8 The Black-Scholes model

In the simplest case, the Black-Scholes model is a continuous time model in which we suppose that the market consists of a single risky asset (with price S_t at time t) and a riskless asset (with price S_t^0 at time t). We suppose that S_t^0 follows the *ordinary differential equation*

$$dS_t^0 = rS_t^0 dt,$$

where, for simplicity, r is a non-negative constant (the instantaneous riskless borrowing rate). Set $S_0^0 = 1$, so that $S_t^0 = e^r t$.

The value of the risky asset is assumed to follow

$$dS_t = S_t \left(\mu dt + \sigma dB_t\right), \qquad 0 \leq t \leq T.$$

That is, the asset price follows *geometric Brownian motion.* We always denote the time of maturity of the option by T. It is easy to check that

$$S_t = S_0 \exp\left(\mu t - \frac{1}{2}\sigma^2 t + \sigma B_t\right).$$

As in the discrete case, out pricing of options will be justified by arbitrage arguments. If we can construct a portfolio that exactly replicates the claim against us, then the value of that portfolio at time zero is the fair price for the option – for any other choice of price, one party can make a risk free profit.

Whereas in the discrete case we adjusted our portfolio at the 'ticks' of the clock, here we are allowed to readjust the portfolio continuously *on the basis of our current knowledge.*

Having constructed our portfolio from the money received from the sale of the option at time zero, we should not have to inject further capital in order to replicate the claim, nor should we be allowed to receive income from the portfolio. That is we shall adopt a self-financing strategy.

Definition 8.1 *A* self-financing strategy *is defined by a pair ϕ of adapted processes $(H_t^0)_{0\leq t\leq T}$, $(H_t)_{0\leq t\leq T}$, denoting the quantities of riskless and risky asset respectively held in the portfolio at time t, satisfying*

1. $$\int_0^T \left|H_t^0\right| dt + \int_0^T \left|H_t\right|^2 dt < \infty$$

   (with probability one),

2. $$H_t^0 S_t^0 + H_t S_t = H_0^0 S_0^0 + H_0 S_0 + \int_0^t H_u^0 dS_u^0 + \int_0^t H_u dS_u$$

   (with probability one) for all $t \in [0, T]$.

Remarks:

Condition 1 is enough to ensure that the integrals in condition 2 make sense.

In differential form, condition 2 says that the value, $V_t(\phi) = H_t^0 S_t^0 + H_t dS_t$ of the portfolio satisfies

$$dV_t(\phi) = H_t^0 dS_t^0 + H_t dS_t,$$

that is changes of value of the portfolio over an infinitesimal time interval are due entirely to changes in value of the assets and not to injection (or removal) of wealth from outside.

Usually one insists on *predictability* of H_t in this definition. That is H_t should depend only on $\mathcal{F}_{t-}$, the information available strictly *before* the time t. However, since Brownian paths are continuous, there is essentially no extra restriction in passing from adapted to predictable processes.

We have made no restriction that H_t^0, H_t should be positive. In particular we can hold negative amounts of stock. This is possible (it is known as short-selling).

As in the discrete setting, we shall seek a probability measure under which the *discounted* stock price is a martingale. It is convenient to have some notation.

Notation. We define $\tilde{S}_t = e^{-rt} S_t$, the discounted price of the risky asset.

Proposition 8.2 *Let $\phi = (H_t^0, H_t)_{0 \le t \le T}$ be an adapted process with values in $\mathbb{R}^2$, satisfying*

$$\int_0^T \left| H_t^0 \right| dt + \int_0^T \left| H_t \right|^2 dt < \infty$$

(with probability one). Set

$$V_t(\phi) = H_t^0 S_t^0 + H_t S_t, \qquad \tilde{V}_t(\phi) = e^{-rt} V_t(\phi).$$

Then ϕ defines a self-financing strategy if and only if

$$\tilde{V}_t(\phi) = \tilde{V}_0(\phi) + \int_0^t H_u d\tilde{S}_u$$

with probability one for all $t \in [0, T]$.

Proof:

Suppose first that ϕ is self-financing. Then

$$\begin{aligned}
d\tilde{V}_t(\phi) &= -re^{-rt} V_t(\phi) dt + e^{-rt} dV_t(\phi) \\
&= -re^{-rt} \left(H_t^0 e^{rt} + H_t S_t \right) dt + e^{-rt} H_t^0 d(e^{rt}) + e^{-rt} H_t dS_t \\
&= H_t \left(-re^{-rt} S_t dt + e^{-rt} dS_t \right) \\
&= H_t d\tilde{S}_t
\end{aligned}$$

as required.

The other direction is similar and is left as an exercise. $\square$

Before going further, we outline our basic strategy.

Suppose that the claim against us that we are trying to replicate is X at time T. (It may depend on $(S_t)_{0 \le t \le T}$ in more complex ways than just through S_T.)

Suppose then that *somehow* we can find a process $(H_t)_{0 \le t \le T}$ such that the claim X, discounted, satisfies

$$\tilde{X} = H_0 + \int_0^T H_u d\tilde{S}_u.$$

Then we can replicate the claim by a portfolio in which we hold H_t units of stock and H_t^0 units of riskless asset, where H_t^0 is chosen so that

$$\tilde{V}_t(\phi) = H_t\tilde{S}_t + H_t^0 e^{-rt} = H_0 + \int_0^t H_u d\tilde{S}_u.$$

By Proposition 8.2, the portfolio is then self-financing, and, moreover, $V_T = X$. The fair price at time zero is then $V_0 = H_0$.

This is fine if we know H_0, but there is a quick and easy way to find the right price without finding the strategy ϕ. Suppose instead that I can find a probability measure, $\mathbb{P}^*$, under which the discounted stock price is a martingale. Then, at least provided $\int_0^T H_u^2 du < \infty$,

$$\int_0^t H_u d\tilde{S}_u$$

will be a mean zero martingale. Then

$$\mathbb{E}\left[\tilde{V}_T(\phi)\right] = H_0 + \mathbb{E}\left[\int_0^t H_u d\tilde{S}_u\right] = H_0.$$

So $H_0 = \mathbb{E}^*\left[\tilde{X}\right]$ is the fair price.

This then is entirely analogous to the pricing formula of Theorem 3.4. If there is a probability measure under which the discounted stock price is a martingale, then the fair time zero price of the claim is $\mathbb{E}^*[\tilde{X}]$, the discounted expected value of the claim under this measure.

We have assumed that the process H_t exists. We prove this later, and for the special case where X depends on $(S_t)_{0\leq t\leq T}$ only through S_T, we find it explicitly. First, if our pricing formula is to be of any use, we should find the *equivalent martingale measure* $\mathbb{P}^*$.

Lemma 8.3 (A probability measure under which $\tilde{S}_t$ is a martingale) *There is a probability measure $\mathbb{P}^*$, equivalent to $\mathbb{P}$, under which the discounted share price $\tilde{S}_t$ is a martingale. Moreover, $\mathbb{P}^* = \mathbb{P}^{(L)}$ of Theorem 7.13 where $\theta_t = (\mu - \sigma)/\sigma$.*

Proof: Recall that

$$dS_t = \mu S_t dt + \sigma S_t dB_t.$$

Thus

$$d\tilde{S}_t = \tilde{S}_t\left(-rdt + \mu dt + \sigma dB_t\right).$$

Consequently, if we set $W_t = B_t + (\mu - r)t/\sigma$,

$$d\tilde{S}_t = \tilde{S}_t \sigma dW_t.$$

Now from Theorem 7.13, under $\mathbb{P}^* = \mathbb{P}^{(L)}$ as in the statement of the Lemma, W_t is a Brownian motion and so $\tilde{S}_t$ is a martingale and moreover,

$$\tilde{S}_t = \tilde{S}_0 \exp\left(\sigma W_t - \sigma^2 t/2\right).$$

$\square$

We can now prove the Fundamental Theorem of option pricing in the Black-Scholes framework.

Theorem 8.4 *In the Black-Scholes model, any option defined by a non-negative $\mathcal{F}_T$-valued random variable X, which is square-integrable under the probability measure $\mathbb{P}^*$ of Lemma 8.3, is replicable, and the value at time $t \leq T$ of any replicating portfolio is given by*

$$V_t = \mathbb{E}^* \left[e^{-r(T-t)} X \middle| \mathcal{F}_t \right].$$

In particular, the fair price at time zero for the option is

$$V_0 = \mathbb{E}^* \left[e^{-rt} X \right] = \mathbb{E}^* \left[\tilde{X} \right].$$

Proof:

In the argument that followed Proposition 8.2 we showed that if we could find a process $(H_t)_{0 \leq t \leq T}$ such that

$$\tilde{X} = H_0 + \int_0^T H_u d\tilde{S}_u,$$

then we could construct a replicating portfolio whose value at time t satisfies

$$\tilde{V}_t(\phi) = H_0 + \int_0^t H_u d\tilde{S}_u, \tag{8}$$

which, by the martingale property of the stochastic integral is precisely

$$\begin{aligned} \tilde{V}_t(\phi) = \mathbb{E}^* & \left[H_0 + \int_0^T H_u d\tilde{S}_u \middle| \mathcal{F}_t \right] \\ &= \mathbb{E}^* \left[\tilde{X} \middle| \mathcal{F}_t \right] = \mathbb{E}^* \left[e^{-rT} X \middle| \mathcal{F}_t \right]. \end{aligned}$$

Undoing the discounting on $[0, t]$ gives

$$V_t(\phi) = \mathbb{E}^* \left[e^{-r(T-t)} X \middle| \mathcal{F}_t \right].$$

Now, any other replicating portfolio has $V_T(\phi) = X$ and, if it is self-financing, also satisfies equation (8) (by Proposition 8.2) and so, if there *is* a replicating, self-financing portfolio, then for any such we obtain the same value of the portfolio.

The proof of the Theorem will be complete if we can show that there *is* an adapted process $(H_t)_{0 \leq t \leq T}$ such that

$$\tilde{X} = H_0 + \int_0^T H_u d\tilde{S}_u.$$

Now, by the tower property of conditional expectations, under $\mathbb{P}^*$,

$$M_t = \mathbb{E}^* \left[e^{-rt} X \middle| \mathcal{F}_t \right]$$

is a martingale. The natural filtration of our original Brownian motion is the same as that for the process W_t defined in Lemma 8.3. That is, M_t is a "Brownian martingale" and by the Martingale Representation Theorem (Theorem 7.12) there exists an $\mathcal{F}_t$-adapted process ϕ_t such that

$$M_t = M_0 + \int_0^t \phi_s dW_s.$$

Now $d\tilde{S}_s = \sigma \tilde{S}_s dW_s$ and so setting

$$H_t = \frac{\phi_t}{\sigma \tilde{S}_t} \quad \text{and} \quad H_t^0 = M_t - H_t \tilde{S}_t,$$

the strategy $(\phi_t) = (H_t^0, H_t)$ is a self-financing replicating strategy as required. □

Remarks:

Notice that the value of the portfolio is always positive.

The Theorem that we have just proved is *very* general. The claim X could be almost arbitrarily complex provided it depends only on the path of the stock price up to time T. We have proved that not only does there exist a fair price, but moreover, we *can hedge* the claim (although we have asserted the existence of a hedging strategy rather than provided a particularly useful expression for it).

The 'fair price' of the value of the claim at time zero takes an unexpectedly simple form – it is just the *expected value* of the discounted claim *under the martingale measure.*

In principle we have identified the three steps to valuing and replicating a claim.

Three steps to replication:

1. Find a measure $\mathbb{P}^*$ under which the discounted asset price $\tilde{S}_t$ is a martingale.

2. Form the process $M_t = \mathbb{E}^* \left[e^{-rt} X \middle| \mathcal{F}_t \right]$.

3. Find an adapted process H_t such that $dM_t = H_t d\tilde{S}_t$.

The value at time zero is $\mathbb{E}_{\mathbb{P}^*} \left[e^{-rt} X \middle| \mathcal{F}_t \right]$. This can be evaluated, at least numerically, even for complex claims X.

Finding H_t is usually rather more involved. However, things are much simpler in certain special cases. In particular, in the next section we value *European claims* in the Black-Scholes framework. Here $X = f(S_T)$, a function of the stock price *at maturity* only.

9 Black-Scholes prices for European options

First we remind ourselves of some examples of European options.

Example 9.1 *1. A European* call *option with maturity T and strike price K gives the holder the right, but not the obligation, to* buy *one unit of stock at price K at time T. The payoff is $f(S_T) = (S_T - K)_+$.*

2. A European put *option with maturity T and strike price K gives the holder the right, but not the obligation, to* sell *one unit of stock at price K at time T. The payoff is $f(S_T) = (K - S_T)_+$.*

3. *A* digital *or* binary *option is a contract whose payoff depends in a discontinuous way on the terminal price of the underlying asset. The simplest examples are* cash-or-nothing *options and* asset-or-nothing *options. The payoff at expiry of a cash-or-nothing call is $X\chi_{S_T>K}$, where X is a prespecified amount of cash. The payoff of an asset-or-nothing call is $S_T\chi_{S_T>K}$.*

There are many more examples. We present our derivation of a pricing formula and hedging strategy in the general case.

As in §8, we assume that we are working within the Black-Scholes framework. That is, stock prices evolve as geometric Brownian motion,

$$dS_t = \mu S_t dt + \sigma S_t dB_t,$$

for some constants μ and σ.

Proposition 9.2 *The value at time t of a European option whose payoff at maturity is $X = f(S_T)$ is $V_t = F(t, S_t)$, where*

$$F(t,x) = e^{-r(T-t)} \int_{-\infty}^{\infty} f\left(x \exp\left((r - \sigma^2/2)(T-t) + \sigma y\sqrt{T-t}\right)\right) \frac{\exp(-y^2/2)}{\sqrt{2\pi}} dy.$$

Proof: From Lemma 8.3 we have that the discounted stock price is a martingale under the measure $\mathbb{P}^*$ where

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = \exp\left(-\left(\frac{\mu - r}{\sigma}\right) B_t - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 t\right),$$

and that under this measure $W_t = B_t + (\mu - r)t/\sigma$ is a Brownian motion. As in the proof of Lemma 8.3 we have

$$d\tilde{S}_t = \sigma \tilde{S}_t dW_t,$$

and so by Itô's formula,

$$d(\log \tilde{S}_t) = \sigma dW_t - \frac{1}{2}\sigma^2 dt.$$

In other words,

$$\tilde{S}_T = \tilde{S}_t \exp\left(\sigma(W_T - W_t) - \frac{1}{2}\sigma^2(T-t)\right).$$

Now from Theorem 8.4,

$$V_t = \mathbb{E}^*\left[e^{-r(T-t)} f(S_T) \middle| \mathcal{F}_t\right],$$

and we have just shown this to be equal to

$$V_t = \mathbb{E}^*\left[e^{-r(T-t)} f\left(S_t e^{-r(T-t)} \exp\left(\sigma(W_T - W_t) - \frac{1}{2}\sigma^2(T-t)\right)\right) \middle| \mathcal{F}_t\right].$$

Under $\mathbb{P}^*$, $W_T - W_t$ is a normally distributed random variable with mean zero and variance $(T-t)$ and so we can evaluate the expectation and after some manipulation we obtain the result. $\square$

For European calls and puts, F can be calculated explicitly.

Example 9.3 (European call) *Suppose* $f(S_T) = (S_T - K)_+$. *Then writing* $\theta = (T-t)$,

$$F(t,x) = \mathbb{E}\left[e^{-r\theta}\left(xe^{\sigma\sqrt{\theta}g - \sigma^2/2} - K\right)_+\right],$$

where $g \sim N(0,1)$.

First we establish for what range of values of g the integrand is non-zero. It is easy to check that

$$xe^{\sigma\sqrt{\theta}g-\sigma^2/2} > Ke^{-r\theta}$$

is equivalent to

$$g > \frac{\log\left(\frac{K}{x}\right) + \frac{\sigma^2}{2}\theta - r\theta}{\sigma\sqrt{\theta}}.$$

Writing

$$d_1 = \frac{\log\left(\frac{x}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)\theta}{\sigma\sqrt{\theta}}$$

and $d_2 = d_1 - \sigma\sqrt{\theta}$, the range of g is $g + d_2 \geq 0$. Using this notation

$$\begin{aligned}
F(t,x) &= \mathbb{E}\left[\left(xe^{\sigma\sqrt{\theta}g-\sigma^2/2} - Ke^{-r\theta}\right)\chi_{g+d_2\geq 0}\right] \\
&= \int_{-d_2}^{\infty}\left(xe^{\sigma\sqrt{\theta}y-\sigma^2/2} - Ke^{-r\theta}\right)\frac{e^{-y^2/2}}{\sqrt{2\pi}}dy \\
&= \int_{-\infty}^{d_2}\left(xe^{\sigma\sqrt{\theta}y-\sigma^2/2} - Ke^{-r\theta}\right)\frac{e^{-y^2/2}}{\sqrt{2\pi}}dy \\
&= x\int_{-\infty}^{d_2} e^{\sigma\sqrt{\theta}y-\sigma^2/2}\frac{e^{-y^2/2}}{\sqrt{2\pi}}dy - Ke^{-r\theta}N(d_2),
\end{aligned}$$

where $N(\cdot)$ is the distribution function for the standard normal distribution. Substituting $z = y + \sigma\sqrt{\theta}$ in the first integral we finally obtain

$$F(t,x) = xN(d_1) - Ke^{-r\theta}N(d_2).$$

□

Using identical notations, one calculates that the price of a put is

$$F(t,x) = Ke^{-r\theta}N(d_2) - xN(-d_1).$$

Remarks: One of the main features of the Black-Scholes model is the fact that the pricing formulae as well as the hedging formulae that we derive below depend only on one non-observable parameter, σ, called the *volatility* by practitioners. The drift parameter μ disappears by the change of probability.

In practice two methods are used to evaluate σ:

1. The historical method: in the present model, $\sigma^2 T$ is the variance of $\log(S_T/S_0)$ and the variables $\log(S_T/S_0), \log(S_{2T}/S_T), \ldots, \log(S_{NT}/S_{(N-1)T})$ are independent and identically distributed. Therefore σ can be estimated by statistical means using asset prices observed in the past (for example by calculating empirical variances).

2. The *implied* volatility: some options are quoted on organised markets. The price of options (calls and puts) is an increasing function of σ, so we can invert the Black-Scholes formula and associate an implied volatility to each option.

In problems concerned with volatility we soon see imperfections in the Black-Scholes model. Important differences between implied and historical volatility are observed. The former appears to depend on the strike price and the time to maturity. (In fact, if one plots a graph of imlied volatility against strike price for options based on the same asset and with the same maturity, one typically sees a volatility 'smile'.) Nonetheless, the model is regarded as a standard reference.

We now turn to the problem of *hedging* European options. That is, how should we construct a portfolio that replicates the claim against us.

From the proof of Theorem 8.4, we want to find an adapted process $(H_t)_{0\leq t\leq T}$, such that

$$e^{-rT}f(S_T) = H_0 + \int_0^T H_u d\tilde{S}_u$$

Any replicating formula must have, at any time t, a discounted value equal to

$$\tilde{V}_t = e^{-rt}F(t, S_t),$$

where $F(t,x)$ is the function defined in Proposition 9.2.

If we set

$$\tilde{F}(t,x) = e^{-rt}F(t, xe^{rt}),$$

then we have $\tilde{V}_t = \tilde{F}(t,\tilde{S}_t)$ and for $t < T$, Itô's formula gives

$$\tilde{F}(t,\tilde{S}_t) = \tilde{F}(0,\tilde{S}_0) + \int_0^t \frac{\partial \tilde{F}}{\partial x}(u,\tilde{S}_u)d\tilde{S}_u + \int_0^t \frac{\partial \tilde{F}}{\partial t}(u,\tilde{S}_u)du + \int_0^t \frac{1}{2}\frac{\partial^2 \tilde{F}}{\partial x^2}(u,\tilde{S}_u)d\langle \tilde{S},\tilde{S}\rangle_u.$$

Now from Lemms 8.3 again, $d\tilde{S}_t = \sigma\tilde{S}_t dW_t$ so this becomes

$$\tilde{F}(t,\tilde{S}_t) = \tilde{F}(0,\tilde{S}_0) + \int_0^t \sigma\frac{\partial \tilde{F}}{\partial x}(u,\tilde{S}_u)\tilde{S}_u dW_u + \int_0^t K_u du.$$

Since $\tilde{F}(t,\tilde{S}_t) = \tilde{V}_t$ is a martingale under the measure $\mathbb{P}^*$, the process K_u is necessarily null. Hence

$$\tilde{F}(t,\tilde{S}_t) = \tilde{F}(0,\tilde{S}_0) + \int_0^t \frac{\partial \tilde{F}}{\partial x}(u,\tilde{S}_u)d\tilde{S}_u.$$

A natural candidate for H_t is then

$$H_t = \frac{\partial \tilde{F}}{\partial x}(t,\tilde{S}_t) = \frac{\partial F}{\partial x}(t,S_t).$$

If we set $H_t^0 = \tilde{F}(t,\tilde{S}_t) - H_t\tilde{S}_t$, then the portfolio (H_t^0, H_t) is self-financing and its discounted value is indeed $\tilde{V}_t = \tilde{F}(t,\tilde{S}_t)$ as required.

Example 9.4 (Hedging a European call)

Using the same notation as in Example 9.3 we have

$$F(t,x) = \mathbb{E}\left[\left(x\exp\left(\sigma\sqrt{\theta}g - \sigma^2\theta/2\right) - K\right)_+\right],$$

where $g \sim N(0,1)$ and $\theta = (T-t)$. Differentiating the integrand with respect to x, we get $\exp\left(\sigma\sqrt{\theta}g - \sigma^2/2\right)$ if the integrand is strictly positive and zero otherwise. Then, again using the notation of Example 9.3,

$$\begin{aligned} \frac{\partial F}{\partial x}(t,x) &= \mathbb{E}\left[\exp\left(\sigma\sqrt{\theta}g - \sigma^2\theta/2\right)\chi_{g+d_2\geq 0}\right] \\ &= \int_{-d_2}^{\infty} \exp\left(\sigma\sqrt{\theta}y - \sigma^2\theta/2 - y^2/2\right)\frac{1}{\sqrt{2\pi}}dy. \end{aligned}$$

Substituting first $u = -y$ and then $z = u + \sigma\sqrt{\theta}$ as before this reduces to $N(d_1)$. So

$$\frac{\partial F}{\partial x}(t,x) = N(d_1).$$

For the European put one calculates

$$\frac{\partial F}{\partial x}(t,x) = -N(-d_1).$$

The quantity $\partial F/\partial x$ is often called the *delta* by practitioners. Question 30 of the problem sheets explains why.