Stochastic Calculus for Finance
Michaelmas Term 1998: Problems for solution

1. Consider the following simple model of stock price movement. The value of the stock at time zero is S_0. At time ΔT, the price has moved to either $S_0 u$, or $S_0 d$. The risk free interest rate is such that \$1 now will be worth $\$e^{r\Delta T}$ at time ΔT.

a. Suppose that $d < e^{r\Delta T} < u$. Show that the market price of a European call option which matures at time ΔT with strike price K is

$$\left(\frac{1 - de^{-r\Delta T}}{u-d}\right)(S_0 u - K)_+ + \left(\frac{ue^{-r\Delta T} - 1}{u-d}\right)(S_0 d - K)_+ .$$

b. What happens if we drop the assumption that $d < e^{r\Delta T} < u$?

2. Suppose that at current exchange rates, £100 is worth 280DM. A speculator believes that by the end of the year there is a probability of 2/3 that the pound will have fallen to 2.60DM, and a 1/3 chance that it will have gained to be worth 3.00DM. He therefore buys a European put option that will give him the right (but not the obligation) to sell £100 for 290DM at the end of the year. He pays 20DM for this option. Assume that the risk free interest rate is zero. Using a single period binary model, either construct a strategy whereby one party is certain to make a profit or prove that this is the fair price.

3. *(Put-Call parity.)* Let us denote by C_t and P_t respectively the prices of a European call and a European put option, each with maturity T and strike K. Assume that the risk free rate of interest is constant, r (so the cost of borrowing \$1 for s units of time is $\$e^{rs}$). Show that for each $t \leq T$,

$$C_t - P_t = S_t - Ke^{-r(T-t)}.$$

4. Suppose that the price of a certain asset has the lognormal distribution. That is $\log(S_T/S_0)$ is normally disrtibuted with mean μ and variance σ^2. Calculate $\mathbb{E}[S_T]$.

5. Find the *risk-neutral probabilities* for the model in Question 2. That is, find the probabilities p, $1-p$ for upward/downward movement of the pound, under which '$\mathbb{E}[e^{-r\Delta T} S_T] = S_0$'. Check that the fair price of the option is then $\mathbb{E}[(K - S_T)_+]$ (since $r = 0$) where the expectation is calculated with respect to these probabilities..

6. Consider two dates T_0, T_1 with $T_0 < T_1$. A *forward start option* is a contract in which the holder receives at time T_0, at no extra cost, an option with expiry date T_1 and strike price equal to S_{T_0} (the asset price at time T_0). Assume that the stock price evolves according to a two-period binary model, in which the asset price at time T_0 is either $S_0 u$ or $S_0 d$, and at time T_1 is one of $S_0 u^2$, $S_0 ud$ and $S_0 d^2$ with

$$d < \min\left(e^{rT_0}, e^{r(T_1 - T_0)}\right) \leq \max\left(e^{rT_0}, e^{r(T_1 - T_0)}\right) < u,$$

where r denotes the risk free interest rate. Find the fair price of such an option at time zero.

7. A *digital option* is one in which the payoff depends in a discontinuous way on the asset price. The simplest example is the *cash-or-nothing option*, in which the payoff to the holder at maturity T is $X\chi_{\{S_T > K\}}$ where X is some prespecified cash sum.

Suppose that an asset price evolves according to the Cox Ross Rubinstein model (CRR model). That is, a multiperiod binary model in which, at each step, the asset price moves from its current value S_n to one of $S_n u$ and $S_n d$. As usual, if ΔT denotes the length of each time step, $d < e^{r\Delta T} < u$.

Find the time zero price of the above option. You may leave your answer as a sum.

8. Suppose that an asset price evolves according to the CRR model described in Question 7. For simplicity suppose that the risk free interest rate is zero and ΔT is 1. Suppose that under the probability $\mathbb{P}$, at each time step, stock prices go up with probability p and down with probability $1 - p$.

The conditional expectation

$$M_n \equiv \mathbb{E}[S_N | \mathcal{F}_n] \qquad 1 \leq n \leq N,$$

is a stochastic process. Check that it is a $\mathbb{P}$-martingale and find the distribution of M_n?

9. Show how to derive the put-call parity relationship of Question 3 from Theorem 3.3 of lectures.

10. Let $\{B_t\}_{t\geq 0}$ be standard Brownian motion. Which of the following are Brownian motions?

a. $\{-B_t\}_{t\geq 0}$,

b. $\{cB_{t/c^2}\}_{t\geq 0}$, where c is a constant,

c. $\{\sqrt{t}B_1\}_{t\geq 0}$,

d. $\{B_{2t} - B_t\}_{t\geq 0}$.

Justify your answers.

11. Suppose that $\{B_t\}_{t\geq 0}$ is standard Brownian motion. Prove that conditional on $B_{t_1} = x_1$, the probability density function of $B_{t_1/2}$ is

$$\sqrt{\frac{2}{\pi t_1}} \exp\left(-\frac{1}{2}\left(\frac{\left(x - \frac{1}{2}x_1\right)^2}{t_1/4}\right)\right).$$

This tells us that the conditional distribution is a normally distributed random variable. What are the mean and variance?

12. Let $\{B_t\}_{t\geq 0}$ be standard Brownian motion. Let T_a be the 'hitting time of level a', that is

$$T_a = \inf\{t \geq 0 : B_t = a\}.$$

Then we shall prove in lectures that

$$\mathbb{E}\left[\exp\left(-\theta T_a\right)\right] = \exp\left(-a\sqrt{2\theta}\right).$$

Use this result to calculate

a. $\mathbb{E}[T_a]$,

b. $\mathbb{P}[T_a < \infty]$.

13. Let $\{B_t\}_{t\geq 0}$ denote standard Brownian motion and define $\{M_t\}_{t\geq 0}$ by

$$M_t = \max_{0\leq s\leq t} B_s.$$

Suppose that $x \geq a$. Calculate

a. $\mathbb{P}[M_t \geq a, B_t \geq x]$,

b. $\mathbb{P}[M_t \geq a, B_t \leq x]$.

14. Let $\{B_t\}_{t\geq 0}$ be standard Brownian motion. Let $T_{a,b}$ denote the hitting time of the sloping line $a + bt$. That is,

$$T_{a,b} = \inf\{t \geq 0 : B_t = a + bt\}.$$

We show in lectures that

$$\mathbb{E}\left[\exp\left(-\theta T_{a,b}\right)\right] = \exp\left(-a\left(b + \sqrt{b^2 + 2\theta}\right)\right).$$

The aim of this question is to calculate the distribution of $T_{a,b}$, without inverting the Laplace transform. In what follows, $\phi(x) = \Phi'(x)$ and

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) dy.$$

a. Find $\mathbb{P}[T_{a,b} < \infty]$.

b. Using the fact that $sB_{1/s}$ has the same distribution as B_s, show that

$$\mathbb{P}\left[T_{a,b} \leq t\right] = \mathbb{P}\left[B_s \geq as + b \text{ for some } s \text{ with } 1/t \leq s < \infty\right].$$

c. By conditioning on the value of $B_{1/t}$, use the previous part to show that

$$\mathbb{P}\left[T_{a,b} \leq t\right] = \int_{-\infty}^{b+a/t} \mathbb{P}\left[T_{b-x+a/t,a} < \infty\right] \phi\left(\sqrt{t}x\right) dx + 1 - \Phi\left(\frac{a+bt}{\sqrt{t}}\right).$$

d. Substitute for the probability in the integral and deduce that

$$\mathbb{P}\left[T_{a,b} \leq t\right] = e^{-2ab}\Phi\left(\frac{bt-a}{\sqrt{t}}\right) + 1 - \Phi\left(\frac{a+bt}{\sqrt{t}}\right).$$

15. Let $\{\mathcal{F}_t\}_{0\leq t\leq T}$ denote the natural filtration associated to a standard Brownian motion $\{B_t\}_{0\leq t\leq T}$. Which of the following are $\mathcal{F}_t$-martingales?

a. $\exp(\sigma B_t)$,

b. cB_{t/c^2}, where c is a constant,

c. $tB_t - \int_0^t B_s ds$,

16. Let $\{\mathcal{F}_t\}_{t\geq 0}$ denote the natural filtration associated to a standard Brownian motion $\{B_t\}_{t\geq 0}$. Define the process S_t by

$$S_t = \exp(\sigma B_t + \mu t).$$

For which values of μ is the process $\{S_t\}_{t\geq 0}$ an $\mathcal{F}_t$-martingale?

17. A function, f, is said to be *Lipschitz continuous* on $[0,T]$ if there exists a constant $C > 0$ such that for any $t, t' \in [0,T]$

$$|f(t) - f(t')| < C|t - t'|.$$

Show that a Lipschitz continuous function has bounded variation and zero quadratic variation.

18. Let $\{B_t\}_{t\geq 0}$ denote standard Brownian motion. For a partition π of $[0,T]$, write $\delta(\pi)$ for the mesh of the partition and $N(\pi)$ for the number of intervals. We write $[t_j, t_{j+1})$ to denote a generic subinterval in the partition. Calculate

a.

$$\lim_{\delta(\pi)\to 0} \sum_0^{N(\pi)-1} B_{t_{j+1}} \left(B_{t_{j+1}} - B_{t_j}\right),$$

b.

$$\int_0^T B_s \circ dB_s \equiv \lim_{\delta(\pi)\to 0} \sum_0^{N(\pi)-1} \frac{1}{2}\left(B_{t_{j+1}} + B_{t_j}\right)\left(B_{t_{j+1}} - B_{t_j}\right).$$

This is the *Stratonovich integral* of B_s with respect to itself.

19. Suppose that S_t is a function of bounded quadratic variation on $[0,T]$, and A_t is a Lipschitz continuous function on $[0,T]$. Using $qv(f)[0,T]$ to denote the quadratic variation of a function f over the interval $[0,T]$, show that

$$qv(S_t + A_t)[0,T] = qv(S_t)[0,T].$$

20. If f is a simple function, prove that the process M_t given by the Itô integral

$$M_t = \int_0^t f(s, B_s) dB_s$$

is a martingale.

21. Verify that

$$\mathbb{E}\left[\left(\int_0^t B_s dB_s\right)^2\right] = \int_0^t \mathbb{E}\left[B_s^2\right] ds.$$

(If you need the moment generating function of B_t, you may assume the result of Question 23.)

22. Use Itô's formula to write down stochastic differential equations for the following quantities. (As usual, $\{B_t\}_{t\geq 0}$ denotes standard Brownian motion.)

a. $Y_t = B_t^3$,

b. $Y_t = \exp\left(\sigma B_t - \frac{1}{2}\sigma^2 t\right)$,

c. $Y_t = tB_t$.

23. Let $\{B_t\}_{t\geq 0}$ denote Brownian motion and define $Z_t = \exp(\alpha B_t)$. Use Itô's formula to write down a stochastic differential equation for Z_t. Hence find an ordinary (deterministic) differential equation for $m(t) \equiv \mathbb{E}[Z_t]$, and solve to show that

$$\mathbb{E}\left[\exp\left(\alpha B_t\right)\right] = \exp\left(\frac{\alpha^2}{2} t\right).$$

24. *(The Ornstein-Uhlenbeck process).* Let $\{B_t\}_{t\geq 0}$ denote standard Brownian motion. The Ornstein-Uhlenbeck process, $\{X_t\}_{t\geq 0}$ is the unique solution to *Langevin's equation*,

$$dX_t = -\alpha X_t dt + dB_t, \qquad X_0 = x.$$

This equation was originally introduced as a simple idealised model for the velocity of a particle suspended in a liquid. Verify that

$$X_t = e^{-\alpha t} x + e^{-\alpha t} \int_0^t e^{\alpha s} dB_s,$$

and use this expression to calculate the mean and variance of X_t.

25. Suppose that under the probability measure $\mathbb{P}$, X_t is a Brownian motion with constant drift μ. Find a measure $\mathbb{P}^*$, equivalent to $\mathbb{P}$, under which X_t is a Brownian motion with drift ν.

26. Suppose that an asset price S_t is such that $dS_t = \mu S_t dt + \sigma S_t dB_t$, where B_t is, as usual, standard Brownian motion. Let r denote the risk free interest rate. The price of a riskless asset then follows $dS_t^0 = rS_t^0 dt$. We write (H_t^0, H_t) for the portfolio consisting of H_t^0 units of the riskless asset, S_t^0, and H_t units of S_t at time t. For each of the following choices of H_t, find H_t^0 so that the portfolio (H_t^0, H_t) is self-financing. (Recall that the value of the portfolio at time t is $V_t = H_t^0 S_t^0 + H_t S_t$, and that the portfolio is self-financing if $dV_t = H_t^0 dS_t^0 + H_t dS_t$.)

a. $H_t = 1$,

b. $H_t = \int_0^t S_u du$,

c. $H_t = S_t$.

27. Let $\mathcal{F}_t$ be the natural filtration associated with Brownian motion (as in the proof of Theorem 8.4 of lectures). Show that if X is an $\mathcal{F}_T$-measurable random variable, then if $\mathbb{P}^*$ is a probability measure equivalent to that of the Brownian motion, then the process

$$M_t \equiv \mathbb{E}^* \left[X | \mathcal{F}_t \right]$$

is a $\mathbb{P}^*$-martingale.

28. Use the Black-Scholes model to value a forward start option (described in Question 3).

29. Suppose that the value of a European call option can be expressed as $V_t = F(t, S_t)$ (as we prove in Proposition 9.2). Then $\tilde{V}_t = e^{-rt} V_t$, and we may define $\tilde{F}$ by

$$\tilde{V}_t = \tilde{F}(t, \tilde{S}_t).$$

Under the risk-neutral measure, the discounted asset price follows $d\tilde{S}_t = \sigma \tilde{S}_t dW_t$, where (under this probability measure) W_t is a standard Brownian motion.

a. Find the stochastic differential equation satisfied by $\tilde{F}(t, \tilde{S}_t)$.

b. Using the fact that $\tilde{V}_t$ is a martingale under the risk-neutral measure, find the partial differential equation satisfied by $\tilde{F}(t, x)$, and hence show that

$$\frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 F}{\partial x^2} + rx\frac{\partial F}{\partial x} - rF = 0.$$

This is the *Black-Scholes equation.*

30. *(Delta-hedging).* The following derivation of the Black-Scholes equation is very popular in the finance literature. We will suppose, as usual, that an asset price follows a geometric Brownian motion. That is, there are parameters μ, σ, such that

$$dS_t = \mu S_t dt + \sigma S_t dB_t.$$

Suppose that we are trying to value a European option based on this asset. Let us denote the value of the option at time t by $V(t, S_t)$. We know that at time T, $V(T, S_T) = f(S_T)$, for some function f.

a. Using Itô's formula express V as the solution to a stochastic differential equation.

b. Suppose that a portfolio, whose value we denote by π, consists of one option and a (negative) quantity $-\delta$ of the asset. *Assuming that the portfolio is self-financing,* find the stochastic differential equation satisfied by π.

c. Find the value of δ for which the portfolio you have constructed is 'instantaneously riskless', that is for which the stochastic term vanishes.

d. An instantaneously riskless portfolio must have the same rate of return as the risk free interest rate. Use this observation to find a (deterministic) partial differential equation for the $V(t, x)$. Notice that this is the Black-Scholes equation obtained in Question 29.

e. Now for the crunch: is the portfolio that you have constructed self-financing?

   (a) Write $U_t = v'(t, S_t)$, where v is a solution of the Black-Scholes equation. Use Itô's formula to write down a stochastic differential equation for U.

   (b) Find an expression for $dV_t \equiv d(U_t S_t) - dv(t, S_t)$.

   (c) If the trading strategy is self-financing then it must satisfy $dV_t = v'(t, S_t)dS_t - dv(t, S_t)$. Use the stochastic differential equation for V_t and the Black-Scholes differential equation (differentiated with respect to the x variable) to show that this is equivalent to

   $$\sigma S_t^2 v''(t, S_t)\left(dB_t + \sigma^{-1}(\mu - r)dt\right) = \sigma S_t^2 v''(t, S_t)dW_t = 0.$$

   Here $W_t = B_t + \sigma^{-1}(\mu - r)t$ is a Brownian motion under the risk neutral measure.

   (d) Deduce that the portfolio is *not* self-financing.

You have shown that the delta-hedging argument is mathematically unsatisfactory in the continuous time setting. It should be emphasized that it is unquestionable in the discrete time setting. All the same, in continuous time it does lead to the right value of the option. The reason is that the additional cost that might be associated with this trading strategy is a martingale under the risk neutral measure.