4 Brownian motion

4.1 Definition of the process

Our discrete models are only a crude approximation to the way in which stock markets actually move. A better model would be one in which stock prices can change at any instant. As early as 1900 Bachelier, in his thesis 'la theorie de la spéculation' proposed Brownian motion as a model of the fluctuations of stock prices. It is remarkable that a large portion of modern finance theory uses *geometric Brownian motion* as the underlying model of the motion of a stock price. In this section we construct and study Brownian motion, the basic building block from which geometric Brownian motion is easily constructed. The importance of Brownian motion in modern probability theory cannot be over-emphasized.

The easiest way to think about Brownian motion is as an 'infinitesimal random walk' and that is often how it arises in applications, so to motivate the formal definition we first study simple random walk.

Suppose that I play a game with a friend which at each stage is equivalent to flipping a coin and if the coin comes up heads my friend pays me a dollar and if it comes up tails, then I pay her a dollar.

Let X_i be the change in my fortune over the ith play. Then $\mathbb{P}[X_i = 1] = \mathbb{P}[X_i = -1] = 1/2$. Moreover, the X_i are *independent* random variables.

Definition 4.1 Simple random walk *is the stochastic process* $S_n = \sum_{i=1}^n X_i$, *where* X_i, $i = 1, 2, \ldots$ *are independent random variables with* $\mathbb{P}[X_i = 1] = \mathbb{P}[X_i = -1] = 1/2$.

In our gambling example, S_n represents my net gain (possibly negative) after n plays.

Note that $\mathbb{E}[S_n] = 0$ and $var(S_n) = n$.

Lemma 4.2 *1. S_n is a martingale.*

2. $cov(S_n, S_m) = n \wedge m$.

Proof:
1.

$$\begin{aligned}
\mathbb{E}[S_n|\mathcal{F}_j] &= \mathbb{E}[S_n|X_1, X_2, \ldots, X_j] \\
&= \mathbb{E}[\sum_1^j X_i + \sum_{j+1}^n X_i|X_1, X_2, \ldots, X_j] \\
&= \sum_1^j X_i + \sum_{j+1}^n \mathbb{E}[X_i] \\
&= \sum_1^j X_i = S_j.
\end{aligned}$$

(In the penultimate line we have used the independence of the X_i's.)
2.

$$\begin{aligned}
cov(S_n, S_m) &= \mathbb{E}[S_n S_m] - \mathbb{E}[S_n]\mathbb{E}[S_m] \\
&= \mathbb{E}\left[\mathbb{E}[S_n S_m | \mathcal{F}_{m\wedge n}]\right] \\
&= \mathbb{E}\left[S_{m\wedge n}\mathbb{E}[S_{m\vee n} | \mathcal{F}_{m\wedge n}]\right] \\
&= \mathbb{E}[S^2_{m\wedge n}] \\
&= var(S_{m\wedge n}) = m \wedge n.
\end{aligned}$$

□

If $1 \leq i < j \leq k < l$, then $S_j - S_i$ is independent of $S_l - S_k$. More generally, if $0 \leq i_1 \leq i_2 \leq \cdots \leq i_n$, then $\{S_{i_r} - S_{i_{r-1}}, 1 \leq r \leq n\}$ are independent. Moreover, if $j - i = l - k = m$, say, then

$$S_j - S_i \stackrel{\mathcal{D}}{=} S_l - S_k \stackrel{\mathcal{D}}{=} S_m.$$

Combining these gives

Lemma 4.3 *The process S_n has stationary, independent increments.*

Recall that we want to think of Brownian motion as infinitesimal random walk. In terms of our gambling game, the time interval between plays is δt and the stake is δx say, and we are thinking of both of these as 'tending to zero'. In order to obtain a nontrivial limit, there has to be a relationship between δt and δx. To see what this must be, we use the Central Limit Theorem.

Theorem 4.4 (Central Limit Theorem) *Let $X_1, X_2, \ldots$ be a sequence of independent identically distributed random variables with finite means μ and finite nonzero variances σ^2 and let $S_n = X_1 + \ldots + X_n$. Then*

$$\frac{S_n - n\mu}{\sqrt{n\sigma^2}}$$

converges in distribution to an $N(0,1)$ random variable as $n \to \infty$.

In our setting, $\mu = \mathbb{E}[X_i] = 0$ and $\sigma^2 = var(X_i) = 1$. Thus,

$$\mathbb{P}\left[\frac{S_n}{\sqrt{n}} \leq x\right] \to \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy \text{ as } n \to \infty.$$

More generally,

$$\mathbb{P}\left[\frac{S_{[nt]}}{\sqrt{n}} \leq x\right] \to \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi t}} e^{-y^2/2t} dy \text{ as } n \to \infty.$$

At time t, in the limit, our net gain since time zero will be normally distributed with mean zero and variance t.

Formally passing to the limit in Lemmas 4.2 and 4.3 suggests that the following is a reasonable definition of Brownian motion.

Definition 4.5 (Brownian motion) *A stochastic process $\{B_t\}_{t\geq 0}$ in continuous time taking real values is a* Brownian motion *(or a* Wiener process*) if, for some real constant σ,*

1. *For each $s \geq 0$ and $t > 0$ the random variable $B_{t+s} - B_s$ has the normal distribution with mean zero and variance $\sigma^2 t$.*
2. *For each $n \geq 1$ and any times $0 \leq t_0 \leq t_1 \leq \cdots \leq t_n$, the random variables $\{B_{t_r} - B_{t_{r-1}}\}$ are independent.*
3. *$B_0 = 0$.*
4. *B_t is continuous in $t \geq 0$.*

Remarks. We consider 1-4 in turn.

1. This condition is self-explanatory. The parameter σ^2 is known as the variance parameter. By scaling of the normal distribution it is immediate that $\{B_{t/\sigma}\}_{t\geq 0}$ is a Brownian motion with variance parameter one.

The process with $\sigma^2 = 1$ is called *standard* Brownian motion. *Unless otherwise stated we shall always assume that $\sigma^2 = 1$.*

2. This says that Brownian motion has independent increments.

Combining 1 and 2 we can write down the *transition probabilities* of the process exactly as

$$\begin{aligned}\mathbb{P}\left[B_{t_n} \leq x | B_{t_i} = x_i, 0 \leq i \leq n-1\right] &= \mathbb{P}\left[B_{t_n} - B_{t_{n-1}} \leq x_n - x_{n-1}\right] \\ &= \int_{-\infty}^{x_n - x_{n-1}} \frac{1}{\sqrt{2\pi(t_n - t_{n-1})}} \exp\left(-\frac{u^2}{2(t_n - t_{n-1})}\right) du.\end{aligned}$$

Notation. We write $p(t,x,y)$ for the *transition density*

$$p(t,x,y) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(x-y)^2}{2t}\right).$$

The joint probability density function of $B_{t_1}, \ldots, B_{t_n}$ can also be written down explicitly as

$$f(x_1, \ldots, x_n) = \prod_1^n p(t_j - t_{j-1}, x_j, x_{j-1}),$$

where we think of $t_0 = 0$, $x_0 = 0$.

For any $s, t > 0$, we have that $cov(B_s, B_t) = s \wedge t$. Since the multivariate normal distribution is determined by its means and covariances and normally distributed random variables are independent if and only if their covariances are zero, it is immediate that (when $\sigma^2 = 1$) conditions 1-3 are equivalent to requiring that for any $n \geq 1$ and $t_1, \ldots, t_n$ the joint distribution of $B_{t_1}, \ldots, B_{t_n}$ is normal with mean zero and $cov(B_{t_i}, B_{t_j}) = t_i \wedge t_j$.

The joint distributions of $B_{t_1}, \ldots, B_{t_n}$ for each $n \geq 1$ and all $t_1, \ldots, t_n$ are called the *finite dimensional distributions* of the process.

Condition 3 is just a convention that is useful for our purposes. Brownian motion started from x at time zero can be obtained as $\{x + B_t\}_{t\geq 0}$.

Finally we turn to condition 4. In a certain sense it is a consequence of 1–3, but we insist once and for all that all paths that our Brownian motion can follow are continuous.

Just because the sample paths of Brownian motion are continuous, it does not mean that they are nice in any other sense. For example, with probability one, a Brownian path will be nowhere differentiable. We return to this in §6.

4.2 Lévy's construction of Brownian motion

We haven't actually proved that Brownian motion exists, although we have hinted that it can be obtained as a limit of random walks. Rather than chasing the technical details of the random walk construction, we present an alternative approach due to Lévy. The idea is that we can simply produce a path of Brownian motion by direct polygonal interpolation. We require just one special computation

Lemma 4.6 *Suppose that $\{B_t\}_{t\geq 0}$ is standard Brownian motion. Conditional on $B_{t_1} = x_1$, the probability density function of $B_{t_1/2}$ is*

$$\sqrt{\frac{2}{\pi t_1}} \exp\left(-\frac{1}{2}\left(\frac{\left(x - \frac{1}{2}x_1\right)^2}{t_1/4}\right)\right).$$

(The proof is Question 11 of the problem sheets.)

Lévy's construction now proceeds as follows. We define inductively a sequence of processes $X_n(t)$. *Without loss of generality* we take the range of t to be $[0,1]$. We need a countable number of *independent* normally distributed random variables with mean zero and variance one. We index them by the dyadic points of $[0,1]$, a generic variable then being denoted $\xi(k2^{-n})$.

Our induction begins with

$$X_1(t) = t\xi(1).$$

Thus X_1 is a linear function on $[0,1]$. The nth process is linear in each interval $[(k-1)2^{-n}, k2^{-n}]$ and is continuous in t. It is thus determined by its values $X_n(k2^{-n})$, $X_n(0) = 0$. Now for the inductive step. We take

$$X_{n+1}\left(2k2^{-(n+1)}\right) = X_n\left(2k2^{-(n+1)}\right) = X_n\left(k2^{-n}\right).$$

The new values are given by Lemma 4.6.

$$X_{n+1}\left((2k-1)2^{-(n+1)}\right) = X_n\left((2k-1)2^{-(n+1)}\right) + 2^{-(n/2+1)}\xi\left((2k-1)2^{-(n+1)}\right).$$

(We have used that if $X \sim N(0,1)$, then $aX + b \sim N(b, a^2)$.)

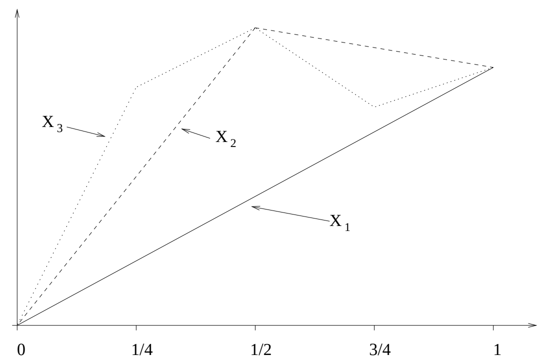

Figure 6.

Lemma 4.7

$$\mathbb{P}\left[\lim_{n\to\infty} X_n(t) \textit{ exists for } 0 \le t \le 1 \textbf{ uniformly } \textit{in } t\right] = 1.$$

Proof: Consider

$$\begin{aligned}
&\mathbb{P}\left[\max_t |X_{n+1}(t) - X_n(t)| \ge 2^{-n/4}\right] \\
&\qquad = \mathbb{P}\left[\max_{1\le k\le 2^n} \xi\left((2k-1)2^{-(n+1)}\right) \ge 2^{n/4+1}\right] \\
&\qquad\qquad \le 2^n \mathbb{P}\left[\xi(1) \ge 2^{n/4+1}\right]
\end{aligned}$$

and since $\exp(-2^{n/2}+1) < 2^{-2n+2}$ for $n \ge 4$, we estimate

$$\mathbb{P}\left[\max_t |X_{n+1}(t) - X_n(t)| \ge 2^{-n/4}\right] \le 2^{-n}$$

if $n \ge 4$. Consider now for $k > n$,

$$\begin{aligned}
&\mathbb{P}\left[\max_t |X_k(t) - X_n(t)| \ge 2^{-n/4+3}\right] \\
&\qquad = 1 - \mathbb{P}\left[\max_t |X_k(t) - X_n(t)| \le 2^{-n/4+3}\right]
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{P}\left[\max_t |X_k(t) - X_n(t)| \le 2^{-n/4+3}\right] \\
&\qquad \ge \mathbb{P}\left[\sum_n^{k-1} \max_t |X_{j+1}(t) - X_j(t)| \le 2^{-n/4+3}\right] \\
&\qquad \ge \mathbb{P}\left[\max_t |X_{j+1}(t) - X_j(t)| \le 2^{-j/4}, j = n, \dots, k-1\right] \\
&\qquad \ge 1 - \sum_{j=n}^{k-1} 2^{-j} = 1 - 2^{-n+1}.
\end{aligned}$$

Finally we have that

$$\mathbb{P}\left[\max_t |X_k(t) - X_n(t)| \ge 2^{-n/4+3}\right] \le 2^{-n+1},$$

for all $k \ge n$. The events on the left are increasing (since the maximum can only increase by the addition of a new vertex) so

$$\mathbb{P}\left[\max_t |X_k(t) - X_n(t)| \ge 2^{-n/4+3} \text{ for some } k > n\right] \le 2^{-n+1}.$$

In particular, for $\epsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left[\text{For some } k > n \text{ and } t \le 1,\ |X_k(t) - X_n(t)| \ge \epsilon\right] = 0,$$

which proves the lemma. □

To complete the proof of existence of the Brownian motion, we must check the following.

Lemma 4.8 *Let $X(t) = \lim_{n\to\infty} X_n(t)$ if the limit exists uniformly and zero otherwise. Then $X(t)$ satisfies the conditions of Definition 4.5 (for t restricted to $[0,1]$).*

Proof: All the properties 1-4 of Definition 4.5 hold for $X_n(t)$ when we restrict to $T_n = \{k2^{-n}\}$. Since we don't change X_k on T_n for $k > n$, they evidently hold for X on $\cup_{n=1}^{\infty} T_n$. Since a uniform limit of continuous functions is continuous, condition 4 holds, and then by approximation of any $0 \le t_1 \le \cdots \le t_n \le 1$ from within the dense set $\cup_{n=1}^{\infty} T_n$, the properties 1-4 must hold without restriction for $t \in [0,1]$. □

5 The reflection principle and scaling

Having proved that Brownian motion actually exists, we illustrate how to calculate the distributions of various quantities related to the process.

It is easy to check that if B_t is a Brownian motion and $s \ge 0$ is any fixed time, then $\{B_{t+s} - B_s\}_{t\ge 0}$ is also a Brownian motion. What is also true is that for certain *random* times, T, the process $\{B_{T+t} - B_T\}_{t\ge 0}$ is again a standard Brownian motion and is *independent* of $\{B_s, 0 \le s \le T\}$.

Definition 5.1 *A* stopping time *T for the process $\{B_t\}_{t\ge 0}$ is a random time such that for each t, the event $\{T \le t\}$ depends only on the history of the process up to and including time t.*

In other words, by observing the Brownian motion up until time t, we can determine whether or not $T \le t$.

We shall encounter stopping times only in the context of *hitting times*. For example, for fixed a, the hitting time of level a is defined by

$$T_a = \inf\{t \ge 0 : B_t = a\}.$$

(We take $T_a = \infty$ if a is never reached.) It is easy to see that T_a is a stopping time since, by continuity of the paths,

$$\{T_a \le t\} = \{B_s = a \text{ for some } s, 0 \le s \le t\},$$

which depends only on $\{B_s, 0 \le s \le t\}$. Notice that, again by continuity, if $T_a < \infty$, then $B_{T_a} = a$.

An example of a random time that is *not* a stopping time is the last time that the process hits some level.

As a warmup we calculate the distribution of T_a.

Lemma 5.2 *Let $a > 0$.*

$$\mathbb{P}_0\left[T_a < t\right] = 2\mathbb{P}_0\left[B_t > a\right].$$

We use the subscript '0' to emphasize that our Brownian motion starts at zero at time zero.

Proof: If $B_t > a$, then by continuity of the Brownian path, $T_a < t$. Moreover, by symmetry, $\mathbb{P}[B_t - B_{T_a} > 0 | T_a < t] = 1/2$. Thus

$$\begin{aligned}\mathbb{P}[B_t > a] &= \mathbb{P}[T_a < t, B_t - B_{T_a} > 0] \\ &= \mathbb{P}[T_a < t]\mathbb{P}[B_t - B_{T_a} > 0 | T_a < t] \\ &= \frac{1}{2}\mathbb{P}[T_a < t].\end{aligned}$$

□

A more refined version of this idea is the following.

Lemma 5.3 (The reflection principle) *Let $\{B_t, t \geq 0\}$ be a standard Brownian motion and let T be a stopping time and define*

$$\tilde{B}_t = \begin{cases} B_t & t \leq T, \\ 2B_T - B_t & t > T. \end{cases}$$

Then $\{\tilde{B}_t, t \geq 0\}$ is a standard Brownian motion.

We won't prove this here. Notice that if $T = T_a$, then the operation $B_t \mapsto \tilde{B}_t$ amounts to reflecting the portion of the path after the first hitting time on a in the line $x = a$. We illustrate it in action. Calculations of this type are used in the analysis of barrier options.

Example 5.4 (Joint distribution of Brownian motion and its maximum) *Let $M_t = \max_{0 \leq s \leq t} B_s$, the maximum level reached by Brownian motion in the time interval $[0, t]$. Then for $a > 0$, $a \geq x$ and all $t \geq 0$,*

$$\mathbb{P}[M_t \geq a, B_t \leq x] = 1 - \Phi\left(\frac{2a - x}{\sqrt{t}}\right),$$

where

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du$$

is the standard normal distribution function.

Notice that $M_t \geq 0$ and is non-decreasing in t and if, for $a > 0$, T_a is defined as above, then $\{M_t \geq a\} = \{T_a \leq t\}$. Taking $T = T_a$ in the reflection principle, for $a \geq 0$, $a \geq x$ and $t \geq 0$,

$$\begin{aligned}\mathbb{P}[M_t \geq a, B_t \leq x] &= \mathbb{P}[T_a \leq t, B_t \leq x] \\ &= \mathbb{P}[T_a \leq t, 2a - x \leq \tilde{B}_t] \\ &= \mathbb{P}[2a - x \leq \tilde{B}_t] \\ &= 1 - \Phi\left(\frac{2a - x}{\sqrt{t}}\right).\end{aligned}$$

Lemma 5.5 (Hitting a sloping line) *Set $T_{a,b} = \inf\{t \geq 0 : B_t = a + bt\}$, where $T_{a,b}$ is taken to be infinite if no such time exists. Then*

$$\mathbb{E}\left[\exp\left(-\theta T_{a,b}\right)\right] = \exp\left(-a\left(b + \sqrt{b^2 + 2\theta}\right)\right).$$

Proof: We *assume* the result for $b = 0$, but there will be a very slick proof of that case in §6.

Fix $\theta > 0$, and for $a > 0$, $b \geq 0$, set

$$\psi(a, b) = \mathbb{E}\left[e^{\left(-\theta T_{a,b}\right)}\right].$$

Now take any two values for a, a_1 and a_2 say, and notice that

$$T_{a_1+a_2,b} = T_{a_1,b} + (T_{a_1+a_2,b} - T_{a_1,b}) \stackrel{\mathcal{D}}{=} T_{a_1,b} + \tilde{T}_{a_2,b},$$

(where $\tilde{T}_{a_2,b}$ is independent of $T_{a_1,b}$ and has the same distribution as $T_{a_2,b}$). In other words,

$$\psi(a_1 + a_2, b) = \psi(a_1, b)\psi(a_2, b),$$

and this implies that

$$\psi(a, b) = e^{(-k(b)a)},$$

for some function $k(b)$.

Since $b \geq 0$, the process must hit level a *before* it can hit the line $a + bt$ and so, by conditioning on T_a, we obtain

$$\begin{aligned}
\psi(a, b) &= \int_0^\infty f_{T_a}(t)\mathbb{E}\left[e^{\left(-\theta T_{a,b}\right)} \middle| T_a = t\right] dt \\
&= \int_0^\infty f_{T_a}(t)e^{-\theta t}\mathbb{E}\left[e^{\left(-\theta T_{bt,b}\right)}\right] dt \\
&= \int_0^\infty f_{T_a}(t)e^{-\theta t}e^{(-k(b)bt)} dt \\
&= \mathbb{E}\left[e^{-(\theta+k(b)b)T_a}\right] \\
&= \exp\left(-a\sqrt{2(\theta + k(b)b)}\right).
\end{aligned}$$

Equating the two expressions we now have for $\psi(a, b)$ gives

$$k^2(b) = 2\theta + 2k(b)b,$$

and since for $\theta > 0$ we must have $\psi(a, b) \leq 1$, this completes the proof. $\square$

Definition 5.6 *For a real constant μ, we refer to the process $B_t^\mu = B_t + \mu t$ as a* Brownian motion with drift *μ.*

In the notation above, $T_{a,b}$ is the first hitting time of the level a by a Brownian motion with drift $-b$.

We conclude this section with the following useful result.

Proposition 5.7 (Transformation and scaling of Brownian motion) *If $\{B_t, t \geq 0\}$ is a standard Brownian motion, then so are*

1. *$\{cB_{t/c^2}, t \geq 0\}$ for any real c,*
2. *$\{tB_{1/t}, t \geq 0\}$ where $tB_{1/t}$ is taken to be zero when $t = 0$,*

3. $\{B_s - B_{s-t}, 0 \leq t \leq s\}$ *for any fixed* $s \geq 0$.

Proof: The proofs of 1–3 are similar. For example in the case of 2, it is clear that $tB_{1/t}$ has continuous sample paths (at least for $t > 0$) and that for any $t_1, \dots, t_n$, the random variables $\{t_1 B_{1/t_1}, \dots, t_n B_{1/t_n}\}$ have a multivariate normal distribution. We must just check that the covariance takes the right form, but

$$\mathbb{E}\left[sB_{1/s}tB_{1/t}\right] = st\mathbb{E}\left[B_{1/s}B_{1/t}\right] = st\left(\frac{1}{s} \wedge \frac{1}{t}\right) = s \wedge t.$$

$\square$