must hold in the triangle graph.

This is a **non-trivial** inequality.

Entropy

We can do exactly the same using an entropic inequality:

It holds for all variables A, B, C taking values in a finite set that:

$$I(A : C) + I(B : C) \leq H(C) + I(A : B).$$

Then note that, since $A_2 \perp\!\!\!\perp B_1$ we have $I(A_2 : B_1) = 0$ so

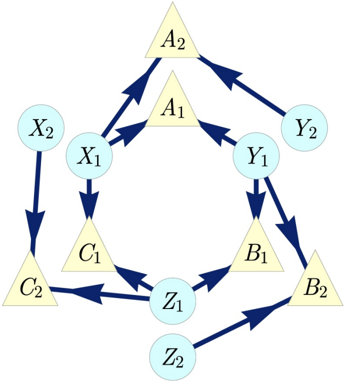
$$I(A : C) + I(B : C) \leq H(C).$$

must hold in the triangle graph.

Again, this is a **non-trivial** inequality.

Being Systematic

We say that a set is **ai-expressible** if it can be written as a disjoint union of injectable sets, where each injectable set is marginally independent.



Example.
The maximal ai-expressible sets are

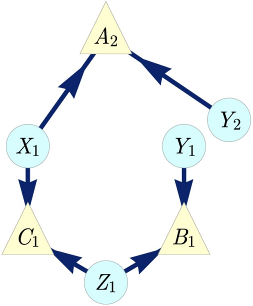
$$\{A_1, B_1, C_1\},$$
$$\{A_1, B_2, C_2\}, \quad \{A_2, B_1, C_2\}, \quad \{A_2, B_2, C_1\},$$
$$\{A_2, B_2, C_2\}.$$

Notice that $P_{A_1 B_2 C_2} = P_{AB} \cdot P_C$, because $A_1, B_2 \perp_d C_2$ in $\mathcal{H}$.

Similarly $P_{A_2 B_2 C_2} = P_A \cdot P_B \cdot P_C$ because they are all d-separated in $\mathcal{H}$.

Marginal Problem

One approach to testing compatibility for a specific distribution is to use linear programming.



Consider the 'cut' inflation. The relevant compatibility conditions are:

$$P_{AB}(a,b) = \sum_{c'} P_{A_2B_1C_1}(a,b,c')$$

$$P_{BC}(b,c) = \sum_{a'} P_{A_2B_1C_1}(a',b,c)$$

$$P_{AC}(a,c) = \sum_{b'} P_{A_2B_1C_1}(a,b',c)$$

So solve $Mv = b$ for v where v consists of entries over $P_{A_2B_1C_1}$ and b of fixed two-way marginals.

Solving LPs is generally easy, so this is very efficient for a **particular distribution**.

Marginal Problem

Another approach is to use **quantifier elimination** to obtain inequalities only over identifiable quantities.

Fourier-Motzkin is the classic method for this.

The general approach employed is to solve the **marginal satisfiability** problem for a collection of variables. This is known to be NP-complete in general, and is in practice often very hard.

However, we can obtain **weaker** constraints by just adapting logical relations.

Logical Relations

Recall the spiral inflation and the distribution $([100] + [010] + [001])/3$ being incompatible with the triangle graph. The following is a (related) logical tautology:

$$\neg\{A_2 = C_1 = 1\} \wedge \neg\{B_2 = A_1 = 1\} \wedge \neg\{C_2 = B_1 = 1\} \wedge \cdots$$
$$\neg\{A_1 = B_1 = C_1 = 0\} \quad \Longrightarrow \quad \neg\{A_2 = B_2 = C_2 = 1\}.$$

Hence we can take the contrapositive

$$\begin{aligned}\{A_2 = B_2 = C_2 = 1\} \implies &\{A_2 = C_1 = 1\} \vee \cdots \\ &\vee \{B_2 = A_1 = 1\} \vee \{C_2 = B_1 = 1\} \vee \{A_1 = B_1 = C_1 = 0\},\end{aligned}$$

and then use a union bound to obtain

$$\begin{aligned} & P_{A_2B_2C_2}(111) \leq P_{A_1B_2}(11) + P_{B_1C_2}(11) + P_{C_1A_2}(11) + P_{A_1B_1C_1}(000) \\ \implies \; & P_A(1)P_B(1)P_C(1) \leq P_{AB}(11) + P_{BC}(11) + P_{CA}(11) + P_{ABC}(000).\end{aligned}$$

Finding Tautologies

We can obtain a list of tautologies by inspecting the matrix M used in the marginal problem.

First we construct a hypergraph with vertices given by maximal ai-expressible sets and edges given by states over all the variables.

In our case, the vertices are the $40 = 5 \times 2^3$ valuations of:

$$\{A_1, B_1, C_1\}, \quad \{A_1, B_2, C_2\}, \quad \{A_2, B_1, C_2\}, \quad \{A_2, B_2, C_1\}, \quad \{A_2, B_2, C_2\}.$$

Edges are the 2^6 possible values of these 6 variables.

Then we construct a second sub-hypergraph by picking an antecedent (e.g. $A_2 = B_2 = C_2 = 1$) removing any vertices and edges not consistent with it (as well as the antecedent itself).

This leaves 14 vertices and 8 edges in our case.

Finding Tautologies

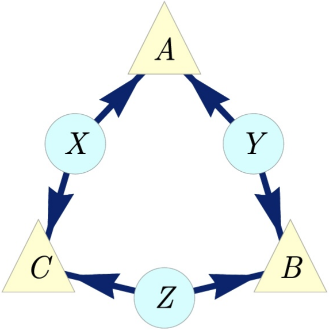
The relevant entries of M are:

vertex		$A_1B_1C_1$ (since $A_2 = B_2 = C_2 = 1$)							
		000	100	010	110	001	101	011	111
$A_1B_1C_1$	000	1	0	0	0	0	0	0	0
	100	0	1	0	0	0	0	0	0
	010	0	0	1	0	0	0	0	0
	110	0	0	0	1	0	0	0	0
	001	0	0	0	0	1	0	0	0
	101	0	0	0	0	0	1	0	0
	011	0	0	0	0	0	0	1	0
	111	0	0	0	0	0	0	0	1
$A_1B_2C_2$	011	1	0	1	0	1	0	1	0
	111	0	1	0	1	0	1	0	1
$A_2B_1C_2$	101	1	1	0	0	1	1	0	0
	111	0	0	1	1	0	0	1	1
$A_2B_2C_1$	110	1	1	1	1	0	0	0	0
	111	0	0	0	0	1	1	1	1

Completeness

In fact, Navascués and Wolfe (2020) show that inflation can **completely solve** the causal compatibility problem without using any of these clever tricks.

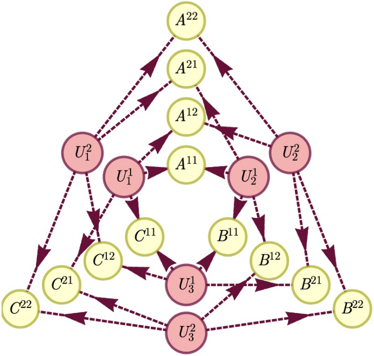
Assume we have a **correlation scenario**, in which all edges are directed from latent nodes to observed ones (e.g. the triangle).



Higher Order Inflation

Define the ***n*th order inflation** $\mathcal{H}^n$ as the graph with:

- hidden variables $U_i^1, \ldots, U_i^n$ for each U_i in $\mathcal{G}$, $i = 1, \ldots, L$,
- observed variables $A_i^{j_1 \ldots j_L}$ where j_k is the **copy number** of the latent version of U_k that points to $A_i^{j_1 \ldots j_L}$.



The 2nd order inflation of the triangle graph.

Bound

Navascués and Wolfe show that if a distribution P is **not in** the marginal model for $\mathcal{G}$, then the nth order inflation will be able to witness it if

$$\inf_{Q \in \mathcal{M}(\mathcal{G})} \|P - Q\|_2 > O\left(\sqrt{\frac{L}{n}}\right)$$

where L is the number of latent variables in $\mathcal{G}$.

Bound

In fact, one can adapt their proof to show that (if $n \geq L$) we have

$$d_{TV}(\tilde{P}^{\otimes 2}, P^{\otimes 2}) \leq \frac{L}{n}.$$

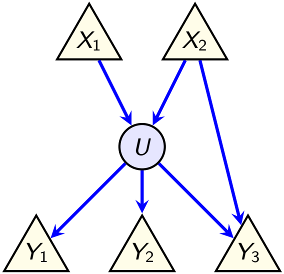
Hence relatively simple algebra shows that if the program has a solution, then there is a distribution $\tilde{P}_n$ in the model such that

$$\|\tilde{P}_n - P\|_2 \leq \sqrt{\frac{L}{n}}.$$

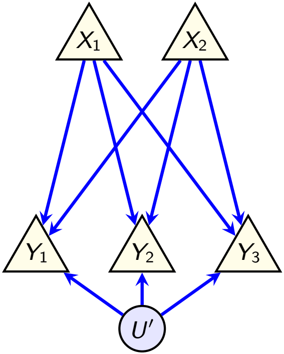
So, as n grows, we get a sequence of solutions which converges to any compatible distribution, and if it is not compatible then there will be an n such that the program has no feasible solution.

Exogenization

WLOG latent vertices have no parents.

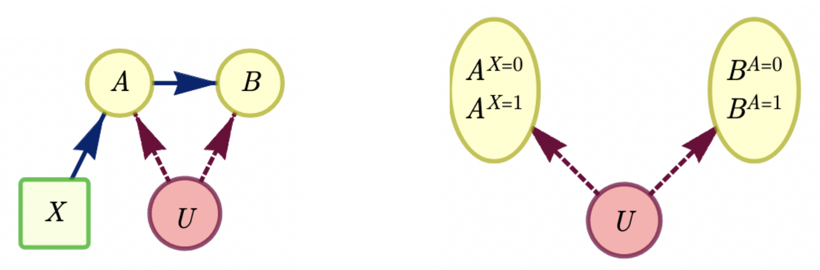


$\stackrel{\mathcal{M}}{=}$



Unpacking

WLOG observed vertices have no observed parents.



We simply replace vertices by their potential outcome vectors.

Now everything is a correlation scenario!

Summary

Inflation is a method that allows one to certify that any distribution **not** in the marginal DAG model for a graph $\mathcal{G}$ is such.

- Testing for compatibility of a **particular** distribution can be performed relatively inexpensively.
- There are **systematic methods** for obtaining a large number of inequalities in this manner, and they are also fairly computationally cheap.

- Computationally it can be very expensive to implement these methods to obtain **symbolic bounds**.
- However, it is unclear whether this is **necessarily** a difficult problem, and it is possible that someone will find a shortcut.

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