Asymptotic Properties of Recursive Particle Maximum Likelihood Estimation

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Abstract—Using stochastic gradient search and the optimal filter derivative, it is possible to perform recursive maximum likelihood estimation in a non-linear state-space model. As the optimal filter and its derivative are analytically intractable for such a model, they need to be approximated numerically. In Poyiadjis *et al.* (G. Poyiadjis, A. Doucet, and S. S. Singh, Biometrika, vol. 98, no. 1, pp. 65–80, 2011), a recursive maximum likelihood algorithm based on a particle approximation to the optimal filter derivative has been proposed and studied through numerical simulations. This algorithm and its asymptotic behavior are here analyzed theoretically. Under regularity conditions, we show that the algorithm accurately estimates maxima of the underlying log-likelihood rate when the number of particles is sufficiently large. We also provide qualitative upper bounds on the estimation error in terms of the number of particles.

Index Terms—Non-linear state-space models, recursive maximum likelihood estimation, sequential Monte Carlo methods, system identification.

I. INTRODUCTION

STATE-space models (also known as continuous-state hidden Markov models) are a class of stochastic processes capable of modeling complex time-series data and stochastic dynamical systems. These models can be viewed as a discrete-time Markov process which can be observed only through noisy measurements of its states.

In many applications, a state-space model depends on a parameter whose value needs to be estimated given a set of state-observations. Due to its practical and theoretical importance, parameter estimation in state-space and hidden Markov models has been extensively studied in the engineering and statistics literature (see e.g. [8], [14] and references cited therein). Among them, the methods based on maximum likelihood principle have gained much attention. Their asymptotic properties (convergence, convergence rate and asymptotic normality) have been analyzed thoroughly in a number of papers (see e.g. [4], [11], [12], [21], [23], [26]–[28]). Unfortunately, to the best of our knowledge, the existing results do not offer much information about recursive (i.e., online) maximum likelihood estimation in non-linear state-space models. However, in a number of different scenarios, the parameter indexing a state-space model needs to

be estimated recursively. For example, this is much more computationally efficient for long observation sequences. In the maximum likelihood approach, this can be achieved using stochastic gradient search and the optimal filter derivative. Since the optimal filter and its derivative are not analytically tractable for a non-linear state-space model, they need to be approximated numerically. In [25], a recursive maximum likelihood algorithm based on a particle approximation to the optimal filter derivative has been proposed and it has been shown experimentally that the algorithm is stable and efficient. We show here that the algorithm proposed in [25] produces asymptotically accurate estimates of maxima to the underlying log-likelihood rate. More specifically, we show that these estimates converge almost surely to a close vicinity of stationary points of the underlying log-likelihood rate. We also provide qualitative upper bounds on the radius of this vicinity. These bounds are expressed in terms of the number of particles used to approximate the filter and its derivative and directly characterize the (asymptotic) error of the recursive particle maximum likelihood algorithm proposed in [25]. The obtained results hold under strong mixing assumptions which are very commonly used in the particle filtering literature (see e.g. [8], [9], [10], [14]). To the best of our knowledge, the results presented here are the first to offer a rigorous analysis of recursive maximum likelihood estimation in non-linear state-space models.

The rest of this paper is organized as follows. In Section II, non-linear state-space models and the corresponding recursive maximum likelihood algorithm are specified. In the same section, the main results of the paper are presented. In Section III, a non-trivial example illustrating the main results is provided. The main results are proved in Sections IV – VI.

II. MAIN RESULTS

A. State-Space Models and Parameter Estimation

To define state-space models, we use the following notation. For a set $\mathcal{Z}$ in a metric space, $\mathcal{B}(\mathcal{Z})$ denotes the collection of Borel subsets of $\mathcal{Z}$. $d_x \geq 1$ and $d_y \geq 1$ are integers, while $\mathcal{X} \in \mathcal{B}(\mathbb{R}^{d_x})$ and $\mathcal{Y} \in \mathcal{B}(\mathbb{R}^{d_y})$. $P(x, dx')$ is a transition kernel on $\mathcal{X}$, while $Q(x, dy)$ is a conditional probability measure on $\mathcal{Y}$ given $x \in \mathcal{X}$. $(\Omega, \mathcal{F}, P)$ is a probability space. A state-space model can be defined as the $\mathcal{X} \times \mathcal{Y}$-valued stochastic process $\{(X_n, Y_n)\}_{n \geq 0}$ on $(\Omega, \mathcal{F}, P)$ which satisfies

$$P\left((X_{n+1}, Y_{n+1}) \in B | X_{0:n}, Y_{0:n}\right)$$
$$= \iint I_B(x, y) Q(x, dy) P(X_n, dx)$$

almost surely for each $B \in \mathcal{B}(\mathcal{X} \times \mathcal{Y})$, $n \geq 0$. $\{X_n\}_{n \geq 0}$ are the unobservable model states, while $\{Y_n\}_{n \geq 0}$ are the

on the asymptotic behavior of the algorithm (Theorem 2.1, below) are expressed in a local form.

The variables appearing in algorithm (1) – (4) have the following meaning. θ_0, $\hat{X}_{0,1}, \ldots, \hat{X}_{0,N}$ and $W_{0,1}, \ldots, W_{0,N}$ are the initial conditions. $\hat{X}_{n,1}, \ldots, \hat{X}_{n,N}$ are particles whose empirical distribution approximates the predictive distribution of X_n given $Y_1, \ldots, Y_{n-1}$ and its derivative (computed using parameter θ_n at time n), while $W_{n,1}, \ldots, W_{n,N}$ are vector-valued weights in the particle approximation to this derivative.[1] θ_n is an estimate of maxima to the log-likelihood rate $l(\theta)$. α_n is the step-size in recursion (1). Recursion (3) is a stochastic gradient search maximizing $l(\theta)$.[2] Recursions (1) and (4) are procedures through which the particle approximations to the optimal filter and its derivative are updated. More details on the recursive particle maximum likelihood algorithm can be found in [25].

C. Convergence Results

To formulate the assumptions under which the analysis of the recursive particle maximum likelihood estimation procedure is carried out, we introduce further notation. $\mathbb{N}_0$ is the set of non-negative integers, while $\mathbb{C}^d$ is the set d-dimensional complex valued vectors. For $\boldsymbol{\alpha} = (a_1, \ldots, a_d) \in \mathbb{N}_0^d$, $\theta = (t_1, \ldots, t_d) \in \mathbb{R}^d$, notation $|\boldsymbol{\alpha}|$ and $\partial_\theta^{\boldsymbol{\alpha}}$ stand for

$$|\boldsymbol{\alpha}| = a_1 + \cdots + a_d, \qquad \partial_\theta^{\boldsymbol{\alpha}} = \frac{\partial_\theta^{|\boldsymbol{\alpha}|}}{\partial t_1^{a_1} \cdots t_d^{a_d}}.$$

For $\eta \in \mathbb{C}^d$, $\|\eta\|$ denotes the Euclidean norm of η. For $\delta \in (0, \infty)$, $\eta \in \mathbb{C}^d$ and $A \subset \mathbb{C}^d$, $V_\delta(A)$, $d(\eta, A)$ denote the δ-vicinity of A and the distance between η, A, i.e.,

$$d(\eta, A) = \inf_{\eta' \in A} \|\eta - \eta'\|,$$

$$V_\delta(A) = \{\eta' \in \mathbb{C}^d : \exists \eta'' \in A, \|\eta' - \eta''\| < \delta\}.$$

Let Q be any compact set satisfying $Q \subset \Theta$. The asymptotic properties of algorithm (1) – (4) are analyzed under the following assumptions.

Assumption 2.1: $\sum_{n=0}^\infty \alpha_n = \infty$, $\sum_{n=0}^\infty \alpha_n^2 < \infty$ and $\sum_{n=0}^\infty |\alpha_n - \alpha_{n+1}| < \infty$.

Assumption 2.2: There exist a probability measure $\pi(dx)$ and real numbers $\rho \in (0, 1)$, $K \in [1, \infty)$ such that

$$|P^n(x, B) - \pi(B)| \le K \rho^n$$

for all $x \in \mathcal{X}$, $B \in \mathcal{B}(\mathcal{X})$, $n \ge 0$.

Assumption 2.3: There exists a real number $\varepsilon_Q \in (0, 1)$ such that

$$\varepsilon_Q \le p_\theta(x'|x) \le \frac{1}{\varepsilon_Q}, \qquad \varepsilon_Q \le q_\theta(y|x) \le \frac{1}{\varepsilon_Q}$$

for all $\theta \in Q$, $x, x' \in \mathcal{X}$, $y \in \mathcal{Y}$.

[1] The empirical measures

$$\frac{1}{N} \sum_{i=1}^N \delta_{\hat{X}_{n,i}}(dx), \qquad \frac{1}{N} \sum_{i=1}^N \left(W_{n,i} - \frac{1}{N} \sum_{j=1}^N W_{n,j}\right) \delta_{\hat{X}_{n,i}}(dx)$$

can be viewed as particle approximations (respectively) to the optimal (one-step) predictor and its gradient at discrete-time n. Here and throughout the paper, $\delta_x(dx')$ denotes the Dirac measure centered at $x \in \mathcal{X}$.

[2] The fraction on the right-hand side of (3) is a Monte Carlo estimate of $l(\theta)$.

Assumption 2.4: There exists a real number $K_{1,Q} \in [1, \infty)$ such that

$$\max\{\|\nabla_\theta p_\theta(x'|x)\|, \|\nabla_\theta q_\theta(y|x)\|\} \le K_{1,Q},$$

$$\max\{|p_\theta(x'|x) - p_{\theta'}(x'|x)|, \|\nabla_\theta p_\theta(x'|x) - \nabla_\theta p_{\theta'}(x'|x)\|\}$$

$$\le K_{1,Q} \|\theta - \theta'\|,$$

$$\max\{|q_\theta(y|x) - q_{\theta'}(y|x)|, \|\nabla_\theta q_\theta(y|x) - \nabla_\theta q_{\theta'}(y|x)\|\}$$

$$\le K_{1,Q} \|\theta - \theta'\|$$

for all $\theta, \theta' \in Q$, $x, x' \in \mathcal{X}$, $y \in \mathcal{Y}$.

Assumption 2.5: $p_\theta(x'|x)$ and $q_\theta(y|x)$ are p-times differentiable in θ for each $\theta \in \Theta$, $x, x' \in \mathcal{X}$, $y \in \mathcal{Y}$, where $p > d$. Moreover, there exists a real number $K_{2,Q} \in [1, \infty)$ such that

$$|\partial_\theta^{\boldsymbol{\alpha}} p_\theta(x'|x)| \le K_{2,Q}, \qquad |\partial_\theta^{\boldsymbol{\alpha}} q_\theta(y|x)| \le K_{2,Q}$$

for all $\theta \in Q$, $x, x' \in \mathcal{X}$, $y \in \mathcal{Y}$, $\boldsymbol{\alpha} \in \mathbb{N}_0^d$ satisfying $|\boldsymbol{\alpha}| \le p$.

Assumption 2.6: $p_\theta(x'|x)$ and $q_\theta(y|x)$ are real-analytic in θ for each $\theta \in \Theta$, $x, x' \in \mathcal{X}$, $y \in \mathcal{Y}$. Moreover, $p_\theta(x'|x)$ and $q_\theta(y|x)$ have (complex-valued) continuations $\hat{p}_\eta(x'|x)$ and $\hat{q}_\eta(y|x)$ with the following properties:

(i) $\hat{p}_\eta(x'|x)$ and $\hat{q}_\eta(y|x)$ map $\eta \in \mathbb{C}^d$, $x, x' \in \mathcal{X}$, $y \in \mathcal{Y}$ to $\mathbb{C}$.

(ii) $\hat{p}_\theta(x'|x) = p_\theta(x'|x)$ and $\hat{q}_\theta(y|x) = q_\theta(y|x)$ for all $\theta \in \Theta$, $x, x' \in \mathcal{X}$, $y \in \mathcal{Y}$.

(iii) There exists a real number $\delta_Q \in (0, 1)$ such that $\hat{p}_\eta(x'|x)$ and $\hat{q}_\eta(y|x)$ are analytic in η for each $\eta \in V_{\delta_Q}(Q)$, $x, x' \in \mathcal{X}$, $y \in \mathcal{Y}$.

(iv) There exists a real number $K_{3,Q} \in (0, 1)$ such that

$$|\hat{p}_\eta(x'|x)| \le K_{3,Q}, \qquad |\hat{q}_\eta(y|x)| \le K_{3,Q}$$

for all $\eta \in V_{\delta_Q}(Q)$, $x, x' \in \mathcal{X}$, $y \in \mathcal{Y}$.

Assumption 2.1 corresponds to the step-size sequence $\{\alpha_n\}_{n \ge 0}$ and its asymptotic properties. This assumption is standard in any asymptotic analysis of stochastic gradient search and stochastic approximation (see e.g., [3], [6], [18]). It holds when $\alpha_n = 1/n^a$ for $n \ge 1$, where $a \in (1/2, 1]$.

Assumption 2.2 is related to the stability of the true system $\{(X_n, Y_n)\}_{n \ge 0}$. It requires $\{X_n\}_{n \ge 0}$ to be uniformly ergodic. Assumption 2.3 implies the stability of the optimal filter for $\{(X_n^{\theta,\lambda}, Y_n^{\theta,\lambda})\}_{n \ge 0}$ (i.e., it ensures that the optimal filter forgets its initial condition exponentially fast). Assumption 2.2 and 2.3 are restrictive from the theoretical point of view as they implicitly require the state and observation spaces $\mathcal{X}$ and $\mathcal{Y}$ to be bounded. However, these assumptions are very commonly used in the literature (see, e.g., [8], [9], [14]).

Assumptions 2.4 – 2.6 are related to the parameterization of the candidate models $\{(X_n^{\theta,\lambda}, Y_n^{\theta,\lambda})\}_{n \ge 0}$ and its analytical properties (i.e. to the analytical properties of conditional densities $p_\theta(x'|x)$ and $q_\theta(y|x)$). The purpose of Assumption 2.4 is to ensure that the Poisson equation associated with algorithm (1) – (4) has a locally Lipschitz solution (see Lemma 5.4). Assumption 2.4 also ensures that the log-likelihood rate $l(\theta)$ is Lipschitz continuously differentiable (see Lemmas 4.1, 6.2). This Poisson equation plays a crucial role in the analysis of the asymptotic error in the Monte Carlo estimation of $\nabla l(\theta)$ (see Lemma 6.1 and (21)), while the Lipschitz continuity of $\nabla l(\theta)$ allows us to analyze algorithm (1) – (4) using the

results on Lipschitz gradient flows (see Theorem 2.1, Part (i)). The purpose of Assumption 2.5 is to provide for $l(\theta)$ to be at least $(d + 1)$-times differentiable (see Lemma 6.2, Part (ii)), while Assumption 2.6 ensures for $l(\theta)$ to be real-analytic (see Lemma 6.2, Part (iii)). These analytical properties of $l(\theta)$ allows us to establish qualitative upper bounds on the asymptotic error in the estimation of maxima to $l(\theta)$ (see Theorem 2.1, Parts (ii), (iii)).

In order to state the main results of the paper, we need further notation. $\mathcal{S}$ and $l(\mathcal{S})$ are the sets of stationary points and critical values of $l(\theta)$ (respectively), i.e.,

$$\mathcal{S} = \{\theta \in \Theta : \nabla l(\theta) = 0\}, \quad l(\mathcal{S}) = \{l(\theta) : \theta \in \mathcal{S}\}. \quad (5)$$

$\pi : \mathbb{R} \times \Theta \to \Theta$ is the solution to the ODE $d\theta/dt = \nabla l(\theta)$ which satisfies the initial condition $\pi(0, \theta) = \theta$ for $\theta \in \Theta$. $\mathcal{R}$ is the set of chain-recurrent points of the ODE $d\theta/dt = \nabla l(\theta)$, i.e., $\theta \in \mathcal{R}$ if and only if for any $\delta, t \in (0, \infty)$, there exist an integer $n \geq 1$, real numbers $t_1, \ldots, t_n \in [t, \infty)$ and vectors $\vartheta_1, \ldots, \vartheta_n \in \Theta$ (each of which can depend on θ, δ, t) such that $\|\vartheta_1 - \theta\| \leq \delta$, $\|\pi(t_n, \vartheta_n) - \theta\| \leq \delta$ and

$$\|\vartheta_{k+1} - \pi(t_k, \vartheta_k)\| \leq \delta$$

for $1 \leq k < n$.

Remark: Chain-recurrent points $\mathcal{R}$ can be interpreted as limit points of slightly perturbed solutions to the ODE $d\theta/dt = \nabla l(\theta)$. Since the piecewise linear interpolation of sequence $\{\theta_n\}_{n\geq 0}$ is such a solution (see (21) and Lemma 6.1; see also [30, Section 5]), the chain-recurrence is closely related to the asymptotic behavior of algorithm (1) – (4). Regarding stationary and chain-recurrent points, the following relationship can be established. If $l(\theta)$ is Lipschitz continuously differentiable, then all stationary points $\mathcal{S}$ are chain-recurrent for the ODE $d\theta/dt = \nabla l(\theta)$ (i.e., $\mathcal{S} \subseteq \mathcal{R}$). If additionally $l(\mathcal{S})$ is of a zero Lebesgue measure (which holds when $l(\theta)$ is d-times continuously differentiable), then all chain-recurrent points $\mathcal{R}$ are stationary for the ODE $d\theta/dt = \nabla l(\theta)$ (i.e., $\mathcal{S} = \mathcal{R}$). However, if $l(\theta)$ is only Lipschitz continuously differentiable, then $\mathcal{S} = \mathcal{R}$ does not necessarily hold and $\mathcal{R} \setminus \mathcal{S} \neq \emptyset$ is quite possible (for details, see [16, Section 4]). For more details on chain-recurrence, see [1], [2], [6].

Let Q be any compact set satisfying $Q \subset \Theta$, while Λ_Q is the event defined by

$$\Lambda_Q = \liminf_{n \to \infty} \{\theta_n \in Q\} = \bigcup_{n=0}^{\infty} \bigcap_{k=n}^{\infty} \{\theta_k \in Q\}. \quad (6)$$

Then, the main results of this paper are summarized in the next theorem.

Theorem 2.1: (i) If Assumptions 2.1 – 2.4 hold, then there exists a non-decreasing function $\psi_Q : [0, \infty) \to [0, \infty)$ depending only on $l(\theta)$, $p_\theta(x'|x)$, $q_\theta(y|x)$) such that $\lim_{t \to 0} \psi_Q(t) = \psi_Q(0) = 0$ and

$$\limsup_{n \to \infty} d(\theta_n, \mathcal{R}) \leq \psi_Q\left(\frac{1}{N}\right)$$

almost surely on Λ_Q.

(ii) If Assumptions 2.1 – 2.5 hold, then there exists a real number $L_{1,Q} \in [1, \infty)$ (independent of N and depending only

on $l(\theta)$, $p_\theta(x'|x)$, $q_\theta(y|x)$) such that

$$\limsup_{n \to \infty} \|\nabla l(\theta_n)\| \leq \frac{L_{1,Q}}{N^{q/2}},$$
$$\limsup_{n \to \infty} l(\theta_n) - \liminf_{n \to \infty} l(\theta_n) \leq \frac{L_{1,Q}}{N^q}$$

almost surely on Λ_Q, where $q = (p - d)/(p - 1)$.

(iii) If Assumptions 2.1 – 2.4 and 2.6 hold, then there exist real numbers $r_Q \in (0, 1)$, $L_{2,Q} \in [1, \infty)$ (independent of N and depending only on $l(\theta)$, $p_\theta(x'|x)$, $q_\theta(y|x)$) such that

$$\limsup_{n \to \infty} d(\theta_n, \mathcal{S}) \leq \frac{L_{2,Q}}{N^{r_Q}},$$
$$\limsup_{n \to \infty} \|\nabla l(\theta_n)\| \leq \frac{L_{2,Q}}{N^{1/2}},$$
$$\limsup_{n \to \infty} d(l(\theta_n), l(\mathcal{S})) \leq \frac{L_{2,Q}}{N}$$

almost surely on Λ_Q.

Theorem 2.1 is proved in Section VI.

Remark: The function $\psi_Q(t)$ and the real numbers $L_{1,Q}$, $L_{2,Q}$ depend on $p_\theta(x'|x)$, $q_\theta(y|x)$ through constants ε_Q, $K_{1,Q}$ (specified in Assumptions 2.3, 2.4). $\psi_Q(t)$ also depends on $l(\theta)$ through a Lipschitz constant of $\nabla l(\theta)$, an upper bound of $\|\nabla l(\theta)\|$ and the geometric properties of $\mathcal{R}$. $L_{1,Q}$, $L_{2,Q}$ depend on $l(\theta)$ through a Lipschitz constant of $\nabla l(\theta)$ and an upper bound of $\|\nabla l(\theta)\|$. Additionally, $L_{1,Q}$, $L_{2,Q}$ also depend on $l(\theta)$ through the Yomdin and Lojasiewicz constants for $l(\theta)$.[3] r_Q is the Lojasiewicz exponent for $l(\theta)$. For further details on how $\psi_Q(t)$, r_Q, $L_{1,Q}$, $L_{2,Q}$ depend on $l(\theta)$, $p_\theta(x'|x)$, $q_\theta(y|x)$, see [30].

As algorithm (1) – (4) is a stochastic gradient search maximizing the log-likelihood rate $l(\theta)$, the asymptotic properties of sequences $\{\theta_n\}_{n\geq 0}$, $\{l(\theta_n)\}_{n\geq 0}$ and $\{\nabla l(\theta_n)\}_{n\geq 0}$ provide a natural way to characterize the asymptotic behavior of this algorithm. If the estimation of $\nabla l(\theta)$ in algorithm (1) – (4) were based on the exact optimal filter instead of a particle approximation, the corresponding estimator would be asymptotically consistent. Then, according to the existing results on stochastic optimization, sequences $\{\theta_n\}_{n\geq 0}$, $\{l(\theta_n)\}_{n\geq 0}$ and $\{\nabla l(\theta_n)\}_{n\geq 0}$ would exhibit the following behavior. If $\nabla l(\theta)$ was estimated using the exact optimal filter and $l(\theta)$ was

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[3]If $l(\theta)$ is p-times differentiable and $p > d$ (which is true under Assumptions 2.2 – 2.5; see Part (ii) of Lemma 6.2), there exists a real number $M_{1,Q} \in [1, \infty)$ such that

$$m(\{l(\theta) : \theta \in Q, \|\nabla l(\theta)\| \leq \varepsilon\}) \leq M_{1,Q}\varepsilon^q$$

for all $\varepsilon \in [1, \infty)$, where $m(\cdot)$ is the Lebesgue measure on $\mathbb{R}^d$ (q is specified in the statement of Theorem 2.1). This result is known as the Yomdin theorem (see [35, Theorem 1.2]), while $M_{1,Q}$ is referred to as the Yomdin constant. If $l(\theta)$ is real-analytic on Θ (which holds under Assumptions 2.2 – 2.4, 2.6; see Part (iii) of Lemma 6.2), there exist real numbers $r_Q \in (0, 1)$, $M_{2,Q}, M_{3,Q} \in [1, \infty)$ such that

$$d(\theta, \mathcal{S}) \leq M_{2,Q}\|\nabla l(\theta)\|^{r_Q}, \quad d(l(\theta), l(\mathcal{S})) \leq M_{3,Q}\|\nabla l(\theta)\|$$

for all $\theta \in Q$. These inequalities are known as the Lojasiewicz inequalities (see [5, Theorem 6.4, Remark 6.5], [17, Theorem ŁI, Page 775]). $M_{2,Q}$, $M_{3,Q}$ are referred to as the Lojasiewicz constants, while r_Q is called the Lojasiewicz exponent.

Lipschitz continuously differentiable, then the limits

$$\lim_{n\to\infty} d(\theta_n, \mathcal{R}) = 0, \tag{7}$$

$$\lim_{n\to\infty} d(l(\theta_n), l(\mathcal{R})) = 0, \tag{8}$$

$$\liminf_{n\to\infty} d(\theta_n, \mathcal{S}) = 0 \tag{9}$$

would hold almost surely on the event $\{\sup_{n\geq 0} \|\theta_n\| < \infty,$ $\inf_{n\geq 0} d(\theta_n, \Theta^c) > 0\}$ (see, e.g., [1, Proposition 4.1, Theorem 5.7]). If $l(\theta)$ was additionally $(d+1)$-times differentiable, then the limits

$$\lim_{n\to\infty} d(\theta_n, \mathcal{S}) = 0, \quad \lim_{n\to\infty} \nabla l(\theta_n) = 0, \tag{10}$$

$$\lim_{n\to\infty} d(l(\theta_n), l(\mathcal{S})) = 0, \quad \limsup_{n\to\infty} l(\theta_n) = \liminf_{n\to\infty} l(\theta_n) \tag{11}$$

would hold almost surely on $\{\sup_{n\geq 0} \|\theta_n\| < \infty,$ $\inf_{n\geq 0} d(\theta_n, \Theta^c) > 0\}$ (see e.g., [1, Corollary 6.7]). Since algorithm (1) – (4) estimates $\nabla l(\theta)$ using a particle approximation, the corresponding estimator is biased. Consequently, the limits (7) – (11) do not hold for algorithm (1) – (4). Instead, the following limits

$$\limsup_{n\to\infty} d(\theta_n, \mathcal{R}), \quad \limsup_{n\to\infty} \|\nabla l(\theta_n)\|, \tag{12}$$

$$\limsup_{n\to\infty} d(l(\theta_n), l(\mathcal{R})), \quad \limsup_{n\to\infty} l(\theta_n) - \liminf_{n\to\infty} l(\theta_n) \tag{13}$$

take strictly positive values. These limits directly depend on the accuracy of the particle approximations to the optimal filter and its derivative.

Theorem 2.1 provides qualitative upper bounds on the limits (12), (13) in terms of the number of particles N and the analytical properties of the log-likelihood rate $l(\theta)$. These bounds are of the almost sure type and based on the strong mixing condition (Assumption 2.3). As such, they can be considered as of the worst-case type. Moreover, these bounds can be rather loose in scenarios for which the strong mixing condition is too conservative or even undesirable (e.g., when $q_\theta(\cdot|x)$ is concentrated, while $p_\theta(\cdot|x)$ is diffuse). This is partly due to the fact that the recursive particle maximum likelihood algorithm analyzed in the paper is based on the bootstrap particle filter (4), which is well-known to perform poorly in such scenarios. We believe that the bounds could be improved using more sophisticated schemes which sample particles relying on a distribution dependent on the observations [24]. These bounds could also be made tighter using the (non-mixing) assumptions on the optimal filter stability adopted in [13]. However, this would require substantial generalization of the existing results on the stability of the optimal filter derivatives and their particle approximations (i.e., of the results of [29], [33]). Since the analysis of the optimal filter derivatives and their particle approximations would be very difficult under non-mixing stability conditions and since the results presented here are already complex, this generalization is left for future research.

D. Outline of Proofs of Convergence Results

An outline/summary of the main steps and key ideas in the proof of Theorem 2.1 is provided here. These steps and ideas can be described as follows.

Step 1: Algorithm (1) – (4) is transformed to stochastic approximation with Markovian dynamics. More specifically, it is rewritten as

$$W_{n+1} = W_n A_{\theta_n}(V_n, V_{n+1}) + B_{\theta_n}(V_n, V_{n+1}), \tag{14}$$

$$\theta_{n+1} = \theta_n + \alpha_n (W_{n+1} C_{\theta_n}(V_{n+1}) + D_{\theta_n}(V_{n+1})). \tag{15}$$

The same algorithm is also rewritten as

$$\theta_{n+1} = \theta_n + \alpha_n H(\theta_n, Z_{n+1}). \tag{16}$$

Here, $\{V_n\}_{n\geq 0}$, $\{W_n\}_{n\geq 0}$ and $\{Z_n\}_{n\geq 0}$ are the stochastic processes defined by $W_n = (W_{n,1}, \ldots, W_{n,N})$ and

$$V_n = (Y_n, X_n, \hat{X}_n), \quad Z_n = (V_n, W_n), \tag{17}$$

while $\hat{X}_n$ is the vector of particles $\hat{X}_n = (\hat{X}_{n,1}, \ldots, \hat{X}_{n,N})$ (W_n is the $d \times N$ matrix whose j-th column is $W_{n,j}$). $A_\theta(v, v')$, $B_\theta(v, v')$, $C_\theta(v)$, $D_\theta(v)$ and $H(\theta, z)$ are suitably chosen functions which are defined precisely in (31) – (35). Equations (14) – (16) are a compact form of (1) – (3), while terms

$$W_{n+1} C_{\theta_n}(V_{n+1}) + D_{\theta_n}(V_{n+1}), \quad H(\theta_n, Z_{n+1}) \tag{18}$$

can be viewed as Monte Carlo estimators of $\nabla l(\theta_n)$. Aggregate process $\{(\theta_n, Z_n)\}_{n\geq 0}$ is a Markov chain, while stochastic processes $\{V_n\}_{n\geq 0}$ and $\{Z_n\}_{n\geq 0}$ can be interpreted as Markov chains controlled by estimates $\{\theta_n\}_{n\geq 0}$ (see (40), (41)).

Step 2: We analyze conditional probability measure of V_{n+1} given $V_n = v$, $\theta_n = \theta$, which is denoted by $T_\theta(v, dv')$ and precisely defined in (38). It is shown that $T_\theta(v, dv')$ is geometrically ergodic with a rate (locally) uniform in θ. It is also established that $T_\theta(v, dv')$ is (locally) Lipschitz continuous in θ. The details are included in Lemma 5.1.

Step 3: We consider the conditional expectations of the products

$$A_{\theta_0}(V_0, V_1) \cdots A_{\theta_{n-1}}(V_{n-1}, V_n) C_{\theta_n}(V_n), \tag{19}$$

$$B_{\theta_0}(V_0, V_1) A_{\theta_1}(V_1, V_2) \cdots A_{\theta_{n-1}}(V_{n-1}, V_n) C_{\theta_n}(V_n) \tag{20}$$

given $\theta_0 = \theta, \ldots, \theta_n = \theta$, $V_0 = v$. These conditional expectations are denoted by $\Phi_\theta^n(v)$, $\Psi_\theta^n(v)$ and defined precisely in (46) – (49). Using results on stochastic matrices (see Appendix A) and the results of Step 2, it is shown that functions $\Phi_\theta^n(v)$, $\Psi_\theta^n(v)$ converge exponentially to zero as $n \to \infty$ with rates (locally) uniform in θ. The same functions are also shown to be (locally) Lipschitz continuous with Lipschitz constants tending exponentially to zero as $n \to \infty$. The details are provided in Lemma 5.3.

Step 4: Function $(\Pi^n H)(\theta, z)$ and its properties are analyzed, where $(\Pi^n H)(\theta, z)$ is the conditional expectation of $H(\theta_n, Z_{n+1})$ given $\theta_0 = \theta, \ldots, \theta_n = \theta$, $Z_0 = z$ (see (87)). $\Pi_\theta(z, dz')$ is the conditional probability measure of Z_{n+1} given $Z_n = z$, $\theta_n = \theta$, which is defined precisely in (39). Relying on the results of Step 3, it is shown that there exists a function $h(\theta)$ such that $(\Pi^n H)(\theta, z)$ converges exponentially to $h(\theta)$ as $n \to \infty$ at a rate (locally) uniform in θ. It is also shown that $(\Pi^n H)(\theta, z) - h(\theta)$ is (locally) Lipschitz continuous in θ with a Lipschitz constant tending

exponentially to zero as $n \to \infty$. The details are included in Lemma 5.4 (see (51), (92), (97)).

Step 5: The Poisson equation associated with algorithm (1) – (4) (i.e., with functions $H(\theta, z)$, $h(\theta)$ and the transition kernel $\Pi_\theta(z, dz')$) and its properties are considered. Relying on the results of Step 4, it is shown that the Poisson equation has a solution and that the solution is (locally) Lipschitz continuous in θ. The details are provided in Lemma 5.4.

Step 6: The weight sequence $\{W_n\}_{n \geq 0}$ and its stability are studied. Using results on stochastic matrices (see Appendix A), it is shown that W_n is (deterministically) bounded in n. The details are contained in Lemma 5.5.

Step 7: The Monte Carlo estimators (18) and their statistical properties are analyzed. By exploiting the results of [33], it is shown that the asymptotic bias of these estimators are inversely proportional to N with a multiplicative constant uniform in θ. The details are included in Lemma 5.6.

Step 8: Algorithm (1) – (4) is transformed to a stochastic gradient search with additive noise. More specifically, it is rewritten as

$$\theta_{n+1} = \theta_n + \alpha_n(\nabla l(\theta_n) + \xi_n). \qquad (21)$$

Moreover, the additive noise ξ_n is decomposed as $\xi_n = \zeta_n + \eta_n$, where

$$\xi_n = H(\theta_n, Z_{n+1}) - h(\theta_n), \qquad \zeta_n = h(\theta_n) - \nabla l(\theta_n).$$

$\nabla l(\theta_n) + \xi_n$ can be interpreted as an estimator of $\nabla l(\theta_n)$, while ζ_n and η_n can be considered as the variance and bias of this estimator. Using results of martingale limit theory and the results of Steps 5 and 6, it is shown that $\{\zeta_n\}_{n \geq 0}$ satisfy the Kushner-Clark noise condition. Relying on the results of Step 7, it is also shown that the asymptotic magnitude of $\{\eta_n\}_{n \geq 0}$ is inversely proportional to N with a deterministic multiplicative constant. The details are provided in Lemma 6.1.

Step 9: Using the results presented in Section IV and the results of [31], [32], Lemma 6.2 is proved. Then, relying on the results obtained at Step 8 and the results of [30], Theorem 2.1 is established.

III. EXAMPLE

To illustrate the main results and their applicability, we use them to study recursive maximum likelihood estimation for the following non-linear state-space model:

$$X_{n+1}^{\theta,\lambda} = A_\theta(X_n^{\theta,\lambda}) + B_\theta(X_n^{\theta,\lambda})V_n, \qquad (22)$$
$$Y_n^{\theta,\lambda} = C_\theta(X_n^{\theta,\lambda}) + D_\theta(X_n^{\theta,\lambda})W_n, \quad n \geq 0. \qquad (23)$$

Here, $\theta \in \Theta$ and $\lambda \in \mathcal{P}(\mathcal{X})$ are the parameters indexing the state-space model (22), (23). $A_\theta(x)$ and $B_\theta(x)$ are functions which map $\theta \in \Theta$, $x \in \mathbb{R}^{d_x}$ (respectively) to $\mathbb{R}^{d_x}$ and $\mathbb{R}^{d_x \times d_x}$, while $C_\theta(x)$ and $D_\theta(x)$ map $\theta \in \Theta$, $x \in \mathbb{R}^{d_x}$ (respectively) to $\mathbb{R}^{d_y}$ and $\mathbb{R}^{d_y \times d_y}$. $X_0^{\theta,\lambda}$ is an $\mathbb{R}^{d_x}$-valued random variable defined on a probability space $(\Omega, \mathcal{F}, P)$ and distributed according to λ. $\{V_n\}_{n \geq 0}$ are $\mathbb{R}^{d_x}$-valued i.i.d. random variables distributed according the probability density $v(x)$ with respect to the Lebesgue measure while $\{W_n\}_{n \geq 0}$ are $\mathbb{R}^{d_y}$-valued i.i.d. random variables distributed according the probability density $w(y)$ with respect to the Lebesgue

measure. We also assume that $X_0^{\theta,\lambda}$, $\{V_n\}_{n \geq 0}$ and $\{W_n\}_{n \geq 0}$ are (jointly) independent.

Let $p_\theta(x'|x)$ and $q_\theta(y|x)$ be the functions defined by

$$p_\theta(x'|x) = \frac{v\left(B_\theta^{-1}(x)(x' - A_\theta(x))\right) 1_\mathcal{X}(x')}{\int_\mathcal{X} v\left(B_\theta^{-1}(x)(x'' - A_\theta(x))\right) dx''}, \qquad (24)$$

$$q_\theta(y|x) = \frac{w\left(D_\theta^{-1}(x)(y - C_\theta(x))\right) 1_\mathcal{Y}(y)}{\int_\mathcal{Y} w\left(D_\theta^{-1}(x)(y' - C_\theta(x))\right) dy'} \qquad (25)$$

for $\theta \in \Theta$, $x, x' \in \mathbb{R}^{d_x}$, $y \in \mathbb{R}^{d_y}$, where $\mathcal{X} \in \mathcal{B}(\mathbb{R}^{d_x})$, $\mathcal{Y} \in \mathcal{B}(\mathbb{R}^{d_y})$. If $\mathcal{X} = \mathbb{R}^{d_x}$, $\mathcal{Y} = \mathbb{R}^{d_y}$, then $p_\theta(x'|x)$ and $q_\theta(y|x)$ reduce to the conditional densities of $X_{n+1}^{\theta,\lambda}$ and $Y_n^{\theta,\lambda}$ (respectively) given $X_n^{\theta,\lambda} = x$. When $\mathcal{X} \neq \mathbb{R}^{d_x}$, $\mathcal{Y} \neq \mathbb{R}^{d_y}$, $p_\theta(x'|x)$ and $q_\theta(y|x)$ can be viewed as a truncation of state-space model (22), (23) to domains $\mathcal{X}$ and $\mathcal{Y}$. Due to the finite precision of digital computers, this kind of truncation is involved (explicitly or implicitly) in the implementation of any numerical approximation to the optimal filter for state-space model (22), (23). In [15], a truncation scheme similar to (24), (25) has been theoretically analyzed and the choice of the corresponding truncation domain has been addressed. In the context of algorithm (1) – (4), the choice of domains $\mathcal{X}$ and $\mathcal{Y}$ is much more complex as it involves many factors such as the stability, accuracy, convergence and convergence rate of algorithm (1) – (4), as well as the stability and accuracy of the optimal filter for model (24), (25). As such, the choice of $\mathcal{X}$ and $\mathcal{Y}$ is beyond the scope of this paper.

In this section, we rely on the following assumptions.

Assumption 3.1: $\mathcal{X}$ and $\mathcal{Y}$ are compact sets with non-empty interiors.

Assumption 3.2: $v(x) > 0$ and $w(y) > 0$ for each $x \in \mathbb{R}^{d_x}$, $y \in \mathbb{R}^{d_y}$. $B_\theta(x)$ and $D_\theta(x)$ are invertible for each $\theta \in \Theta$, $x \in \mathbb{R}^{d_x}$.

Assumption 3.3: $v(x)$ and $w(y)$ are differentiable for each $x \in \mathbb{R}^{d_x}$, $y \in \mathbb{R}^{d_y}$. The first order derivatives of $v(x)$ and $w(y)$ are locally Lipschitz continuous on $\mathbb{R}^{d_x}$, $\mathbb{R}^{d_y}$. $A_\theta(x)$, $B_\theta(x)$, $C_\theta(x)$ and $D_\theta(x)$ are differentiable in θ for each $\theta \in \Theta$, $x \in \mathbb{R}^{d_x}$. The first order derivatives in θ of $A_\theta(x)$, $B_\theta(x)$, $C_\theta(x)$ and $D_\theta(x)$ are locally Lipschitz continuous in (θ, x) on $\Theta \times \mathbb{R}^{d_x}$.

Assumption 3.4: $v(x)$ and $w(y)$ are p-times differentiable for each $x \in \mathbb{R}^{d_x}$, $y \in \mathbb{R}^{d_y}$, where $p > d$. The p-th order derivatives of $v(x)$ and $w(y)$ are locally bounded on $\mathbb{R}^{d_x}$, $\mathbb{R}^{d_y}$. $A_\theta(x)$, $B_\theta(x)$, $C_\theta(x)$ and $D_\theta(x)$ are p-times differentiable in θ for each $\theta \in \Theta$, $x \in \mathbb{R}^{d_x}$. The p-th order derivatives in θ of $A_\theta(x)$, $B_\theta(x)$, $C_\theta(x)$ and $D_\theta(x)$ are locally bounded in (θ, x) on $\Theta \times \mathbb{R}^{d_x}$.

Assumption 3.5: $v(x)$ and $w(y)$ are real-analytic for each $x \in \mathbb{R}^{d_x}$, $y \in \mathbb{R}^{d_y}$. $A_\theta(x)$, $B_\theta(x)$, $C_\theta(x)$ and $D_\theta(x)$ are real-analytic in (θ, x) for each $\theta \in \Theta$, $x \in \mathbb{R}^{d_x}$.

Regarding Assumptions 2.3–2.6 and 3.1–3.5, the following relationships can be established. Assumptions 3.1–3.3 imply Assumptions 2.3 and 2.4, while Assumptions 3.4 and 3.5 are particular cases of Assumptions 2.5 and 2.6 (respectively). For the proof of these relationships, see [31, Corollary 4.1], [32, Corollary 4.1] (and the arguments used therein). Assumptions 3.1 – 3.5 are relevant for several practically important classes of state-space models and cover, for example, stochastic

volatility models, dynamic probit models and their truncated versions. For other models satisfying (22), (23) and Assumptions 3.1 – 3.5, see [8], [9], [14] and references cited therein.

As a direct consequence of the relationships between Assumptions 2.3 – 2.6 and 3.1 – 3.5, we get the following corollary to Theorem 2.1.

Corollary 3.1: (i) If Assumptions 2.1, 2.2 and 3.1 – 3.3 are satisfied, then the conclusions of Part (i) of Theorem 2.1 hold.

(ii) If Assumptions 2.1, 2.2 and 3.1 – 3.4 are fulfilled, then the conclusions of Part (ii) of Theorem 2.1 hold.

(iii) If Assumptions 2.1, 2.2, 3.1 – 3.3 and 3.5 are satisfied, then the conclusions of Part (iii) of Theorem 2.1 hold.

IV. RESULTS RELATED TO OPTIMAL FILTER AND LOG-LIKELIHOOD RATE

In this section, we study the stability and analytical properties of the optimal filter and its derivative as well as some regularity properties of the log-likelihood rate. The results presented here are a prerequisite for Lemmas 5.6 and 6.2. Note that we only consider here the results which are essential for the proof of Theorem 2.1 and not well-covered in the existing literature on optimal filtering.

Throughout this section and the whole paper, we use the following notation. Q stands for any compact set satisfying $Q \subset \Theta$. $\mathcal{M}_s(\mathcal{X})$ is the collection of signed measures on $\mathcal{X}$, while $\mathcal{M}_s^d(\mathcal{X})$ is the set of d-dimensional vector measures on $\mathcal{X}$. For $\xi \in \mathcal{M}_s(\mathcal{X})$, $|\xi|(dx)$ and $\|\xi\|$ denote (respectively) the total variation and the total variation norm of ξ. For $\zeta \in \mathcal{M}_s^d(\mathcal{X})$, $|\zeta|(dx)$ and $\|\zeta\|$ denote (respectively) the total variation and the total variation norm of ζ induced by l_1 vector norm.[4] $r_\theta(x'|y, x)$ is the function defined by

$$r_\theta(x'|y, x) = p_\theta(x'|x) q_\theta(y|x)$$

for $\theta \in \Theta$, $x, x' \in \mathcal{X}$, $y \in \mathcal{Y}$, while $h_{\theta,y}(x|\xi, \zeta)$ and $H_{\theta,y}(\xi, \zeta)$ are defined for $\xi \in \mathcal{P}(\mathcal{X})$, $\zeta \in \mathcal{M}_s^d(\mathcal{X})$ as

$$h_{\theta,y}(x|\xi, \zeta) = \frac{\int r_\theta(x|y, x') \zeta(dx') + \int \nabla_\theta r_\theta(x|y, x') \xi(dx')}{\int q_\theta(y|x') \xi(dx')},$$

$$H_{\theta,y}(\xi, \zeta) = \int h_{\theta,y}(x|\xi, \zeta) \mu(dx). \quad (26)$$

$r_{\theta,\boldsymbol{y}}^{m:n}(x'|x)$ is the function recursively defined by

$$r_{\theta,\boldsymbol{y}}^{m:m+1}(x'|x) = r_\theta(x'|y_m, x),$$

$$r_{\theta,\boldsymbol{y}}^{m:n+1}(x'|x) = \int r_{\theta,\boldsymbol{y}}^{n:n+1}(x'|x'') r_{\theta,\boldsymbol{y}}^{m:n}(x''|x) \mu(dx'')$$

for $n > m \geq 0$ and a sequence $\boldsymbol{y} = \{y_n\}_{n \geq 0}$ in $\mathcal{Y}$. $h_{\theta,\boldsymbol{y}}^{m:n}(x|\xi, \zeta)$ and $H_{\theta,\boldsymbol{y}}^{m:n}(\xi, \zeta)$ are the functions defined by

$$h_{\theta,\boldsymbol{y}}^{m:n}(x|\xi, \zeta) = \frac{\int r_{\theta,\boldsymbol{y}}^{m:n}(x|x') \zeta(dx') + \int \nabla_\theta r_{\theta,\boldsymbol{y}}^{m:n}(x|x') \xi(dx')}{\iint r_{\theta,\boldsymbol{y}}^{m:n}(x''|x') \xi(dx') \mu(dx'')},$$

$$H_{\theta,\boldsymbol{y}}^{m:n}(\xi, \zeta) = \int h_{\theta,\boldsymbol{y}}^{m:n}(x|\xi, \zeta) \mu(dx),$$

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[4]If $\zeta \in \mathcal{M}_s^d(\mathcal{X})$, then $|\zeta|(dx) = \sum_{i=1}^d |e_i^T \zeta|(dx)$ and $\|\zeta\| = \sum_{i=1}^d \|e_i^T \zeta\|$, where e_i is the i-th standard unit vector in $\mathbb{R}^d$.

while $f_{\theta,\boldsymbol{y}}^{m:n}(x|\xi)$ and $g_{\theta,\boldsymbol{y}}^{m:n}(x|\xi, \zeta)$ are defined as

$$f_{\theta,\boldsymbol{y}}^{m:n}(x|\xi) = \frac{\int r_{\theta,\boldsymbol{y}}^{m:n}(x|x') \xi(dx')}{\iint r_{\theta,\boldsymbol{y}}^{m:n}(x''|x') \xi(dx') \mu(dx'')},$$

$$g_{\theta,\boldsymbol{y}}^{m:n}(x|\xi, \zeta) = h_{\theta,\boldsymbol{y}}^{m:n}(x|\xi, \zeta) - f_{\theta,\boldsymbol{y}}^{m:n}(x|\xi) H_{\theta,\boldsymbol{y}}^{m:n}(\xi, \zeta).$$

$F_{\theta,\boldsymbol{y}}^{m:m}(dx|\xi)$, $F_{\theta,\boldsymbol{y}}^{m:n}(dx|\xi)$ and $G_{\theta,\boldsymbol{y}}^{m:m}(dx|\xi, \zeta)$, $G_{\theta,\boldsymbol{y}}^{m:n}(dx|\xi, \zeta)$ are the measures defined for $B \in \mathcal{B}(\mathcal{X})$ by $F_{\theta,\boldsymbol{y}}^{m:m}(B|\xi) = \xi(B)$, $G_{\theta,\boldsymbol{y}}^{m:m}(B|\xi, \zeta) = \zeta(B)$ and

$$F_{\theta,\boldsymbol{y}}^{m:n}(B|\xi) = \int_B f_{\theta,\boldsymbol{y}}^{m:n}(x|\xi) \mu(dx), \quad (27)$$

$$G_{\theta,\boldsymbol{y}}^{m:n}(B|\xi, \zeta) = \int_B g_{\theta,\boldsymbol{y}}^{m:n}(x|\xi, \zeta) \mu(dx). \quad (28)$$

Throughout this paper, the measures $F_{\theta,\boldsymbol{y}}^{m:n}(dx|\xi)$ and $G_{\theta,\boldsymbol{y}}^{m:n}(dx|\xi, \zeta)$ are also denoted by $F_{\theta,\boldsymbol{y}}^{m:n}(\xi)$ and $G_{\theta,\boldsymbol{y}}^{m:n}(\xi, \zeta)$ (short-hand notation). Then, it is easy to show that $F_{\theta,\boldsymbol{y}}^{m:n}(\xi)$ and $G_{\theta,\boldsymbol{y}}^{m:n}(\xi, \zeta)$ are the optimal (one-step) predictor and its gradient, i.e.,

$$F_{\theta,\boldsymbol{y}}^{0:n}(B|\lambda) = P\left(X_n^{\theta,\lambda} \in B \,\middle|\, Y_{0:n-1}^{\theta,\lambda} = y_{0:n-1}\right),$$

$$G_{\theta,\boldsymbol{y}}^{0:n}(B|\lambda, \mathbf{0}) = \nabla_\theta F_{\theta,\boldsymbol{y}}^{0:n}(B|\lambda)$$

for each $\lambda \in \mathcal{P}(\mathcal{X})$, $n \geq 1$. Here, $\mathbf{0}$ is the d-dimensional zero-measure (i.e., $\mathbf{0} \in \mathcal{M}_s^d(\mathcal{X})$, $\|\mathbf{0}\| = 0$).

Lemma 4.1: Let Assumptions 2.2 – 2.4 hold. Then, the following is true:

(i) $l(\theta)$ is well-defined and differentiable on Θ.

(ii) $\nabla l(\theta)$ is locally Lipschitz continuous on Θ and satisfies

$$\nabla l(\theta) = \lim_{n \to \infty} E\left(H_{\theta,Y_n}\left(F_{\theta,\boldsymbol{Y}}^{0:n}(\xi), G_{\theta,\boldsymbol{Y}}^{0:n}(\xi, \zeta)\right)\right) \quad (29)$$

for all $\theta \in \Theta$, $\xi \in \mathcal{P}(\mathcal{X})$, $\zeta \in \mathcal{M}_s^d(\mathcal{X})$, where $\boldsymbol{Y} = \{Y_n\}_{n \geq 0}$.

(iii) There exists a real number $C_{1,Q} \in [1, \infty)$ (depending only on $p_\theta(x'|x)$, $q_\theta(y|x)$) such that

$$\left\|G_{\theta,\boldsymbol{y}}^{0:n}(\xi, \zeta)\right\| \leq C_{1,Q}(1 + \|\zeta\|)$$

for all $\theta \in Q$, $\xi \in \mathcal{P}(\mathcal{X})$, $\zeta \in \mathcal{M}_s^d(\mathcal{X})$, $n \geq 0$ and any sequence $\boldsymbol{y} = \{y_n\}_{n \geq 0}$.

Lemma 4.1 is proved in Appendix B.

V. RESULTS RELATED TO SEQUENTIAL MONTE CARLO APPROXIMATIONS

In this section, we study the asymptotic properties of the particles $\{\hat{X}_{n,i} : n \geq 0, 1 \leq i \leq N\}$ and their weights $\{W_{n,i} : n \geq 0, 1 \leq i \leq N\}$. Using these properties, we show that the Poisson equation associated with algorithm (1) – (4) has a Lipschitz continuous solution (see Lemma 5.4). The results presented here are needed to analyze the error in the Monte Carlo estimation of $\nabla l(\theta)$ (see Lemma 6.1 and (21)).

Throughout this section, we use the following notation. $\mathcal{V}$ and $\mathcal{Z}$ are the sets defined by $\mathcal{V} = \mathcal{Y} \times \mathcal{X} \times \mathcal{X}^N$ and $\mathcal{Z} = \mathcal{V} \times \mathbb{R}^{d \times N}$. e is the N-dimensional vector whose all elements are one (i.e., $e = (1, \ldots, 1)^T \in \mathbb{R}^N$). I is the $N \times N$ unit matrix, while Λ is the $N \times N$ matrix defined as

$$\Lambda = I - \frac{ee^T}{N}. \quad (30)$$

$A_\theta(v, v')$ and $B_\theta(v, v')$ are respectively $\mathbb{R}^{N \times N}$ and $\mathbb{R}^{d \times N}$-valued functions defined by

$$A_\theta^{i,j}(v, v') = \frac{r_\theta(x_j'|y, x_i)}{\sum_{k=1}^N r_\theta(x_j'|y, x_k)}, \tag{31}$$

$$B_\theta^j(v, v') = \frac{\sum_{k=1}^N \nabla_\theta r_\theta(x_j'|y, x_k)}{\sum_{k=1}^N r_\theta(x_j'|y, x_k)} \tag{32}$$

for $\theta \in \Theta$, $x, x' \in \mathcal{X}$, $y, y' \in \mathcal{Y}$, $\hat{x} = (x_1, \ldots, x_N) \in \mathcal{X}^N$, $\hat{x}' = (x_1', \ldots, x_N') \in \mathcal{X}^N$, $1 \le i, j \le N$ and $v = (y, x, \hat{x})$, $v' = (y', x', \hat{x}')$, where $A_\theta^{i,j}(v, v')$ and $B_\theta^j(v, v')$ are the (i, j)-entry of $A_\theta(v, v')$ and the j-th column of $B_\theta(v, v')$ (respectively). $C_\theta(v)$ and $D_\theta(v)$ are respectively $\mathbb{R}^N$ and $\mathbb{R}^d$-valued functions defined by

$$C_\theta^i(v) = \frac{q_\theta(y|x_i)}{\sum_{k=1}^N q_\theta(y|x_k)} - \frac{1}{N}, \tag{33}$$

$$D_\theta(v) = \frac{\sum_{k=1}^N \nabla_\theta q_\theta(y|x_k)}{\sum_{k=1}^N q_\theta(y|x_k)}, \tag{34}$$

where $C_\theta^i(v)$ is the i-th element of $C_\theta(v)$. $H(\theta, z)$ is the function defined by

$$H(\theta, z) = W C_\theta(v) + D_\theta(v) \tag{35}$$

for $v \in \mathcal{V}$, $W \in \mathbb{R}^{d \times N}$ and $z = (v, W)$. Then, it is straightforward to verify

$$e^T A_\theta(v, v') = e^T, \qquad e^T C_\theta(v) = 0 \tag{36}$$

for all $\theta \in \Theta$, $v, v' \in \mathcal{V}$, where $A_\theta(v, v')$, $C_\theta(v)$ are defined in (31), (33).

We rely here on the following notation, too. $s_\theta(x|y, \hat{x})$ is the function defined by

$$s_\theta(x|y, \hat{x}) = \frac{\sum_{k=1}^N p_\theta(x|x_k) q_\theta(y|x_k)}{\sum_{k=1}^N q_\theta(y|x_k)}. \tag{37}$$

For $\hat{x} = (x_1, \ldots, x_N) \in \mathcal{X}^N$, $S_\theta(d\hat{x}'|y, \hat{x})$ is the conditional probability measure on $\mathcal{X}^N$ defined for $B \in \mathcal{B}(\mathcal{X}^N)$ as

$$S_\theta(B|y, \hat{x}) = \int \cdots \int I_B(x_1', \ldots, x_N') \left(\prod_{k=1}^N s_\theta(x_k'|y, \hat{x}) \right) \cdot \mu(dx_1') \cdots \mu(dx_N'),$$

where I_B denotes the indicator of B. $T_\theta(v, dv')$ is the kernel on $\mathcal{V}$ defined for $B \in \mathcal{B}(\mathcal{V})$ and $v = (y, x, \hat{x})$ by

$$T_\theta(v, B) = \iiint I_B(y', x', \hat{x}') Q(x', dy') \cdot P(x, dx') S_\theta(d\hat{x}'|y, \hat{x}). \tag{38}$$

$\Pi_\theta(z, dz')$ is the kernel on $\mathcal{Z}$ defined for $B \in \mathcal{B}(\mathcal{Z})$, $W \in \mathbb{R}^{d \times N}$ and $z = (v, W)$ as

$$\Pi_\theta(z, B) = \int I_B \left(v', W A_\theta(v, v') + B_\theta(v, v') \right) T_\theta(v, dv'). \tag{39}$$

Then, it is straightforward to verify that $\{V_n\}_{n \ge 0}$ and $\{Z_n\}_{n \ge 0}$ defined in (17) satisfy

$$P(V_{n+1} \in A|\theta_0, V_0, \ldots, \theta_n, V_n) = T_{\theta_n}(V_n, A), \tag{40}$$

$$P(Z_{n+1} \in B|\theta_0, Z_0, \ldots, \theta_n, Z_n) = \Pi_{\theta_n}(Z_n, B) \tag{41}$$

almost surely for each $A \in \mathcal{B}(\mathcal{V})$, $B \in \mathcal{B}(\mathcal{Z})$, $n \ge 0$.

Using functions $A_\theta(v, v')$, $B_\theta(v, v')$ and $S_\theta(d\hat{x}'|y, \hat{x})$, we introduce the following notation. $\{\hat{X}_{n,i}^\theta : n \ge 0, 1 \le i \le N\}$ are $\mathcal{X}$-valued random variables generated through the sequential Monte Carlo scheme

$$\hat{X}_{n+1,i}^\theta \sim s_\theta \left(x|Y_n, (\hat{X}_{n,1}^\theta, \ldots, \hat{X}_{n,N}^\theta) \right) \mu(dx), \tag{42}$$

while $\hat{X}_n^\theta$, V_n^θ are the random variables defined by

$$\hat{X}_n^\theta = (\hat{X}_{n,1}^\theta, \ldots, \hat{X}_{n,N}^\theta), \qquad V_n^\theta = (Y_n, X_n, \hat{X}_n^\theta).$$

In (42), $\{\hat{X}_{n+1,i}^\theta : 1 \le i \le N\}$ are sampled independently from one another and independently of $\{X_k : 0 \le k \le n\}$, $\{Y_k, \hat{X}_{k,i}^\theta : 0 \le k < n, 1 \le i \le N\}$, while $\{\hat{X}_{0,i}^\theta : 1 \le i \le N\}$ are selected independently of (X_0, Y_0). $\{W_n^\theta\}_{n \ge 0}$ are $d \times N$ random matrices generated by the recursion

$$W_{n+1}^\theta = W_n^\theta A_\theta(V_n^\theta, V_{n+1}^\theta) + B_\theta(V_n^\theta, V_{n+1}^\theta), \tag{43}$$

while Z_n^θ is the random variable defined by $Z_n^\theta = (V_n^\theta, W_n^\theta)$. In (43), W_0^θ is selected independently of (X_0, Y_0). Then, it can easily be shown that $\{V_n^\theta\}_{n \ge 0}$ and $\{Z_n^\theta\}_{n \ge 0}$ are Markov chains whose transition kernels are $T_\theta(v, dv')$ and $\Pi_\theta(z, dz')$ (respectively).

Using functions $A_\theta(v, v')$, $B_\theta(v, v')$, $C_\theta(v)$, $D_\theta(v)$ (defined in (31) – (34)) and stochastic process $\{V_n^\theta\}_{n \ge 0}$, we introduce the following notation. $T_\theta^n(v, dv')$ and $\tau_\theta(dv)$ are (respectively) the n-th step transition kernel and the invariant probability measure of $\{V_n^\theta\}_{n \ge 0}$ (the existence and uniqueness of $\tau_\theta(dv)$ are guaranteed by Lemma 5.1). $\tilde{T}_\theta^n(v, dv')$ is the kernel on $\mathcal{V}$ defined for $B \in \mathcal{B}(\mathcal{V})$ by

$$\tilde{T}_\theta^n(v, B) = T_\theta^n(v, B) - \tau_\theta(B). \tag{44}$$

$\tilde{A}_\theta^0(v)$ and $\Phi_\theta^0(v)$ are the functions defined by

$$\tilde{A}_\theta^0(v) = I, \qquad \Phi_\theta^0(v) = C_\theta(v). \tag{45}$$

$\tilde{A}_\theta^n(v_0, \ldots, v_n)$ and $\Phi_\theta^n(v)$ are the functions defined for $v, v_0, \ldots, v_n \in \mathcal{V}$, $n \ge 1$ by

$$\tilde{A}_\theta^n(v_0, \ldots, v_n) = A_\theta(v_0, v_1) \cdots A_\theta(v_{n-1}, v_n), \tag{46}$$

$$\Phi_\theta^n(v) = E \left(\tilde{A}_\theta^n(V_0^\theta, \ldots, V_n^\theta) C_\theta(V_n^\theta) \Big| V_0^\theta = v \right). \tag{47}$$

$\tilde{B}_\theta^n(v_0, \ldots, v_n)$ and $\Psi_\theta^n(v)$ are the functions defined by

$$\tilde{B}_\theta^n(v_0, \ldots, v_n) = B_\theta(v_0, v_1) \tilde{A}_\theta^{n-1}(v_1, \ldots, v_n), \tag{48}$$

$$\Psi_\theta^n(v) = E \left(\tilde{B}_\theta^n(V_0^\theta, \ldots, V_n^\theta) C_\theta(V_n^\theta) \Big| V_0^\theta = v \right). \tag{49}$$

$h(\theta)$ is the function defined by

$$h(\theta) = \int D_\theta(v) \tau_\theta(dv) + \sum_{n=1}^\infty \int \Psi_\theta^n(v) \tau_\theta(dv). \tag{50}$$

Then, for each $\theta \in \Theta$, $v \in \mathcal{V}$, $n \ge 1$, it is straightforward to verify

$$\Psi_\theta^n(v) = E \left(B_\theta(V_0^\theta, V_1^\theta) \Phi_\theta^n(V_1^\theta) \Big| V_0^\theta = v \right). \tag{51}$$

Remark: Throughout this and subsequent sections, the following convention is applied. Diacritic $\tilde{}$ is used to denote a locally defined quantity, i.e., a quantity whose definition holds

only within the proof where the quantity appears. We also recall here that Q stands for any compact set satisfying $Q \subset \Theta$.

Lemma 5.1: Let Assumptions 2.2 – 2.4 hold. Then, the following is true:

(i) $\{V_n^\theta\}_{n \geq 0}$ is geometrically ergodic for each $\theta \in \Theta$.

(ii) There exist real numbers $\rho_{1,Q} \in (0,1)$, $C_{2,Q} \in [1, \infty)$ (possibly depending on N) such that

$$|\tilde{T}_\theta^n(v, B)| \leq C_{2,Q}\rho_{1,Q}^n, \tag{52}$$

$$|\tilde{T}_\theta^n(v, B) - \tilde{T}_{\theta'}^n(v, B)| \leq C_{2,Q}\rho_{1,Q}^n\|\theta - \theta'\|, \tag{53}$$

$$\max\{|\tau_\theta(B) - \tau_{\theta'}(B)|, |T_\theta(v, B) - T_{\theta'}(v, B)|\}$$
$$\leq C_{2,Q}\|\theta - \theta'\| \tag{54}$$

for all $\theta, \theta' \in Q$, $v \in \mathcal{V}$, $B \in \mathcal{B}(\mathcal{V})$, $n \geq 0$.

Proof: Using Assumption 2.2 and [22, Theorem 16.0.2], we conclude that there exist an integer $n_0 \geq 1$, a real number $\gamma \in (0,1)$ and a probability measure $\xi(dx)$ on $\mathcal{X}$ such that $P^{n_0}(x, A) \geq \gamma\xi(A)$ for all $x \in \mathcal{X}$, $A \subseteq \mathcal{B}(\mathcal{X})$.

Throughout the proof, the following notation is used. θ, θ' are any elements of Q. $x, x_1, \ldots, x_N$ are any elements of $\mathcal{X}$, while $\hat{x} = (x_1, \ldots, x_N)$. y is any element of $\mathcal{Y}$, while $v = (y, x, \hat{x})$. B is any element of $\mathcal{B}(\mathcal{V})$, while n is any non-negative integer. $\zeta(d\hat{x})$ is the probability measure on $\mathcal{X}^N$ defined for $A \in \mathcal{B}(\mathcal{X}^N)$ by

$$\zeta(A) = \left(\frac{1}{\mu(\mathcal{X})}\right)^N \int \cdots \int I_A(x_1, \ldots, x_N)\mu(dx_1)\cdots\mu(dx_N).$$

Let $\beta_Q = (\varepsilon_Q\mu(\mathcal{X}))^N$ (ε_Q is specified in Assumption 2.3, while $\mu(dx)$ is defined in Subsection II-A). Relying on Assumption 2.3, we deduce

$$\varepsilon_Q \leq s_\theta(x|y, \hat{x}) \leq \frac{1}{\varepsilon_Q}. \tag{55}$$

Consequently, for $A \in \mathcal{B}(\mathcal{X}^N)$, we get

$$S_\theta(A|y, \hat{x}) \geq \varepsilon_Q^N \int \cdots \int I_A(x_1', \ldots, x_N')\mu(dx_1')\cdots\mu(dx_N')$$
$$= \beta_Q\zeta(A).$$

Hence, we have

$$T_\theta(v, B) = \iiint I_B(y', x', \hat{x}')Q(x', dy')$$
$$\cdot P(x, dx')S_\theta(d\hat{x}'|y, \hat{x})$$
$$\geq \beta_Q \iiint I_B(y', x', \hat{x}')Q(x, dy')P(x, dx')\zeta(d\hat{x}').$$

Therefore, we get

$$T_\theta^{n+1}(v, B) = E\left(T_\theta(V_n^\theta, B)|V_0^\theta = v\right)$$
$$\geq \beta_Q E\left(\iiint I_B(y', x', \hat{x}')Q(x', dy')\right.$$
$$\left.\cdot P(X_n, dx')\zeta(d\hat{x}')\middle| Y_0 = y, X_0 = x, \hat{X}_0^\theta = \hat{x}\right)$$
$$= \beta_Q \iiint I_B(y', x', \hat{x}')Q(x', dy')P^{n+1}(x, dx')\zeta(d\hat{x}').$$

Since $P^{n_0}(x, A) \geq \gamma\xi(A)$ for any $A \subseteq \mathcal{B}(\mathcal{X})$, we get

$$T_\theta^{n_0}(v, B) \geq \beta_Q\gamma \iiint I_B(y', x', \hat{x}')Q(x', dy')\xi(dx')\zeta(d\hat{x}'). \tag{56}$$

Let $\rho_{1,Q} = (1 - \beta_Q\gamma)^{1/(2n_0)}$. As v is any element in $\mathcal{V}$, [22, Theorem 16.0.2]) and (56) imply that $\{V_n^\theta\}_{n \geq 0}$ is geometrically ergodic. The same arguments also imply

$$|\tilde{T}_\theta^n(v, B)| = |T_\theta^n(v, B) - \tau_\theta(B)| \leq \rho_{1,Q}^{2n}. \tag{57}$$

Since Q is any compact set in Θ, we conclude that (i) is true.

Let $\tilde{C}_{1,Q} = 3\varepsilon_Q^{-2}K_{1,Q}$, $\tilde{C}_{2,Q} = \varepsilon_Q^{-N}\tilde{C}_{1,Q}N$, $\tilde{C}_{3,Q} = (\mu(\mathcal{X}))^N\tilde{C}_{2,Q}$ ($K_{1,Q}$ is specified in Assumption 2.4). Owing to Assumptions 2.3, 2.4, we have

$$|s_\theta(x|y, \hat{x}) - s_{\theta'}(x|y, \hat{x})|$$
$$\leq \frac{\sum_{i=1}^N |p_\theta(x|x_i) - p_{\theta'}(x|x_i)|q_\theta(y|x_i)}{\sum_{i=1}^N q_\theta(y|x_i)}$$
$$+ \frac{\sum_{i=1}^N p_{\theta'}(x|x_i)|q_\theta(y|x_i) - q_{\theta'}(y|x_i)|}{\sum_{i=1}^N q_\theta(y|x_i)}$$
$$+ \frac{s_{\theta'}(x|y, \hat{x})\sum_{i=1}^N |q_\theta(y|x_i) - q_{\theta'}(y|x_i)|}{\sum_{i=1}^N q_\theta(y|x_i)}$$
$$\leq \frac{3K_{1,Q}\|\theta - \theta'\|}{\varepsilon_Q^2} = \tilde{C}_{1,Q}\|\theta - \theta'\|.$$

Consequently, for any $x_1', \ldots, x_N' \in \mathcal{X}$, (55) yields

$$\left|\prod_{i=1}^N s_\theta(x_i'|y, \hat{x}) - \prod_{i=1}^N s_{\theta'}(x_i'|y, \hat{x})\right|$$
$$\leq \sum_{i=1}^N \left(\prod_{j=1}^{i-1} s_\theta(x_j'|y, \hat{x})\right)\left(\prod_{j=i+1}^N s_{\theta'}(x_j'|y, \hat{x})\right)$$
$$\cdot |s_\theta(x_i'|y, \hat{x}) - s_{\theta'}(x_i'|y, \hat{x})|$$
$$\leq \frac{\tilde{C}_{1,Q}N\|\theta - \theta'\|}{\varepsilon_Q^{N-1}} \leq \tilde{C}_{2,Q}\|\theta - \theta'\|.$$

Here and throughout the paper, we use the convention that the product $\prod_{i=k}^l$ is one whenever $k > l$. Hence, we have

$$|S_\theta(B|y, \hat{x}) - S_{\theta'}(B|y, \hat{x})|$$
$$\leq \int \cdots \int I_B(x_1, \ldots, x_N)\left|\prod_{i=1}^N s_\theta(x_i|y, \hat{x}) - \prod_{i=1}^N s_{\theta'}(x_i|y, \hat{x})\right|$$
$$\cdot \mu(dx_1)\cdots\mu(dx_N)$$
$$\leq \tilde{C}_{2,Q}(\mu(\mathcal{X}))^N\|\theta - \theta'\| = \tilde{C}_{3,Q}\|\theta - \theta'\|.$$

Therefore, we get

$$|T_\theta(v, B) - T_{\theta'}(v, B)| \leq \iiint I_B(y', x', \hat{x}')Q(x', dy')$$
$$\cdot P(x, dx')|S_\theta - S_{\theta'}|(d\hat{x}'|y, \hat{x})$$
$$\leq \tilde{C}_{3,Q}\|\theta - \theta'\|. \tag{58}$$

Here, $|S_\theta - S_{\theta'}|(d\hat{x}'|y, \hat{x})$ denotes the total variation of the signed measure $S_\theta(d\hat{x}'|y, \hat{x}) - S_{\theta'}(d\hat{x}'|y, \hat{x})$.

Let $\tilde{C}_{4,Q} \in [1,\infty)$ be an upper bound of sequence $\{n\rho_{1,Q}^{n-1}\}_{n\geq 1}$, while $C_{2,Q} = 2\tilde{C}_{3,Q}\tilde{C}_{4,Q}(1-\rho_{1,Q})^{-1}$. Using (57), (58), we conclude

$$
\begin{aligned}
&|T_\theta^{n+1}(v,B) - T_{\theta'}^{n+1}(v,B)| \\
&= \left| \sum_{i=0}^n \iint \tilde{T}_\theta^i(v'',B)(T_\theta - T_{\theta'})(v',dv'')T_{\theta'}^{n-i}(v,dv') \right| \\
&\leq \sum_{i=0}^n \iint |\tilde{T}_\theta^i(v'',B)||T_\theta - T_{\theta'}|(v',dv'')T_{\theta'}^{n-i}(v,dv') \\
&\leq \tilde{C}_{3,Q}\|\theta - \theta'\| \sum_{i=0}^n \rho_{1,Q}^{2i} \leq C_{2,Q}\|\theta - \theta'\|. \quad (59)
\end{aligned}
$$

Similarly, we deduce

$$
\begin{aligned}
&|\tilde{T}_\theta^{n+1}(v,B) - \tilde{T}_{\theta'}^{n+1}(v,B)| \\
&= \left| \sum_{i=0}^n \iint \tilde{T}_\theta^i(v'',B)(T_\theta - T_{\theta'})(v',dv'')\tilde{T}_{\theta'}^{n-i}(v,dv') \right| \\
&\leq \sum_{i=0}^n \iint |\tilde{T}_\theta^i(v'',B)||T_\theta - T_{\theta'}|(v',dv'')|\tilde{T}_{\theta'}^{n-i}|(v,dv') \\
&\leq \tilde{C}_{3,Q}\rho_{1,Q}^{2n}(n+1)\|\theta - \theta'\| \leq C_{2,Q}\rho_{1,Q}^{n+1}\|\theta - \theta'\|. \quad (60)
\end{aligned}
$$

Combining (57), (59), we get

$$
\begin{aligned}
|\tau_\theta(B) - \tau_{\theta'}(B)| &\leq |T_\theta^n(v,B) - T_{\theta'}^n(v,B)| + |\tilde{T}_\theta^n(v,B)| \\
&\quad + |\tilde{T}_{\theta'}^n(v,B)| \\
&\leq C_{2,Q}\|\theta - \theta'\| + 2\rho_{1,Q}^n. \quad (61)
\end{aligned}
$$

Letting $n \to \infty$ in (61) and using (57), (58), (60), we conclude that (52) – (54) hold. $\qquad\square$

Lemma 5.2: Let Assumptions 2.3 and 2.4 hold. Then, the following is true:

(i) There exists a real number $\rho_{2,Q} \in (0,1)$ (independent of N and depending only on $p_\theta(x'|x)$, $q_\theta(y|x)$) such that $A_\theta^{i,j}(v,v') \geq \rho_{2,Q}/N$ for all $\theta \in Q$, $v,v' \in \mathcal{V}$, $1 \leq i,j \leq N$.

(ii) There exists a real number $C_{3,Q} \in [1,\infty)$ (possibly depending on N) such that

$$
\max\{\|A_\theta(v,v')\|, \|B_\theta(v,v')\|, \|C_\theta(v)\|, \|D_\theta(v)\|\} \leq C_{3,Q}, \quad (62)
$$
$$
\max\{\|A_\theta(v,v') - A_{\theta'}(v,v')\|, \|B_\theta(v,v') - B_{\theta'}(v,v')\|\} \\
\leq C_{3,Q}\|\theta - \theta'\|, \quad (63)
$$
$$
\max\{\|C_\theta(v) - C_{\theta'}(v)\|, \|D_\theta(v) - D_{\theta'}(v)\|\} \\
\leq C_{3,Q}\|\theta - \theta'\| \quad (64)
$$

for all $\theta, \theta' \in Q$, $v, v' \in \mathcal{V}$.

Proof: Throughout the proof, the following notation is used. θ, θ' are any elements of Q. x, x', $x_1, x'_1, \ldots, x_N, x'_N$ are any elements of $\mathcal{X}$, while $\hat{x} = (x_1, \ldots, x_N)$, $\hat{x}' = (x'_1, \ldots, x'_N)$. y, y' are any elements of $\mathcal{Y}$, while $v = (y, x, \hat{x})$, $v' = (y', x', \hat{x}')$. i, j are any integers satisfying $1 \leq i, j \leq N$.

Let $\rho_{2,Q} = \varepsilon_Q^4$ (ε_Q is specified in Assumption 2.3). Owing to Assumption 2.3, we have $\varepsilon_Q^2 \leq r_\theta(x'|y,x) \leq 1/\varepsilon_Q^2$. Therefore, we get $A_\theta^{i,j}(v,v') \geq \varepsilon_Q^4/N = \rho_{2,Q}/N$. Hence, (i) is true.

Due to Assumptions 2.3 and 2.4, we have

$$
\begin{aligned}
|r_\theta(x'|y,x) - r_{\theta'}(x'|y,x)| &\leq |p_\theta(x'|x) - p_{\theta'}(x'|x)|q_\theta(y|x) \\
&\quad + p_{\theta'}(x'|x)|q_\theta(y|x) - q_{\theta'}(y|x)| \\
&\leq \frac{2K_{1,Q}\|\theta - \theta'\|}{\varepsilon_Q}.
\end{aligned}
$$

Then, we get

$$
\begin{aligned}
&|A_\theta^{i,j}(v,v') - A_{\theta'}^{i,j}(v,v')| \\
&\leq \frac{|r_\theta(x'_j|y,x_i) - r_{\theta'}(x'_j|y,x_i)|}{\sum_{k=1}^N r_\theta(x'_j|y,x_k)} \\
&\quad + A_{\theta'}^{i,j}(v,v')\frac{\sum_{k=1}^N |r_\theta(x'_j|y,x_k) - r_{\theta'}(x'_j|y,x_k)|}{\sum_{k=1}^N r_\theta(x'_j|y,x_k)} \\
&\leq \frac{2K_{1,Q}\|\theta - \theta'\|}{\varepsilon_Q^3}\left(\frac{1}{N} + A_{\theta'}^{i,j}(v,v')\right). \quad (65)
\end{aligned}
$$

Since $\sum_{i=1}^N A_\theta^{i,j}(v,v') = 1$ (due to (36)), (65) implies

$$
\begin{aligned}
\sum_{i=1}^N |A_\theta^{i,j}(v,v') - A_{\theta'}^{i,j}(v,v')| \\
\leq \frac{2K_{1,Q}\|\theta - \theta'\|}{\varepsilon_Q^3}\left(1 + \sum_{i=1}^N A_{\theta'}^{i,j}(v,v')\right) \\
= \frac{4K_{1,Q}\|\theta - \theta'\|}{\varepsilon_Q^3}. \quad (66)
\end{aligned}
$$

It is straightforward to verify

$$
B_\theta^j(v,v') = \sum_{i=1}^N A_\theta^{i,j}(v,v')\frac{\nabla_\theta r_\theta(x'_j|y,x_i)}{r_\theta(x'_j|y,x_i)}. \quad (67)
$$

Moreover, using Assumptions 2.3 and 2.4, we conclude

$$
\begin{aligned}
\left\|\frac{\nabla_\theta r_\theta(x'|y,x)}{r_\theta(x'|y,x)}\right\| &\leq \left\|\frac{\nabla_\theta p_\theta(x'|x)}{p_\theta(x'|x)}\right\| + \left\|\frac{\nabla_\theta q_\theta(y|x)}{q_\theta(y|x)}\right\| \\
&\leq \frac{2K_{1,Q}}{\varepsilon_Q}. \quad (68)
\end{aligned}
$$

Relying on the same assumptions, we deduce

$$
\begin{aligned}
&\left\|\frac{\nabla_\theta r_\theta(x'|y,x)}{r_\theta(x'|y,x)} - \frac{\nabla_\theta r_{\theta'}(x'|y,x)}{r_{\theta'}(x'|y,x)}\right\| \\
&\leq \frac{\|\nabla_\theta p_\theta(x'|x) - \nabla_\theta p_{\theta'}(x'|x)\|}{p_\theta(x'|x)} \\
&\quad + \left\|\frac{\nabla_\theta p_{\theta'}(x'|x)}{p_{\theta'}(x'|x)}\right\|\frac{|p_\theta(x'|x) - p_{\theta'}(x'|x)|}{p_\theta(x'|x)} \\
&\quad + \frac{\|\nabla_\theta q_\theta(y|x) - \nabla_\theta q_{\theta'}(y|x)\|}{q_\theta(y|x)} \\
&\quad + \left\|\frac{\nabla_\theta q_{\theta'}(y|x)}{q_{\theta'}(y|x)}\right\|\frac{|q_\theta(y|x) - q_{\theta'}(y|x)|}{q_\theta(y|x)} \\
&\leq \frac{4K_{1,Q}^2\|\theta - \theta'\|}{\varepsilon_Q^2}. \quad (69)
\end{aligned}
$$

Then, (36), (67), (68) imply

$$
\begin{aligned}
\|B_\theta^j(v,v')\| &\leq \sum_{i=1}^N A_\theta^{i,j}(v,v')\left\|\frac{\nabla_\theta r_\theta(x'_j|y,x_i)}{r_\theta(x'_j|y,x_i)}\right\| \\
&\leq \frac{2K_{1,Q}}{\varepsilon_Q}. \quad (70)
\end{aligned}
$$

Similarly, (36), (66) – (69) yield

$$
\begin{aligned}
& \|B_\theta^j(v,v') - B_{\theta'}^j(v,v')\| \\
& \leq \sum_{i=1}^N |A_\theta^{i,j}(v,v') - A_{\theta'}^{i,j}(v,v')| \left\| \frac{\nabla_\theta r_\theta(x_j'|y,x_i)}{r_\theta(x_j'|y,x_i)} \right\| \\
& \quad + \sum_{i=1}^N A_{\theta'}^{i,j}(v,v') \left\| \frac{\nabla_\theta r_\theta(x_j'|y,x_i)}{r_\theta(x_j'|y,x_i)} - \frac{\nabla_\theta r_{\theta'}(x_j'|y,x_i)}{r_{\theta'}(x_j'|y,x_i)} \right\| \\
& \leq \frac{12 K_{1,Q}^2 \|\theta - \theta'\|}{\varepsilon_Q^4}.
\end{aligned} \tag{71}
$$

Due to Assumptions 2.3, 2.4, we have

$$
|C_\theta^i(v)| \leq \max\left\{ \frac{1}{N}, \frac{q_\theta(y|x_i)}{\sum_{k=1}^N q_\theta(y|x_k)} \right\} \leq 1, \tag{72}
$$

$$
\|D_\theta(v)\| \leq \frac{\sum_{k=1}^N \|\nabla_\theta q_\theta(y|x_k)\|}{\sum_{k=1}^N q_\theta(y|x_k)} \leq \frac{K_{1,Q}}{\varepsilon_Q}. \tag{73}
$$

Combining Assumptions 2.3, 2.4 and (72), we get

$$
\begin{aligned}
|C_\theta^i(v) - C_{\theta'}^i(v)| \leq & \frac{|q_\theta(y|x_i) - q_{\theta'}(y|x_i)|}{\sum_{k=1}^N q_\theta(y|x_k)} \\
& + \frac{|C_{\theta'}^i(v)| \sum_{k=1}^N |q_\theta(y|x_k) - q_{\theta'}(y|x_k)|}{\sum_{k=1}^N q_\theta(y|x_k)} \\
\leq & \frac{2 K_{1,Q} \|\theta - \theta'\|}{\varepsilon_Q}.
\end{aligned} \tag{74}
$$

Moreover, using Assumptions 2.3, 2.4 and (73), we get

$$
\begin{aligned}
\|D_\theta(v) - D_{\theta'}(v)\| \leq & \frac{\sum_{k=1}^N \|\nabla_\theta q_\theta(y|x_k) - \nabla_\theta q_{\theta'}(y|x_k)\|}{\sum_{k=1}^N q_\theta(y|x_k)} \\
& + \frac{\|D_{\theta'}(v)\| \sum_{k=1}^N |q_\theta(y|x_k) - q_{\theta'}(y|x_k)|}{\sum_{k=1}^N q_\theta(y|x_k)} \\
\leq & \frac{2 K_{1,Q}^2 \|\theta - \theta'\|}{\varepsilon_Q^2}.
\end{aligned} \tag{75}
$$

Let $C_{3,Q} = 12 \varepsilon_Q^{-4} K_{1,Q}^2 N$. Then, relying on (36), (66) – (75), we deduce that (62) – (64) hold. Hence, (ii) is true. □

Lemma 5.3: Let Assumptions 2.3 and 2.4 hold. Then, there exist real numbers $\rho_{3,Q} \in (0,1)$, $C_{4,Q} \in [1,\infty)$ (possibly depending on N) such that

$$
\begin{aligned}
& \max\{\|\Phi_\theta^n(v)\|, \|\Psi_\theta^n(v)\|\} \leq C_{4,Q} \rho_{3,Q}^n, \\
& \max\{\|\Phi_\theta^n(v) - \Phi_{\theta'}^n(v)\|, \|\Psi_\theta^n(v) - \Phi_{\theta'}^n(v)\|\} \\
& \quad \leq C_{4,Q} \rho_{3,Q}^n \|\theta - \theta'\|
\end{aligned}
$$

for all $\theta, \theta' \in Q$, $v \in \mathcal{V}$, $n \geq 1$.

Proof: Throughout the proof, the following notation is used. θ, θ' are any elements of Q. v is any element of $\mathcal{V}$, while $\{v_n\}_{n\geq 0}$ is any sequence in $\mathcal{V}$. n is any positive integer.

Let $\rho_{3,Q} = (1 - \rho_{2,Q})^{1/2}$, $\tilde{C}_{1,Q} = 4 \rho_{3,Q}^{-2} N$, $\tilde{C}_{2,Q} = 2\tilde{C}_{1,Q} C_{3,Q}^2$, $\tilde{C}_{3,Q} = \rho_{3,Q}^{-2} \tilde{C}_{2,Q} C_{3,Q}$ ($\rho_{2,Q}$, $C_{3,Q}$ are specified in Lemma 5.2). Owing to Lemmas 5.2, A1.2 (see Appendix) and (36), we have

$$
\begin{aligned}
\left\| \tilde{A}_\theta^n(v_0,\dots,v_n) C_\theta(v_n) \right\| \leq & \tilde{C}_{1,Q} \rho_{3,Q}^{2n} \|C_\theta(v_n)\| \\
\leq & \tilde{C}_{1,Q} C_{3,Q} \rho_{3,Q}^{2n} \\
\leq & \tilde{C}_{2,Q} \rho_{3,Q}^{2n}.
\end{aligned} \tag{76}
$$

Since $A_\theta^0(v) = I$ (due to (45)), Lemma 5.2 and (76) imply

$$
\begin{aligned}
\left\| \tilde{B}_\theta^n(v_0,\dots,v_n) C_\theta(v_n) \right\| \leq & \left\| \tilde{A}_\theta^{n-1}(v_1,\dots,v_n) C_\theta(v_n) \right\| \\
& \cdot \|B_\theta(v_0,v_1)\| \\
\leq & \tilde{C}_{2,Q} C_{3,Q} \rho_{3,Q}^{2(n-1)} \\
= & \tilde{C}_{3,Q} \rho_{3,Q}^{2n}.
\end{aligned} \tag{77}
$$

Moreover, due to Lemmas 5.2, A1.2 (see Appendix A), we have

$$
\begin{aligned}
& \left\| \tilde{A}_\theta^n(v_0,\dots,v_n) C_\theta(v_n) - \tilde{A}_{\theta'}^n(v_0,\dots,v_n) C_{\theta'}(v_n) \right\| \\
& \leq \tilde{C}_{1,Q} \rho_{3,Q}^{2n} \left(\|C_\theta(v_n)\| + \|C_{\theta'}(v_n)\| \right) \\
& \quad \cdot \sum_{k=0}^{n-1} \|A_\theta(v_k, v_{k+1}) - A_{\theta'}(v_k, v_{k+1})\| \\
& \quad + \tilde{C}_{1,Q} \rho_{3,Q}^{2n} \|C_\theta(v_n) - C_{\theta'}(v_n)\| \\
& \leq 2\tilde{C}_{1,Q} C_{3,Q}^2 \rho_{3,Q}^{2n} (n+1) \|\theta - \theta'\| \\
& = \tilde{C}_{2,Q} \rho_{3,Q}^{2n} (n+1) \|\theta - \theta'\|.
\end{aligned} \tag{78}
$$

Combining this with Lemma 5.2 and (76), we get

$$
\begin{aligned}
& \left\| \tilde{B}_\theta^n(v_0,\dots,v_n) C_\theta(v_n) - \tilde{B}_{\theta'}^n(v_0,\dots,v_n) C_{\theta'}(v_n) \right\| \\
& \leq \|B_\theta(v_0,v_1) - B_{\theta'}(v_0,v_1)\| \left\| \tilde{A}_\theta^{n-1}(v_1,\dots,v_n) C_\theta(v_n) \right\| \\
& \quad + \left\| \tilde{A}_\theta^{n-1}(v_1,\dots,v_n) C_\theta(v_n) - \tilde{A}_{\theta'}^{n-1}(v_1,\dots,v_n) C_{\theta'}(v_n) \right\| \\
& \quad \cdot \|B_{\theta'}(v_0,v_1)\| \\
& \leq \tilde{C}_{2,Q} C_{3,Q} \rho_{3,Q}^{2(n-1)} (n+1) \|\theta - \theta'\| \\
& = \tilde{C}_{3,Q} \rho_{3,Q}^{2n} (n+1) \|\theta - \theta'\|.
\end{aligned} \tag{79}
$$

Let $U_\theta^n(dv_1,\dots,dv_n|v)$ be the conditional probability measure defined for $B \in \mathcal{B}(\mathcal{V}^n)$ by

$$
U_\theta^n(B|v) = E\left(I_B(V_1^\theta,\dots,V_n^\theta) \middle| V_0^\theta = v \right).
$$

Moreover, let $u_{\theta,\theta'}^n(v)$ be the function defined by

$$
u_{\theta,\theta'}^n(v) = \sup_{B \in \mathcal{B}(\mathcal{V}^n)} |U_\theta^n(B|v) - U_{\theta'}^n(B|v)|.
$$

Then, for $B \in \mathcal{B}(\mathcal{V}^{n+1})$, we have

$$
U_\theta^{n+1}(B|v) = \iint \cdots \int I_B(v_1,\dots,v_n,v_{n+1}) T_\theta(v_n, dv_{n+1}) \\
\cdot U_\theta^n(dv_1,\dots,dv_n|v).
$$

Consequently, Lemma 5.1 implies

$$
\begin{aligned}
& \left| U_\theta^{n+1}(B|v) - U_{\theta'}^{n+1}(B|v) \right| \\
& \leq \iint \cdots \int I_B(v_1,\dots,v_n,v_{n+1}) |T_\theta - T_{\theta'}|(v_n, dv_{n+1}) \\
& \quad \cdot U_\theta^n(dv_1,\dots,dv_n|v) \\
& \quad + \iint \cdots \int I_B(v_1,\dots,v_n,v_{n+1}) T_{\theta'}(v_n, dv_{n+1}) \\
& \quad \quad \cdot |U_\theta^n - U_{\theta'}^n|(dv_1,\dots,dv_n|v) \\
& \leq C_{2,Q} \|\theta - \theta'\| + u_{\theta,\theta'}^n(v)
\end{aligned}
$$

($C_{2,Q}$ is specified in Lemma 5.1). For $B \in \mathcal{B}(\mathcal{V})$, Lemma 5.1 also yields

$$
\begin{aligned}
\left| U_\theta^1(B|v) - U_{\theta'}^1(B|v) \right| &= |T_\theta(v, B) - T_{\theta'}(v, B)| \\
&\leq C_{2,Q} \|\theta - \theta'\|.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
u_{\theta,\theta'}^1(v) &\leq C_{2,Q} \|\theta - \theta'\|, \\
u_{\theta,\theta'}^{n+1}(v) &\leq u_{\theta,\theta'}^n(v) + C_{2,Q} \|\theta - \theta'\|. \quad (80)
\end{aligned}
$$

Then, iterating (80), we conclude

$$
u_{\theta,\theta'}^n(v) \leq C_{2,Q} n \|\theta - \theta'\|. \quad (81)
$$

Let $\tilde{C}_{4,Q} \in [1, \infty)$ be an upper bound of sequence $\{n\rho_{3,Q}^{n-1}\}_{n \geq 1}$, while $C_{4,Q} = 4\tilde{C}_{3,Q}\tilde{C}_{4,Q}C_{2,Q}$. It is straightforward to verify

$$
\begin{aligned}
\Phi_\theta^n(v) &= \int \cdots \int \tilde{A}_\theta^n(v, v_1, \ldots, v_n) C_\theta(v_n) \\
&\quad \cdot U_\theta^n(dv_1, \ldots, dv_n|v), \quad (82) \\
\Psi_\theta^n(v) &= \int \cdots \int \tilde{B}_\theta^n(v, v_1, \ldots, v_n) C_\theta(v_n) \\
&\quad \cdot U_\theta^n(dv_1, \ldots, dv_n|v). \quad (83)
\end{aligned}
$$

Combining this with (76), (77), we get

$$
\begin{aligned}
\|\Phi_\theta^n(v)\| &\leq \int \cdots \int \left\| \tilde{A}_\theta^n(v, v_1, \ldots, v_n) C_\theta(v_n) \right\| \\
&\quad \cdot U_\theta^n(dv_1, \ldots, dv_n|v) \\
&\leq \tilde{C}_{2,Q} \rho_{3,Q}^{2n} \leq C_{4,Q} \rho_{3,Q}^n, \\
\|\Psi_\theta^n(v)\| &\leq \int \cdots \int \left\| \tilde{B}_\theta^n(v, v_1, \ldots, v_n) C_\theta(v_n) \right\| \\
&\quad \cdot U_\theta^n(dv_1, \ldots, dv_n|v) \\
&\leq \tilde{C}_{3,Q} \rho_{3,Q}^{2n} \leq C_{4,Q} \rho_{3,Q}^n.
\end{aligned}
$$

Moreover, (78), (82), (81) imply

$$
\begin{aligned}
\|\Phi_\theta^n(v) - \Phi_{\theta'}^n(v)\| &\leq \int \cdots \int \left\| \tilde{A}_\theta^n(v, v_1, \ldots, v_n) C_\theta(v_n) \right. \\
&\quad \left. - \tilde{A}_{\theta'}^n(v, v_1, \ldots, v_n) C_{\theta'}(v_n) \right\| \\
&\quad \cdot U_\theta^n(dv_1, \ldots, dv_n|v) \\
&\quad + \int \cdots \int \left\| \tilde{A}_{\theta'}^n(v, v_1, \ldots, v_n) C_{\theta'}(v_n) \right\| \\
&\quad \cdot |U_\theta^n - U_{\theta'}^n|(dv_1, \ldots, dv_n|v) \\
&\leq 2\tilde{C}_{2,Q} C_{2,Q} \rho_{3,Q}^{2n}(n+1) \|\theta - \theta'\| \\
&\leq C_{4,Q} \rho_{3,Q}^n \|\theta - \theta'\|.
\end{aligned}
$$

Similarly, (79), (83), (81) yield

$$
\begin{aligned}
\|\Psi_\theta^n(v) - \Psi_{\theta'}^n(v)\| &\leq \int \cdots \int \left\| \tilde{B}_\theta^n(v, v_1, \ldots, v_n) C_\theta(v_n) \right. \\
&\quad \left. - \tilde{B}_{\theta'}^n(v, v_1, \ldots, v_n) C_{\theta'}(v_n) \right\| \\
&\quad \cdot U_\theta^n(dv_1, \ldots, dv_n|v) \\
&\quad + \int \cdots \int \left\| \tilde{B}_{\theta'}^n(v, v_1, \ldots, v_n) C_{\theta'}(v_n) \right\| \\
&\quad \cdot |U_\theta^n - U_{\theta'}^n|(dv_1, \ldots, dv_n|v) \\
&\leq 2\tilde{C}_{3,Q} C_{2,Q} \rho_{3,Q}^{2n}(n+1) \|\theta - \theta'\| \\
&\leq C_{4,Q} \rho_{3,Q}^n \|\theta - \theta'\|.
\end{aligned}
$$

$\square$

Lemma 5.4: Let Assumptions 2.2 – 2.4 hold. Then, the following is true:

(i) $h(\theta)$ is well-defined on Θ.

(ii) $h(\theta) = \lim_{n \to \infty} E\left(H(\theta, Z_n^\theta)\right)$ for each $\theta \in \Theta$ satisfying $E(\|W_0^\theta \Lambda\|) < \infty$.

(iii) There exists a function $\tilde{H}(\theta, z)$ mapping $\theta \in \Theta$, $z \in \mathcal{Z}$ to $\mathbb{R}^d$ such that

$$
H(\theta, z) - h(\theta) = \tilde{H}(\theta, z) - (\Pi\tilde{H})(\theta, z) \quad (84)
$$

for all $\theta \in \Theta$, $z \in \mathcal{Z}$. Here, $(\Pi\tilde{H})(\theta, z)$ denotes $\int \tilde{H}(\theta, z') \Pi_\theta(z, dz')$.

(iv) There exists a real number $C_{5,Q} \in [1, \infty)$ (possibly depending on N) such that

$$
\begin{aligned}
&\max\{\|H(\theta, z)\|, \|\tilde{H}(\theta, z)\|, \|(\Pi\tilde{H})(\theta, z)\|\} \\
&\leq C_{5,Q}(1 + \|W\Lambda\|), \\
&\|(\Pi\tilde{H})(\theta, z) - (\Pi\tilde{H})(\theta', z)\| \leq C_{5,Q} \|\theta - \theta'\|(1 + \|W\Lambda\|)
\end{aligned}
$$

for all $\theta, \theta' \in Q$, $v \in \mathcal{V}$, $W \in \mathbb{R}^{d \times N}$ and $z = (v, W)$ (Λ is defined in (30)).

Proof: Throughout the proof, the following notation is used. θ, θ' are any elements of Q. v, W are any elements of $\mathcal{V}$, $\mathbb{R}^{d \times N}$ (respectively), while $z = (v, W)$. n, k are any positive integers.

Owing to (36), we have

$$
e^T \tilde{A}_\theta^n(V_0^\theta, \ldots, V_n^\theta) C_\theta(V_n^\theta) = e^T C_\theta(V_n^\theta) = 0.
$$

Therefore, we get

$$
\begin{aligned}
&\Lambda \tilde{A}_\theta^n(V_0^\theta, \ldots, V_n^\theta) C_\theta(V_n^\theta) \\
&= \tilde{A}_\theta^n(V_0^\theta, \ldots, V_n^\theta) C_\theta(V_n^\theta) - \frac{e}{N} e^T \tilde{A}_\theta^n(V_0^\theta, \ldots, V_n^\theta) C_\theta(V_n^\theta) \\
&= \tilde{A}_\theta^n(V_0^\theta, \ldots, V_n^\theta) C_\theta(V_n^\theta). \quad (85)
\end{aligned}
$$

Moreover, iterating (43), it is straightforward to verify

$$
W_n^\theta = W_0^\theta \tilde{A}_\theta^n(V_0^\theta, \ldots, V_n^\theta) + \sum_{k=0}^{n-1} \tilde{B}_\theta^{n-k}(V_k^\theta, \ldots, V_n^\theta). \quad (86)
$$

Combining this with (85), we conclude

$$
\begin{aligned}
H(\theta, Z_n^\theta) &= D_\theta(V_n^\theta) + W_0^\theta \tilde{A}_\theta^n(V_0^\theta, \ldots, V_n^\theta) C_\theta(V_n^\theta) \\
&\quad + \sum_{k=0}^{n-1} \tilde{B}_\theta^{n-k}(V_k^\theta, \ldots, V_n^\theta) C_\theta(V_n^\theta) \\
&= D_\theta(V_n^\theta) + W_0^\theta \Lambda \tilde{A}_\theta^n(V_0^\theta, \ldots, V_n^\theta) C_\theta(V_n^\theta) \\
&\quad + \sum_{k=0}^{n-1} \tilde{B}_\theta^{n-k}(V_k^\theta, \ldots, V_n^\theta) C_\theta(V_n^\theta).
\end{aligned}
$$

Consequently, we have

$$
\begin{aligned}
(\Pi^n H)(\theta, z) &= E\left(H(\theta, Z_n^\theta)\big| Z_0^\theta = z\right)\\
&= E\left(D_\theta(V_n^\theta)\big| V_0^\theta = v\right)\\
&\quad + E\left(W\Lambda \tilde{A}_\theta^n(V_0^\theta, \dots, V_n^\theta) C_\theta(V_n^\theta)\big| V_0^\theta = v\right)\\
&\quad + \sum_{k=0}^{n-1} E\left(E\left(\tilde{B}_\theta^{n-k}(V_k^\theta, \dots, V_n^\theta) C_\theta(V_n^\theta)\big| V_k^\theta\right)\bigg| V_0^\theta = v\right)\\
&= (T^n D)_\theta(v) + W\Lambda \Phi_\theta^n(v) + \sum_{k=0}^{n-1} E\left(\Psi_\theta^{n-k}(V_k^\theta)\big| V_0^\theta = v\right)\\
&= (T^n D)_\theta(v) + W\Lambda \Phi_\theta^n(v) + \sum_{k=0}^{n-1} (T^k \Psi^{n-k})_\theta(v). \quad (87)
\end{aligned}
$$

Here, $(\Pi^n H)(\theta, z)$ denotes $\int H(\theta, z')\Pi_\theta^n(z, dz')$, while $(T^n D)_\theta(v)$, $(T^k \Psi^l)_\theta(v)$ stand for $\int D_\theta(v')T_\theta^n(v, dv')$, $\int \Psi_\theta^l(v')T_\theta^k(v, dv')$ (respectively).

Let $\beta_Q = \max\{\rho_{1,Q}^{1/2}, \rho_{3,Q}^{1/2}\}$, $\tilde{C}_{1,Q} = 4C_{2,Q}C_{3,Q}C_{4,Q}$, $\tilde{C}_{2,Q} = 2\tilde{C}_{1,Q}(1 - \beta_Q)^{-1}$ ($\rho_{1,Q}$, $\rho_{3,Q}$, $C_{2,Q}$, $C_{3,Q}$, $C_{4,Q}$ are specified in Lemmas 5.1 – 5.3). Owing to Lemma 5.3, we have

$$
\int \|\Psi_\theta^n(v)\| \tau_\theta(dv) \le C_{4,Q}\rho_{3,Q}^n \le \tilde{C}_{1,Q}\beta_Q^{2n}. \quad (88)
$$

Consequently, Lemma 5.2 yields

$$
\begin{aligned}
&\int \|D_\theta(v)\| \tau_\theta(dv) + \sum_{n=1}^\infty \int \|\Psi_\theta^n(v)\| \tau_\theta(dv)\\
&\le C_{3,Q} + \tilde{C}_{1,Q}\sum_{n=1}^\infty \beta_Q^{2n} \le 2\tilde{C}_{1,Q}(1 - \beta_Q)^{-1} \le \tilde{C}_{2,Q} < \infty.
\end{aligned}
$$

Hence, $h(\theta)$ is well-defined and satisfies $\|h(\theta)\| \le \tilde{C}_{2,Q}$. Since Q is any compact set in Θ, we conclude that (i) holds. Moreover, using (87), we deduce

$$
\begin{aligned}
(\Pi^n H)(\theta, z) - h(\theta) &= (\tilde{T}^n D)_\theta(v) + W\Lambda \Phi_\theta^n(v)\\
&\quad + \sum_{k=0}^{n-1} (\tilde{T}^k \Psi^{n-k})_\theta(v)\\
&\quad - \sum_{k=n+1}^\infty \int \Psi_\theta^k(v') \tau_\theta(dv'). \quad (89)
\end{aligned}
$$

Here, $(\tilde{T}^n D)_\theta(v)$, $(\tilde{T}^k \Psi^l)_\theta(v)$ denote $\int D_\theta(v')\tilde{T}_\theta^n(v, dv')$, $\int \Psi_\theta^l(v')\tilde{T}_\theta^k(v, dv')$ (respectively).

Let $\tilde{C}_{3,Q} \in [1, \infty)$ be an upper bound of sequence $\{n\beta_Q^{n-1}\}_{n\ge 1}$, while $\tilde{C}_{4,Q} = 2\tilde{C}_{2,Q}\tilde{C}_{3,Q}(1 - \beta_Q)^{-1}$, $C_{5,Q} = \tilde{C}_{4,Q}(1 - \beta_Q)^{-1}$. Owing to Lemmas 5.1, 5.2, we have

$$
\begin{aligned}
\|(\tilde{T}^n D)_\theta(v)\| &\le \int \|D_\theta(v')\| |\tilde{T}_\theta^n|(v, dv')\\
&\le C_{2,Q}C_{3,Q}\rho_{1,Q}^n \le \tilde{C}_{1,Q}\beta_Q^{2n}. \quad (90)
\end{aligned}
$$

Similarly, due to Lemmas 5.1, 5.3, we have

$$
\begin{aligned}
\|(\tilde{T}^{n-k} \Psi^k)_\theta(v)\| &\le \int \|\Psi_\theta^k(v')\| |\tilde{T}_\theta^{n-k}|(v, dv')\\
&\le C_{2,Q}C_{4,Q}\rho_{1,Q}^{n-k}\rho_{3,Q}^k \le \tilde{C}_{1,Q}\beta_Q^{2n}. \quad (91)
\end{aligned}
$$

Combining Lemma 5.3 and (88), (89) – (91), we get

$$
\begin{aligned}
\|(\Pi^n H)(\theta, z) - h(\theta)\| &\le \|(\tilde{T}^n D)_\theta(v)\| + \|\Phi_\theta^n(v)\| \|W\Lambda\|\\
&\quad + \sum_{k=1}^n \|(\tilde{T}^{n-k}\Psi^k)_\theta(v)\|\\
&\quad + \sum_{k=n+1}^\infty \|\Psi_\theta^k(v')\| \tau_\theta(dv')\\
&\le \tilde{C}_{1,Q}\beta_Q^{2n}(n+1) + C_{4,Q}\rho_{3,Q}^n \|W\Lambda\|\\
&\quad + \tilde{C}_{1,Q}\sum_{k=n+1}^\infty \beta_Q^{2k}\\
&\le \tilde{C}_{1,Q}\beta_Q^{2n}(n+1)(1 + \|W\Lambda\|)\\
&\quad + \tilde{C}_{1,Q}\beta_Q^{2n}(1 - \beta_Q)^{-1}\\
&\le \tilde{C}_{4,Q}\beta_Q^n(1 + \|W\Lambda\|). \quad (92)
\end{aligned}
$$

Since $\|h(\theta)\| \le \tilde{C}_{2,Q}$ and $\Lambda C_\theta(v) = C_\theta(v)$ (due to (36)), Lemma 5.2 yields

$$
\begin{aligned}
\|(\Pi^0 H)(\theta, z) - h(\theta)\| &\le \|C_\theta(v)\| \|W\Lambda\| + \|D_\theta(v)\| + \|h(\theta)\|\\
&\le C_{3,Q}(1 + \|W\Lambda\|) + \tilde{C}_{2,Q}\\
&\le \tilde{C}_{4,Q}(1 + \|W\Lambda\|).
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
\sum_{n=0}^\infty \|(\Pi^n H)(\theta, z) - h(\theta)\| &\le \tilde{C}_{4,Q}(1 + \|W\Lambda\|) \sum_{n=0}^\infty \beta_Q^n\\
&\le C_{5,Q}(1 + \|W\Lambda\|). \quad (93)
\end{aligned}
$$

Owing to Lemmas 5.1, 5.2, we have

$$
\begin{aligned}
&\|(\tilde{T}^n D)_\theta(v) - (\tilde{T}^n D)_{\theta'}(v)\|\\
&\le \int \|D_\theta(v') - D_{\theta'}(v')\| |\tilde{T}_\theta^n|(v, dv')\\
&\quad + \int \|D_{\theta'}(v')\| |\tilde{T}_\theta^n - \tilde{T}_{\theta'}^n|(v, dv')\\
&\le 2C_{2,Q}C_{3,Q}\rho_{1,Q}^n \|\theta - \theta'\| \le \tilde{C}_{1,Q}\beta_Q^{2n} \|\theta - \theta'\|. \quad (94)
\end{aligned}
$$

Moreover, Lemmas 5.1, 5.3 imply

$$
\begin{aligned}
&\|(\tilde{T}^{n-k}\Psi^k)_\theta(v) - (\tilde{T}^{n-k}\Psi^k)_{\theta'}(v)\|\\
&\le \int \|\Psi_\theta^k(v') - \Psi_{\theta'}^k(v')\| |\tilde{T}_\theta^{n-k}|(v, dv')\\
&\quad + \int \|\Psi_{\theta'}^k(v')\| |\tilde{T}_\theta^{n-k} - \tilde{T}_{\theta'}^{n-k}|(v, dv')\\
&\le 2C_{2,Q}C_{4,Q}\rho_{1,Q}^{n-k}\rho_{3,Q}^k \|\theta - \theta'\| \le \tilde{C}_{1,Q}\beta_Q^{2n}\|\theta - \theta'\| \quad (95)
\end{aligned}
$$

for $n \ge k$. The same lemmas also yield

$$
\begin{aligned}
&\left\| \int \Psi_\theta^n(v)\tau_\theta(dv) - \int \Psi_{\theta'}^n(v)\tau_{\theta'}(dv) \right\|\\
&\le \int \|\Psi_\theta^n(v) - \Psi_{\theta'}^n(v)\| \tau_\theta(dv) + \int \|\Psi_{\theta'}^n(v)\| |\tau_\theta - \tau_{\theta'}|(dv)\\
&\le 2C_{2,Q}C_{4,Q}\rho_{3,Q}^n \|\theta - \theta'\| \le \tilde{C}_{1,Q}\beta_Q^{2n}\|\theta - \theta'\|. \quad (96)
\end{aligned}
$$

Combining Lemma 5.3 and (89), (94) – (96), we get

$$\|((\Pi^n H)(\theta, z) - h(\theta)) - ((\Pi^n H)(\theta', z) - h(\theta'))\|$$
$$\leq \|(\tilde{T}^n D)_\theta(v) - (\tilde{T}^n D)_{\theta'}(v)\| + \|\Phi_\theta^n(v) - \Phi_{\theta'}^n(v)\| \|W\Lambda\|$$
$$+ \sum_{k=1}^n \|(\tilde{T}^{n-k}\Psi^k)_\theta(v) - (\tilde{T}^{n-k}\Psi^k)_{\theta'}(v)\|$$
$$+ \sum_{k=n+1}^\infty \left\|\int \Psi_\theta^k(v')\tau_\theta(dv') - \int \Psi_{\theta'}^k(v')\tau_{\theta'}(dv')\right\|$$
$$\leq \tilde{C}_{1,Q}\beta_Q^{2n}(n+1)\|\theta - \theta'\| + C_{4,Q}\rho_{3,Q}^n\|W\Lambda\|\|\theta - \theta'\|$$
$$+ \tilde{C}_{1,Q}\|\theta - \theta'\| \sum_{k=n+1}^\infty \beta_Q^{2k}$$
$$\leq \tilde{C}_{1,Q}\beta_Q^{2n}(n+1)(1 + \|W\Lambda\|)\|\theta - \theta'\|$$
$$+ \tilde{C}_{1,Q}\beta_Q^{2n}(1 - \beta_Q)^{-1}\|\theta - \theta'\|$$
$$\leq \tilde{C}_{4,Q}\beta_Q^n\|\theta - \theta'\|(1 + \|W\Lambda\|). \tag{97}$$

Hence, we have

$$\sum_{n=1}^\infty \|((\Pi^n H)(\theta, z) - h(\theta)) - ((\Pi^n H)(\theta', z) - h(\theta'))\|$$
$$\leq \tilde{C}_{4,Q}\|\theta - \theta'\|(1 + \|W\Lambda\|)\sum_{n=1}^\infty \beta_Q^n$$
$$\leq C_{5,Q}\|\theta - \theta'\|(1 + \|W\Lambda\|). \tag{98}$$

Let $\tilde{H}(\theta, z)$ be the function defined by

$$\tilde{H}(\theta, z) = \sum_{n=0}^\infty ((\Pi^n H)(\theta, z) - h(\theta)).$$

Then, (93) implies that $\tilde{H}(\theta, z)$, $(\Pi\tilde{H})(\theta, z)$ are well-defined and satisfy

$$(\Pi\tilde{H})(\theta, z) = \sum_{n=1}^\infty ((\Pi^n H)(\theta, z) - h(\theta)).$$

Consequently, (84) holds. Since Q is any compact set in Θ, we conclude that (iii) also holds. Moreover, using (93), (98), we deduce that (iv) is also true.

When $E(\|W_0^\theta\Lambda\|) < \infty$, (92) implies

$$\|E(H(\theta, Z_n^\theta)) - h(\theta)\| = \|E((\Pi^n H)(\theta, Z_0^\theta) - h(\theta))\|$$
$$\leq E(\|(\Pi^n H)(\theta, Z_0^\theta) - h(\theta)\|)$$
$$\leq \tilde{C}_{4,Q}\beta_Q^n(1 + E(\|W_0^\theta\Lambda\|)).$$

Therefore, $h(\theta) = \lim_{n\to\infty} E(H(\theta, Z_n^\theta))$ if $E(\|W_0^\theta\Lambda\|) < \infty$. As Q is any compact set in Θ, we conclude that (ii) holds. $\square$

Lemma 5.5: Let Assumptions 2.3 and 2.4 hold. Then, there exists a real number $C_{6,Q} \in [1, \infty)$ (possibly depending on N) such that

$$\|W_n\Lambda\|I_{\{\tau_Q \geq n\}} \leq C_{6,Q}(1 + \|W_0\Lambda\|)$$

for $n \geq 1$, where τ_Q is the stopping time defined by

$$\tau_Q = \inf(\{n \geq 0 : \theta_n \notin Q\} \cup \{\infty\})$$

(Λ is specified in (30)).

Proof: Throughout the proof, the following notation is used. n is any positive integer. A_n and B_n are the random matrices defined by

$$A_n = A_{\theta_{n-1}}(V_{n-1}, V_n), \qquad B_n = B_{\theta_{n-1}}(V_{n-1}, V_n).$$

$A_{k,k}$ and $A_{k,l}$ are the random matrices defined by

$$A_{k,k} = I, \qquad A_{k,l} = A_{k+1}\cdots A_l$$

for $l > k \geq 0$. Then, iterating (14), we get

$$W_n = W_0 A_{0,n} + \sum_{j=1}^n B_j A_{j,n}. \tag{99}$$

Moreover, (36) implies $e^T A_{k,l}\Lambda = e^T\Lambda = 0$ for $l \geq k \geq 0$. Consequently, we have

$$\Lambda A_{k,l}\Lambda = A_{k,l}\Lambda - \frac{e}{N}e^T A_{k,l}\Lambda = A_{k,l}\Lambda.$$

Combining this with (99), we get

$$W_n\Lambda = W_0 A_{0,n}\Lambda + \sum_{j=1}^n B_j A_{j,n}\Lambda$$
$$= W_0\Lambda A_{0,n}\Lambda + \sum_{j=1}^n B_k A_{j,n}\Lambda. \tag{100}$$

Let $\beta_Q = 1 - \rho_{2,Q}$, $\tilde{C}_{1,Q} = 4\beta_Q^{-1}N$, $\tilde{C}_{2,Q} = \tilde{C}_{1,Q}C_{3,Q}$, $C_{6,Q} = \tilde{C}_{2,Q}(1 - \beta_Q)^{-1}$ ($\rho_{2,Q}$, $C_{3,Q}$ are specified in Lemma 5.2). Since $\theta_0, \ldots, \theta_{n-1} \in Q$ on $\{\tau_Q \geq n\}$, Lemmas 5.2, A1.2 (see Appendix A) and (36) imply

$$\|A_{k,n}\Lambda\|I_{\{\tau_Q \geq n\}}$$
$$= \|A_{\theta_k}(V_k, V_{k+1})\cdots A_{\theta_{n-1}}(V_{n-1}, V_n)\Lambda\|I_{\{\tau_Q \geq n\}}$$
$$\leq \tilde{C}_{1,Q}\beta_Q^{n-k} \tag{101}$$

for $n > k \geq 0$. Consequently, Lemma 5.2 yields

$$\|B_k A_{k,n}\Lambda\|I_{\{\tau_Q \geq n\}}$$
$$= \|B_{\theta_{k-1}}(V_{k-1}, V_k)A_{k,n}\Lambda\|I_{\{\tau_Q \geq n\}}$$
$$\leq \|B_{\theta_{k-1}}(V_{k-1}, V_k)\|\|A_{k,n}\Lambda\|I_{\{\tau_Q \geq n\}}$$
$$\leq \tilde{C}_{1,Q}C_{3,Q}\beta_Q^{n-k} \leq \tilde{C}_{2,Q}\beta_Q^{n-k}$$

for $n \geq k \geq 1$. Combining this with (100), we get

$$\|W_n\Lambda\|I_{\{\tau_Q \geq n\}} \leq \|W_0\Lambda\|\|A_{0,n}\Lambda\|I_{\{\tau_Q \geq n\}}$$
$$+ \sum_{j=1}^n \|B_j A_{j,n}\Lambda\|I_{\{\tau_Q \geq n\}}$$
$$\leq \tilde{C}_{1,Q}\beta_Q^n\|W_0\Lambda\| + \tilde{C}_{2,Q}\sum_{j=1}^n \beta_Q^{n-j}$$
$$\leq \tilde{C}_{2,Q}(1 - \beta_Q)^{-1}(1 + \|W_0\Lambda\|)$$
$$= C_{6,Q}(1 + \|W_0\Lambda\|).$$

$\square$

Lemma 5.6: Let Assumptions 2.2 – 2.4 hold. Then, there exists a real number $M_Q \in [1, \infty)$ (independent of N and depending only on $p_\theta(x'|x)$, $q_\theta(y|x)$) such that

$$\|h(\theta) - \nabla l(\theta)\| \leq \frac{M_Q}{N} \tag{102}$$

for all $\theta \in Q$.

Proof: Throughout the proof, the following notation is used. $H'_{\theta,y}(\xi,\zeta)$ and $H''_{\theta,y}(\xi,\zeta)$ are the functions defined by

$$H'_{\theta,y}(\xi,\zeta) = \frac{\int \nabla_\theta q_\theta(y|x)\xi(dx)}{\int q_\theta(y|x)\xi(dx)},$$

$$H''_{\theta,y}(\xi,\zeta) = \frac{\int q_\theta(y|x)\zeta(dx)}{\int q_\theta(y|x)\xi(dx)}$$

for $\theta \in \Theta$, $y \in \mathcal{Y}$, $\xi \in \mathcal{P}(\mathcal{X})$, $\zeta \in \mathcal{M}_s^d(\mathcal{X})$. $A_{1,n}^\theta$, $A_{2,n}^\theta$ and $A_{3,n}^\theta$ are the random variables defined by

$$A_{1,n}^\theta = \int q_\theta(Y_n|x)\xi_n^\theta(dx),$$

$$A_{2,n}^\theta = \int q_\theta(Y_n|x)\tilde{\xi}_n^\theta(dx),$$

$$A_{3,n}^\theta = \int q_\theta(Y_n|x)F_{\theta,\boldsymbol{Y}}^{0:n}(dx|\xi_0^\theta)$$

for $n \geq 0$. $B_{1,n}^\theta$, $B_{2,n}^\theta$ and $B_{3,n}^\theta$ are the random variables defined by

$$B_{1,n}^\theta = \int \nabla_\theta q_\theta(Y_n|x)\xi_n^\theta(dx),$$

$$B_{2,n}^\theta = \int \nabla_\theta q_\theta(Y_n|x)\tilde{\xi}_n^\theta(dx),$$

$$B_{3,n}^\theta = \int \nabla_\theta q_\theta(Y_n|x)F_{\theta,\boldsymbol{Y}}^{0:n}(dx|\xi_0^\theta).$$

$C_{1,n}^\theta$, $C_{2,n}^\theta$ and $C_{3,n}^\theta$ are the random variables defined by

$$C_{1,n}^\theta = \int q_\theta(Y_n|x)\zeta_n^\theta(dx),$$

$$C_{2,n}^\theta = \int q_\theta(Y_n|x)\tilde{\zeta}_n^\theta(dx),$$

$$C_{3,n}^\theta = \int q_\theta(Y_n|x)G_{\theta,\boldsymbol{Y}}^{0:n}(dx|\xi_0^\theta,\zeta_0).$$

For $1 \leq i \leq N$, $W_{n,i}^\theta$ is the i-th column of W_n^θ. $\xi_n^\theta(dx)$ and $\zeta_n^\theta(dx)$ are the (empirical) measures defined by

$$\xi_n^\theta(B) = \frac{1}{N}\sum_{i=1}^N \delta_{\hat{X}_{n,i}^\theta}(B), \tag{103}$$

$$\zeta_n^\theta(B) = \frac{1}{N}\sum_{i=1}^N \left(W_{n,i}^\theta - \frac{1}{N}\sum_{j=1}^N W_{n,j}^\theta\right)\delta_{\hat{X}_{n,i}^\theta}(B) \tag{104}$$

for $B \in \mathcal{B}(\mathcal{X})$. $\tilde{\xi}_n^\theta(dx)$ and $\tilde{\zeta}_n^\theta(dx)$ are the (random) measures defined by

$$\tilde{\xi}_n^\theta(B) = \xi_n^\theta(B) - F_{\theta,\boldsymbol{Y}}^{0:n}(B|\xi_0^\theta),$$

$$\tilde{\zeta}_n^\theta(B) = \zeta_n^\theta(B) - G_{\theta,\boldsymbol{Y}}^{0:n}(B|\xi_0^\theta,\zeta_0^\theta)$$

($\boldsymbol{Y}$, $F_{\theta,\boldsymbol{y}}^{0:n}(dx|\xi)$, $G_{\theta,\boldsymbol{y}}^{0:n}(dx|\xi,\zeta)$ are defined in the statement of Lemma 4.1 and (27), (28)). Throughout the proof, we assume (without loss of generality) that $\hat{X}_0^\theta = \hat{x}_0$, $W_0^\theta = \boldsymbol{0}$ for each $\theta \in \Theta$, where $\hat{x}_0 \in \mathcal{X}^N$ is a deterministic vector and $\boldsymbol{0}$ is the $d \times N$ zero matrix. Consequently, $\xi_0^\theta(dx)$, $\zeta_0^\theta(dx)$ are deterministic, independent of θ and satisfy $\|\zeta_0^\theta\| = 0$.

In the rest of the proof, let θ, y be any elements of Q, $\mathcal{Y}$ (respectively), while ξ, ζ are any elements of $\mathcal{P}(\mathcal{X})$, $\mathcal{M}_s^d(\mathcal{X})$. Moreover, let n be any non-negative integer, while $\varphi : \mathcal{X} \to$ $[-1,1]$ is any function. Then, relying on [33, Theorem 2.1, Proposition 5.1], we conclude that there exists a real number $\tilde{C}_{1,Q} \in [1,\infty)$ (independent of N and depending only on $p_\theta(x'|x)$, $q_\theta(y|x)$) such that $\|\zeta_n^\theta\| \leq \tilde{C}_{1,Q}$ and

$$\left|E\left(\int \varphi(x)\tilde{\xi}_n^\theta(dx)\Big|\boldsymbol{Y}\right)\right| \leq \frac{\tilde{C}_{1,Q}}{N}, \tag{105}$$

$$\left\|E\left(\int \varphi(x)\tilde{\zeta}_n^\theta(dx)\Big|\boldsymbol{Y}\right)\right\| \leq \frac{\tilde{C}_{1,Q}}{N} \tag{106}$$

almost surely. Similarly, using [10, Theorem 3.1], [20, Theorem 5.8] (or [33, Proposition 6.4]), we deduce that there exists a real number $\tilde{C}_{2,Q} \in [1,\infty)$ (independent of N and depending only on $p_\theta(x'|x)$, $q_\theta(y|x)$) such that

$$E\left(\left|\int \varphi(x)\tilde{\xi}_n^\theta(dx)\right|^2\Big|\boldsymbol{Y}\right) \leq \frac{\tilde{C}_{2,Q}}{N}, \tag{107}$$

$$E\left(\left\|\int \varphi(x)\tilde{\zeta}_n^\theta(dx)\right\|^2\Big|\boldsymbol{Y}\right) \leq \frac{\tilde{C}_{2,Q}}{N} \tag{108}$$

almost surely.

It is straightforward to verify

$$H_{\theta,y}(\xi,\zeta) = H'_{\theta,y}(\xi,\zeta) + H''_{\theta,y}(\xi,\zeta) \tag{109}$$

($H_{\theta,y}(\xi,\zeta)$ is defined in (26)). It is also easy to show

$$\frac{1}{A_{1,n}^\theta} = \frac{1}{A_{3,n}^\theta} - \frac{A_{2,n}^\theta}{|A_{3,n}^\theta|^2} + \frac{1}{A_{1,n}^\theta}\left|\frac{A_{2,n}^\theta}{A_{3,n}^\theta}\right|^2. \tag{110}$$

Relying on (109), we conclude

$$\begin{aligned}
H(\theta, Z_n^\theta) &= \frac{\sum_{i=1}^N q_\theta(Y_n|\hat{X}_{n,i}^\theta)\left(W_{n,i}^\theta - N^{-1}\sum_{j=1}^N W_{n,j}^\theta\right)}{\sum_{i=1}^N q_\theta(Y_n|\hat{X}_{n,i}^\theta)} \\
&\quad + \frac{\sum_{i=1}^N \nabla_\theta q_\theta(Y_n|\hat{X}_{n,i}^\theta)}{\sum_{i=1}^N q_\theta(Y_n|\hat{X}_{n,i}^\theta)} \\
&= \frac{\int q_\theta(Y_n|x)\zeta_n^\theta(dx) + \int \nabla_\theta q_\theta(Y_n|x)\xi_n^\theta(dx)}{\int q_\theta(Y_n|x)\xi_n^\theta(dx)} \\
&= H_{\theta,Y_n}(\xi_n^\theta, \zeta_n^\theta). \tag{111}
\end{aligned}$$

Using (110), we also deduce

$$\begin{aligned}
&H'_{\theta,Y_n}(\xi_n^\theta, \zeta_n^\theta) - H'_{\theta,Y_n}(F_{\theta,\boldsymbol{Y}}^{0:n}(\xi_0^\theta), G_{\theta,\boldsymbol{Y}}^{0:n}(\xi_0^\theta, \zeta_0^\theta)) \\
&= \frac{B_{2,n}^\theta}{A_{3,n}^\theta} - \frac{A_{2,n}^\theta B_{2,n}^\theta}{|A_{3,n}^\theta|^2} - \frac{A_{2,n}^\theta B_{3,n}^\theta}{|A_{3,n}^\theta|^2} + \frac{B_{1,n}^\theta}{A_{1,n}^\theta}\left|\frac{A_{2,n}^\theta}{A_{3,n}^\theta}\right|^2, \tag{112}
\end{aligned}$$

$$\begin{aligned}
&H''_{\theta,Y_n}(\xi_n^\theta, \zeta_n^\theta) - H''_{\theta,Y_n}(F_{\theta,\boldsymbol{Y}}^{0:n}(\xi_0^\theta), G_{\theta,\boldsymbol{Y}}^{0:n}(\xi_0^\theta, \zeta_0^\theta)) \\
&= \frac{C_{2,n}^\theta}{A_{3,n}^\theta} - \frac{A_{2,n}^\theta C_{2,n}^\theta}{|A_{3,n}^\theta|^2} - \frac{A_{2,n}^\theta C_{3,n}^\theta}{|A_{3,n}^\theta|^2} + \frac{C_{1,n}^\theta}{A_{1,n}^\theta}\left|\frac{A_{2,n}^\theta}{A_{3,n}^\theta}\right|^2. \tag{113}
\end{aligned}$$

Moreover, due to Assumption 2.3, we have

$$A_{1,n}^\theta = \int q_\theta(Y_n|x)\xi_n^\theta(dx) \geq \varepsilon_Q, \tag{114}$$

$$A_{3,n}^\theta = \int q_\theta(Y_n|x)F_{\theta,\boldsymbol{Y}}^{0:n}(dx|\xi_0^\theta) \geq \varepsilon_Q. \tag{115}$$

Similarly, owing to Assumption 2.4, we have

$$\|B_{1,n}^\theta\| \le \int \|\nabla_\theta q_\theta(Y_n|x)\|\xi_n^\theta(dx) \le K_{1,Q}, \tag{116}$$

$$\|B_{3,n}^\theta\| \le \int \|\nabla_\theta q_\theta(Y_n|x)\|F_{\theta,\boldsymbol{Y}}^{0:n}(dx|\xi_0^\theta) \le K_{1,Q}. \tag{117}$$

Since $\|\zeta_n^\theta\| \le \tilde{C}_{1,Q}$, Assumption 2.3 and Lemma 4.1 yield

$$\|C_{1,n}^\theta\| \le \int q_\theta(Y_n|x)|\zeta_n^\theta|(dx) \le \frac{\tilde{C}_{1,Q}}{\varepsilon_Q}, \tag{118}$$

$$\|C_{3,n}^\theta\| \le \int q_\theta(Y_n|x)|G_{\theta,\boldsymbol{Y}}^{0:n}|(dx|\xi_0^\theta,\zeta_0^\theta) \le \frac{C_{1,Q}}{\varepsilon_Q}. \tag{119}$$

Let $\tilde{C}_{3,Q} = \varepsilon_Q^{-4}\sqrt{d}\tilde{C}_{1,Q}C_{1,Q}K_{1,Q}$. Due to Assumptions 2.3, 2.4, we have

$$0 \le \varepsilon_Q q_\theta(Y_n|x) \le 1, \qquad \left\|\frac{\nabla_\theta q_\theta(Y_n|x)}{K_{1,Q}}\right\| \le 1.$$

Then, using (105), (106), we conclude

$$\left|E\left(A_{2,n}^\theta\,\middle|\,\boldsymbol{Y}\right)\right| = \left|E\left(\int q_\theta(Y_n|x)\tilde{\xi}_n^\theta(dx)\,\middle|\,\boldsymbol{Y}\right)\right|$$
$$\le \frac{\tilde{C}_{1,Q}}{\varepsilon_Q N}, \tag{120}$$

$$\left\|E\left(B_{2,n}^\theta\,\middle|\,\boldsymbol{Y}\right)\right\| = \left\|E\left(\int \nabla_\theta q_\theta(Y_n|x)\tilde{\xi}_n^\theta(dx)\,\middle|\,\boldsymbol{Y}\right)\right\|$$
$$\le \frac{\sqrt{d}\tilde{C}_{1,Q}K_{1,Q}}{N}, \tag{121}$$

$$\left\|E\left(C_{2,n}^\theta\,\middle|\,\boldsymbol{Y}\right)\right\| = \left\|E\left(\int q_\theta(Y_n|x)\tilde{\zeta}_n^\theta(dx)\,\middle|\,\boldsymbol{Y}\right)\right\|$$
$$\le \frac{\tilde{C}_{1,Q}}{\varepsilon_Q N} \tag{122}$$

almost surely. As $A_{3,n}^\theta$ is measurable with respect to $\boldsymbol{Y}$, (115), (121), (122) imply

$$\left\|E\left(\frac{B_{2,n}^\theta}{A_{3,n}^\theta}\right)\right\| \le E\left(\frac{\|E\left(B_{2,n}^\theta\,|\,\boldsymbol{Y}\right)\|}{A_{3,n}^\theta}\right)$$
$$\le \frac{\sqrt{d}\tilde{C}_{1,Q}K_{1,Q}}{\varepsilon_Q N} \le \frac{\tilde{C}_{3,Q}}{N}, \tag{123}$$

$$\left\|E\left(\frac{C_{2,n}^\theta}{A_{3,n}^\theta}\right)\right\| \le E\left(\frac{\|E\left(C_{2,n}^\theta\,|\,\boldsymbol{Y}\right)\|}{A_{3,n}^\theta}\right)$$
$$\le \frac{\tilde{C}_{1,Q}}{\varepsilon_Q^2 N} \le \frac{\tilde{C}_{3,Q}}{N}. \tag{124}$$

Since $A_{3,n}^\theta$, $B_{3,n}^\theta$, $C_{3,n}^\theta$ are measurable with respect to $\boldsymbol{Y}$, (115), (117), (119), (120) yield

$$\left\|E\left(\frac{A_{2,n}^\theta B_{3,n}^\theta}{|A_{3,n}^\theta|^2}\right)\right\| \le E\left(\frac{|E\left(A_{2,n}^\theta\,|\,\boldsymbol{Y}\right)|\,\|B_{3,n}^\theta\|}{|A_{3,n}^\theta|^2}\right)$$
$$\le \frac{\tilde{C}_{1,Q}K_{1,Q}}{\varepsilon_Q^3 N} \le \frac{\tilde{C}_{3,Q}}{N}, \tag{125}$$

$$\left\|E\left(\frac{A_{2,n}^\theta C_{3,n}^\theta}{|A_{3,n}^\theta|^2}\right)\right\| \le E\left(\frac{|E\left(A_{2,n}^\theta\,|\,\boldsymbol{Y}\right)|\,\|C_{3,n}^\theta\|}{|A_{3,n}^\theta|^2}\right)$$
$$\le \frac{\tilde{C}_{1,Q}C_{1,Q}}{\varepsilon_Q^4 N} \le \frac{\tilde{C}_{3,Q}}{N}. \tag{126}$$

Let $\tilde{C}_{4,Q} = \sqrt{d}\varepsilon_Q^{-4}\tilde{C}_{2,Q}K_{1,Q}$, $\tilde{C}_{5,Q} = \varepsilon_Q^{-2}\tilde{C}_{1,Q}\tilde{C}_{4,Q}K_{1,Q}$. Relying on Assumptions 2.3, 2.4 and (107), (108), we deduce

$$E\left(|A_{2,n}^\theta|^2\,\middle|\,\boldsymbol{Y}\right) = E\left(\left|\int q_\theta(Y_n|x)\tilde{\xi}_n^\theta(dx)\right|^2\,\middle|\,\boldsymbol{Y}\right)$$
$$\le \frac{\tilde{C}_{2,Q}}{\varepsilon_Q^2 N}, \tag{127}$$

$$E\left(\|B_{2,n}^\theta\|^2\,\middle|\,\boldsymbol{Y}\right) = E\left(\left\|\int \nabla_\theta q_\theta(Y_n|x)\tilde{\xi}_n^\theta(dx)\right\|^2\,\middle|\,\boldsymbol{Y}\right)$$
$$\le \frac{d\tilde{C}_{2,Q}K_{1,Q}^2}{N}, \tag{128}$$

$$E\left(\|C_{2,n}^\theta\|^2\,\middle|\,\boldsymbol{Y}\right) = E\left(\left\|\int q_\theta(Y_n|x)\tilde{\zeta}_n^\theta(dx)\right\|^2\,\middle|\,\boldsymbol{Y}\right)$$
$$\le \frac{\tilde{C}_{2,Q}}{\varepsilon_Q^2 N} \tag{129}$$

almost surely. Then, Hölder inequality and (115), (127) – (129) imply

$$\left\|E\left(\frac{A_{2,n}^\theta B_{2,n}^\theta}{|A_{3,n}^\theta|^2}\right)\right\| \le \left(E\left(\frac{E\left(|A_{2,n}^\theta|^2\,|\,\boldsymbol{Y}\right)}{|A_{3,n}^\theta|^2}\right)\right)^{1/2}$$
$$\cdot\left(E\left(\frac{E\left(\|B_{2,n}^\theta\|^2\,|\,\boldsymbol{Y}\right)}{|A_{3,n}^\theta|^2}\right)\right)^{1/2}$$
$$\le \frac{\sqrt{d}\tilde{C}_{2,Q}K_{1,Q}}{\varepsilon_Q^3 N} \le \frac{\tilde{C}_{4,Q}}{N}, \tag{130}$$

$$\left\|E\left(\frac{A_{2,n}^\theta C_{2,n}^\theta}{|A_{3,n}^\theta|^2}\right)\right\| \le \left(E\left(\frac{E\left(|A_{2,n}^\theta|^2\,|\,\boldsymbol{Y}\right)}{|A_{3,n}^\theta|^2}\right)\right)^{1/2}$$
$$\cdot\left(E\left(\frac{E\left(\|C_{2,n}^\theta\|^2\,|\,\boldsymbol{Y}\right)}{|A_{3,n}^\theta|^2}\right)\right)^{1/2}$$
$$\le \frac{\tilde{C}_{2,Q}}{\varepsilon_Q^4 N} \le \frac{\tilde{C}_{4,Q}}{N}. \tag{131}$$

Moreover, due to (115), (127), we have

$$E\left(\left|\frac{A_{2,n}^\theta}{A_{3,n}^\theta}\right|^2\right) = E\left(\frac{E\left(|A_{2,n}^\theta|^2\,|\,\boldsymbol{Y}\right)}{|A_{3,n}^\theta|^2}\right) \le \frac{\tilde{C}_{2,Q}}{\varepsilon_Q^4 N} \le \frac{\tilde{C}_{4,Q}}{N}. \tag{132}$$

Owing to (114), (116), (118), (132), we also have

$$\left\|E\left(\frac{B_{1,n}^\theta}{A_{1,n}^\theta}\left|\frac{A_{2,n}^\theta}{A_{3,n}^\theta}\right|^2\right)\right\| \le E\left(\frac{\|B_{1,n}^\theta\|}{A_{1,n}^\theta}\left|\frac{A_{2,n}^\theta}{A_{3,n}^\theta}\right|^2\right)$$
$$\le \frac{K_{1,Q}}{\varepsilon_Q}E\left(\left|\frac{A_{2,n}^\theta}{A_{3,n}^\theta}\right|^2\right) \le \frac{\tilde{C}_{4,Q}K_{1,Q}}{\varepsilon_Q N} \le \frac{\tilde{C}_{5,Q}}{N}, \tag{133}$$

$$\left\|E\left(\frac{C_{1,n}^\theta}{A_{1,n}^\theta}\left|\frac{A_{2,n}^\theta}{A_{3,n}^\theta}\right|^2\right)\right\| \le E\left(\frac{\|C_{1,n}^\theta\|}{A_{1,n}^\theta}\left|\frac{A_{2,n}^\theta}{A_{3,n}^\theta}\right|^2\right)$$
$$\le \frac{\tilde{C}_{1,Q}}{\varepsilon_Q^2}E\left(\left|\frac{A_{2,n}^\theta}{A_{3,n}^\theta}\right|^2\right) \le \frac{\tilde{C}_{1,Q}\tilde{C}_{4,Q}}{\varepsilon_Q^2 N} \le \frac{\tilde{C}_{5,Q}}{N}. \tag{134}$$

Let $M_Q = 4(\tilde{C}_{3,Q} + \tilde{C}_{4,Q} + \tilde{C}_{5,Q})$. Then, (112), (123), (125), (130), (133) imply

$$\left\| E\left(H'_{\theta,Y_n}(\xi_n^\theta, \zeta_n^\theta) - H'_{\theta,Y_n}(F_{\theta,\mathbf{Y}}^{0:n}(\xi_0^\theta), G_{\theta,\mathbf{Y}}^{0:n}(\xi_0^\theta, \zeta_0^\theta)) \right)\right\|$$
$$\leq \frac{2\tilde{C}_{3,Q} + \tilde{C}_{4,Q} + \tilde{C}_{5,Q}}{N} \leq \frac{M_Q}{2N}.$$

Similarly, (113), (124), (126), (131), (134) yield

$$\left\| E\left(H''_{\theta,Y_n}(\xi_n^\theta, \zeta_n^\theta) - H''_{\theta,Y_n}(F_{\theta,\mathbf{Y}}^{0:n}(\xi_0^\theta), G_{\theta,\mathbf{Y}}^{0:n}(\xi_0^\theta, \zeta_0^\theta)) \right)\right\|$$
$$\leq \frac{2\tilde{C}_{3,Q} + \tilde{C}_{4,Q} + \tilde{C}_{5,Q}}{N} \leq \frac{M_Q}{2N}.$$

Combining this with (109), (111), we get

$$\left\| E\left(H(\theta, Z_n^\theta) - H_{\theta,Y_n}(F_{\theta,\mathbf{Y}}^{0:n}(\xi_0^\theta), G_{\theta,\mathbf{Y}}^{0:n}(\xi_0^\theta, \zeta_0^\theta)) \right)\right\|$$
$$\leq \left\| E\left(H'_{\theta,Y_n}(\xi_n^\theta, \zeta_n^\theta) - H'_{\theta,Y_n}(F_{\theta,\mathbf{Y}}^{0:n}(\xi_0^\theta), G_{\theta,\mathbf{Y}}^{0:n}(\xi_0^\theta, \zeta_0^\theta)) \right)\right\|$$
$$+ \left\| E\left(H''_{\theta,Y_n}(\xi_n^\theta, \zeta_n^\theta) - H''_{\theta,Y_n}(F_{\theta,\mathbf{Y}}^{0:n}(\xi_0^\theta), G_{\theta,\mathbf{Y}}^{0:n}(\xi_0^\theta, \zeta_0^\theta)) \right)\right\|$$
$$\leq \frac{M_Q}{N}.$$

Hence, we have

$$\|h(\theta) - \nabla l(\theta)\|$$
$$\leq \left\| E\left(H(\theta, Z_n^\theta) \right) - h(\theta)\right\|$$
$$+ \left\| E\left(H_{\theta,Y_n}(F_{\theta,\mathbf{Y}}^{0:n}(\xi_0^\theta), G_{\theta,\mathbf{Y}}^{0:n}(\xi_0^\theta, \zeta_0^\theta)) \right) - \nabla l(\theta)\right\|$$
$$+ \left\| E\left(H(\theta, Z_n^\theta) - H_{\theta,Y_n}(F_{\theta,\mathbf{Y}}^{0:n}(\xi_0^\theta), G_{\theta,\mathbf{Y}}^{0:n}(\xi_0^\theta, \zeta_0^\theta)) \right)\right\|$$
$$\leq \left\| E\left(H(\theta, Z_n^\theta) \right) - h(\theta)\right\|$$
$$+ \left\| E\left(H_{\theta,Y_n}(F_{\theta,\mathbf{Y}}^{0:n}(\xi_0^\theta), G_{\theta,\mathbf{Y}}^{0:n}(\xi_0^\theta, \zeta_0^\theta)) \right) - \nabla l(\theta)\right\| + \frac{M_Q}{N}.$$

Then, letting $n \to \infty$ and using Lemmas 4.1, 5.4, we conclude that (102) holds. □

VI. PROOF OF MAIN RESULTS

In this section, we study the Monte Carlo estimation of the log-likelihood rate gradient $\nabla l(\theta)$ (see Lemma 6.1). We also study the analytical properties of $l(\theta)$ (see Lemma 6.2). Using these results (together with the results of [30]–[31]), we prove Theorem 2.1.

Throughout the section, the following notation is used. $\{\zeta_n\}_{n\geq 0}$, $\{\eta_n\}_{n\geq 0}$ and $\{\xi_n\}_{n\geq 0}$ are the stochastic processes defined by

$$\zeta_n = H(\theta_n, Z_{n+1}) - h(\theta_n),$$
$$\eta_n = h(\theta_n) - \nabla l(\theta_n),$$
$$\xi_n = \zeta_n + \eta_n$$

for $n \geq 0$ ($H(\theta, z)$, $h(\theta)$, $\{Z_n\}_{n\geq 0}$ are specified in (35), (17), (50)). Then, using (15), it is easy to show that (21) holds for each $n \geq 0$.

Remark: Due to (21), algorithm (1) – (4) is stochastic gradient search which maximizes log-likelihood rate $l(\theta)$, while $\{\xi_n\}_{n\geq 0}$ can be interpreted as noise in the (Monte Carlo) estimation of $\nabla l(\theta)$. We also recall here that Q is any compact set satisfying $Q \subset \Theta$.

Lemma 6.1: Let Assumptions 2.1 – 2.4 hold. Then, relations

$$\lim_{n\to\infty}\sup_{k\geq n}\left\|\sum_{i=n}^{k}\alpha_i\zeta_i\right\| = 0, \qquad \limsup_{n\to\infty}\|\eta_n\| \leq \frac{M_Q}{N} \quad (135)$$

hold almost surely on Λ_Q (Λ_Q is defined in (6)).

Proof: Let τ_Q be the stopping time defined in Lemma 5.5. Moreover, let $\tilde{\Lambda}_Q$ be the event defined by $\tilde{\Lambda}_Q = \bigcap_{n=0}^{\infty}\{\theta_n \in Q\}$. Hence, on Λ_Q, $\theta_n \in Q$ for all, but finitely many $n \geq 0$. Then, using Lemma 5.6, we conclude that the second part of (135) holds almost surely on Λ_Q.

Throughout the rest of the proof, the following notation is used. $\mathcal{F}_k$ is the σ-algebra defined by $\mathcal{F}_k = \sigma\{\theta_0, Z_0, \cdots, \theta_k, Z_k\}$ for $k \geq 0$. n is any positive integer, while $\zeta_{1,n}$, $\zeta_{2,n}$ and $\zeta_{3,n}$ are the random variables defined by

$$\zeta_{1,n} = \tilde{H}(\theta_n, Z_{n+1}) - (\Pi\tilde{H})(\theta_n, Z_n),$$
$$\zeta_{2,n} = (\Pi\tilde{H})(\theta_n, Z_n) - (\Pi\tilde{H})(\theta_{n-1}, Z_n),$$
$$\zeta_{3,n} = -(\Pi\tilde{H})(\theta_n, Z_{n+1}).$$

Then, for $l \geq k > 1$, it is straightforward to verify

$$\sum_{i=k}^{l}\alpha_i\zeta_i = \sum_{i=k}^{l}\alpha_i\zeta_{1,i} + \sum_{i=k}^{l}\alpha_i\zeta_{2,i} + \sum_{i=k}^{l}(\alpha_i - \alpha_{i+1})\zeta_{3,i}$$
$$+ \alpha_{l+1}\zeta_{3,l} - \alpha_k\zeta_{3,k-1}. \quad (136)$$

As a direct consequence of Lemmas 5.4, 5.5, we have

$$\|\zeta_{1,n}\|I_{\{\tau_Q>n\}} \leq C_{5,Q}(2 + \|W_n\Lambda\| + \|W_{n+1}\Lambda\|)I_{\{\tau_Q>n\}}$$
$$\leq 4C_{5,Q}C_{6,Q}(1 + \|W_0\Lambda\|).$$

Since W_0 is measurable with respect to $\mathcal{F}_0$, Assumption 2.1 yields

$$E\left(\sum_{n=1}^{\infty}\alpha_n^2\|\zeta_{1,n}\|^2 I_{\{\tau_Q>n\}} \middle| \mathcal{F}_0 \right)$$
$$\leq 16C_{5,Q}^2C_{6,Q}^2(1 + \|W_0\Lambda\|)^2\left(\sum_{n=0}^{\infty}\alpha_n^2\right) < \infty \quad (137)$$

almost surely. As $\{\tau_Q > n\} \in \mathcal{F}_n$, we also have

$$E\left(\zeta_{1,n}I_{\{\tau_Q>n\}} \middle| \mathcal{F}_n \right)$$
$$= \left(E\left(\tilde{H}(\theta_n, Z_{n+1}) \middle| \mathcal{F}_n \right) - (\Pi\tilde{H})(\theta_n, Z_n) \right) I_{\{\tau_Q>n\}} = 0$$

almost surely. Then, Doob theorem and (137) imply that $\sum_{n=1}^{\infty}\alpha_n\zeta_{1,n}I_{\{\tau_Q>n\}}$ is almost surely convergent. Since $\tilde{\Lambda}_Q \subseteq \{\tau_Q > n\}$, $\sum_{n=1}^{\infty}\alpha_n\zeta_{1,n}$ converges almost surely on $\tilde{\Lambda}_Q$.

Due to Lemmas 5.4, 5.5 and (15), we have

$$\|\zeta_{2,n}\|I_{\{\tau_Q>n\}} \leq C_{5,Q}\|\theta_n - \theta_{n-1}\|(1 + \|W_n\Lambda\|)I_{\{\tau_Q>n\}}$$
$$= C_{5,Q}\alpha_{n-1}\|H(\theta_{n-1}, Z_n)\|$$
$$\cdot (1 + \|W_n\Lambda\|)I_{\{\tau_Q>n\}}$$
$$\leq C_{5,Q}^2\alpha_{n-1}(1 + \|W_n\Lambda\|)^2 I_{\{\tau_Q>n\}}$$
$$\leq 4C_{5,Q}^2C_{6,Q}^2\alpha_{n-1}(1 + \|W_0\Lambda\|)^2.$$

Combining this with Assumption 2.1, we get

$$\sum_{n=1}^{\infty}\alpha_n\|\zeta_{2,n}\|I_{\{\tau_Q>n\}}$$
$$\leq 4C_{5,Q}^2C_{6,Q}^2(1 + \|W_0\Lambda\|)^2\left(\sum_{n=0}^{\infty}\alpha_n\alpha_{n+1}\right)$$
$$\leq 2C_{5,Q}^2C_{6,Q}^2(1 + \|W_0\Lambda\|)^2\left(\sum_{n=0}^{\infty}\alpha_n^2\right) < \infty.$$

Hence, $\sum_{n=0}^{\infty} \alpha_n \zeta_{2,n} I_{\{\tau_Q > n\}}$ converges almost surely. Therefore, $\sum_{n=0}^{\infty} \alpha_n \zeta_{2,n}$ is convergent almost surely on $\tilde{\Lambda}_Q$.

As a direct consequence of Lemmas 5.4, 5.5, we have

$$\|\zeta_{3,n}\| I_{\{\tau_Q > n\}} \leq C_{5,Q}(1 + \|W_{n+1}\Lambda\|) I_{\{\tau_Q > n\}}$$
$$\leq 2C_{5,Q} C_{6,Q}(1 + \|W_0 \Lambda\|).$$

Consequently, Assumption 2.1 yields

$$\sum_{n=1}^{\infty} \alpha_{n+1}^2 \|\zeta_{3,n}\|^2 I_{\{\tau_Q > n\}}$$
$$\leq 4C_{5,Q}^2 C_{6,Q}^2 (1 + \|W_0 \Lambda\|)^2 \left(\sum_{n=0}^{\infty} \alpha_n^2\right) < \infty, \qquad (138)$$

$$\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| \|\zeta_{3,n}\| I_{\{\tau_Q > n\}}$$
$$\leq 2C_{5,Q} C_{6,Q}(1 + \|W_0 \Lambda\|) \left(\sum_{n=0}^{\infty} |\alpha_n - \alpha_{n+1}|\right) < \infty. \qquad (139)$$

Therefore, we have

$$\lim_{n \to \infty} \alpha_{n+1} \zeta_{3,n} I_{\{\tau_Q > n\}} = 0$$

almost surely. Hence, $\lim_{n \to \infty} \alpha_{n+1} \zeta_{3,n} = 0$ almost surely on $\tilde{\Lambda}_Q$. Moreover, due to (139),

$$\sum_{n=1}^{\infty} (\alpha_n - \alpha_{n+1}) \zeta_{3,n} I_{\{\tau_Q > n\}}$$

is almost surely convergent. Thus, $\sum_{n=1}^{\infty} (\alpha_n - \alpha_{n+1}) \zeta_{3,n}$ converges almost surely on $\tilde{\Lambda}_Q$. Since $\sum_{n=1}^{\infty} \alpha_n \zeta_{1,n}$, $\sum_{n=1}^{\infty} \alpha_n \zeta_{2,n}$ are almost surely convergent on $\tilde{\Lambda}_Q$, (136) implies that $\sum_{n=0}^{\infty} \alpha_n \zeta_n$ converges almost surely on $\tilde{\Lambda}_Q$, too. As Q is any compact set in Θ, we conclude that $\sum_{n=0}^{\infty} \alpha_n \zeta_n$ is almost surely convergent on $\{\sup_{n \geq 0} \|\theta_n\| < \infty, \inf_{n \geq 0} d(\theta_n, \Theta^c) > 0\}$. Consequently, the first part of (137) holds almost surely on Λ_Q. $\qquad \square$

Lemma 6.2: Let Assumptions 2.2 – 2.4 hold. Then, the following is true:

(i) $l(\theta)$ is well-defined for each $\theta \in \Theta$. Moreover, $l(\theta)$ is Lipschitz continuously differentiable on Θ.

(ii) If Assumption 2.5 also holds, then $l(\theta)$ is p-times differentiable on Θ.

(iii) If Assumption 2.6 also holds, then $l(\theta)$ is real-analytic on Θ.

Proof: (i) See Lemma 4.1. (ii) See [32, Theorem 3.1]. (iii) See [31, Theorem 2.1]. $\qquad \square$

Proof of Theorem 2.1: Let $\eta = \limsup_{n \to \infty} \|\eta_n\|$. Then, Lemma 6.1 yields $\eta \leq M_Q/N$ almost surely on Λ_Q. Moreover, due to Assumption 2.1 and Lemmas 6.1, 6.2, Algorithm (21) satisfies all conditions which [30, Theorem 2.1] is based on. Consequently, [30, Theorem 2.1] implies that there exist a function $\psi_Q(t)$ and real numbers r_Q, $L_{1,Q}$, $L_{2,Q}$ with the properties specified in the statement Theorem 2.1. $\qquad \square$

APPENDIX A

In this section, we present results on stochastic matrices which are needed for the proof of Lemmas 5.3 and 5.5. Here, we rely on the following notation. $\|\cdot\|$ denotes the Euclidean vector norm and Frobenius matrix norm, while $\|\cdot\|_1$ stands for the l_1 vector norm. $N \geq 1$ is an integer. $\mathcal{P}^N$ is the set of N-dimensional (column) probability vectors, while $\mathcal{P}^{N \times N}$ is the set of $N \times N$ (column) stochastic matrices (i.e., $A \in \mathcal{P}^{N \times N}$ if and only if the columns of A are elements of $\mathcal{P}^N$). e is the N-dimensional vector whose all elements are one. For $1 \leq i \leq N$, e_i is the i-th standard unit vector in $\mathbb{R}^N$ (i.e., e_i is the element of $\mathcal{P}^N$ whose i-th element is one). I is the $N \times N$ unit matrix. Λ is the matrix defined by $\Lambda = I - ee^T/N$. For $A \in \mathcal{P}^{N \times N}$, $\tau(A)$ is the (Dobrushin) ergodicity coefficient, i.e.,

$$\tau(A) = \frac{1}{2} \max_{1 \leq j, j' \leq N} \sum_{i=1}^{N} |A_{i,j} - A_{i,j'}|,$$

where $A_{i,j}$ is the (i,j) entry of A.

Lemma A1.1: (i) If $A \in \mathcal{P}^{N \times N}$, then we have

$$\tau(A) = 1 - \min_{1 \leq j, j' \leq N} \sum_{i=1}^{N} \min\{A_{i,j}, A_{i,j'}\},$$

where $A_{i,j}$ is the (i,j) entry of A.

(ii) If $A \in \mathcal{P}^{N \times N}$ and $z, z' \in \mathcal{P}^N$, then we have

$$\|A(z - z')\|_1 \leq \tau(A) \|z - z'\|_1.$$

Moreover, if $A, A' \in \mathcal{P}^{N \times N}$, then $\tau(AA') \leq \tau(A)\tau(A')$.

Proof: (i) See [7, Definition 15.2.1, Equation (15.9)]. (ii) See [7, Theorems 15.2.4, 15.2.5]. $\qquad \square$

Lemma A1.2: Let $\{A_n\}_{n \geq 1}$, $\{B_n\}_{n \geq 1}$ and $\{C_n\}_{n \geq 1}$ be sequences in $\mathcal{P}^{N \times N}$. Moreover, let $a, b, c \in \mathbb{R}^N$. Assume the following:

(i) There exists a real number $\alpha \in (0,1)$ such that $\min\{A_{n,i,j}, B_{n,i,j}, C_{n,i,j}\} \geq \alpha/N$ for each $1 \leq i, j \leq N$, $n \geq 1$, where $A_{n,i,j}$, $B_{n,i,j}$, $C_{n,i,j}$ are the (i,j) entries of A_n, B_n, C_n (respectively).

(ii) $e^T a = e^T b = e^T c = 0$.

Then, we have

$$\|A_1 \cdots A_n \Lambda\| \leq K\beta^n,$$
$$\|A_1 \cdots A_n a\| \leq K\beta^n \|a\|,$$
$$\|B_1 \cdots B_n b - C_1 \cdots C_n c\|$$
$$\leq K\beta^n (\|b\| + \|c\|) \sum_{i=1}^{n} \|B_i - C_i\| + K\beta^n \|b - c\|$$

for each $n \geq 1$, where $\beta = 1 - \alpha$ and $K = 4\beta^{-1}N$.

Proof: Throughout the proof, the following notation is used. n, k, l are any integers satisfying $n \geq 1$, $l \geq k \geq 0$. $\tilde{A}_{k,k}$, $\tilde{B}_{k,k}$, $\tilde{C}_{k,k}$ and $\tilde{A}_{k,m}$, $\tilde{B}_{k,m}$, $\tilde{C}_{k,m}$ are the matrices defined by $\tilde{A}_{k,k} = \tilde{B}_{k,k} = \tilde{C}_{k,k} = I$ and

$$\tilde{A}_{k,m} = A_{k+1} \cdots A_m,$$
$$\tilde{B}_{k,m} = B_{k+1} \cdots B_m,$$
$$\tilde{C}_{k,m} = C_{k+m} \cdots C_m$$

for $m > k \geq 0$. Then, using Lemma A1.1, we conclude

$$\tau\big(\tilde{A}_{k,m}\big) \leq \tau(A_{k+1}) \cdots \tau(A_m) \leq \beta^{m-k}.$$

Relying on the same lemma, we deduce

$$\tau(A_n) = 1 - \min_{1 \leq j,j' \leq N} \sum_{i=1}^{N} \min\{A_{n,i,j}, A_{n,i,j'}\} \leq 1 - \alpha = \beta.$$

Noticing $e_i, e/N \in \mathcal{P}^N$ and applying Lemma A1.1 again, we get

$$\left\|\tilde{A}_{k,l}\left(e_i - \frac{e}{N}\right)\right\|_1 \leq \tau(\tilde{A}_{k,l}) \left\|e_i - \frac{e}{N}\right\|_1 \leq 2\beta^{l-k} \quad (140)$$

for $1 \leq i \leq N$. Since $\tilde{A}_{k,l}\left(e_i - \frac{e}{N}\right)$ is the i-th column of $\tilde{A}_{k,l}\Lambda$, (140) yields

$$\left\|\tilde{A}_{k,l}\Lambda\right\| \leq N^{1/2} \max_{1 \leq i \leq N} \left\|\tilde{A}_{k,l}\left(e_i - \frac{e}{N}\right)\right\|_1 \leq 2N^{1/2}\beta^{l-k}. \quad (141)$$

Hence, we get

$$\left\|A_1 \cdots A_n \Lambda\right\| = \left\|\tilde{A}_{0,n}\Lambda\right\| \leq 2N^{1/2}\beta^n \leq K\beta^n.$$

Moreover, we have

$$\tilde{A}_{k,l}\Lambda a = \tilde{A}_{k,l}a - \frac{\tilde{A}_{k,l}e}{N}e^T a = A_{k,l}a.$$

Consequently, (141) implies

$$\left\|\tilde{A}_{k,l}a\right\| = \left\|\tilde{A}_{k,l}\Lambda a\right\| \leq \left\|\tilde{A}_{k,l}\Lambda\right\|\|a\| \leq 2N^{1/2}\beta^{l-k}\|a\|. \quad (142)$$

Thus, we get

$$\left\|A_1 \cdots A_n a\right\| = \left\|\tilde{A}_{0,n}a\right\| \leq 2N^{1/2}\beta^n\|a\| \leq K\beta^n\|a\|.$$

Since $e^T B_n = e^T C_n = e^T$, we have

$$\Lambda(B_n - C_n) = B_n - C_n - \frac{e}{N}(e^T B_n - e^T C_n) = B_n - C_n.$$

Therefore, we get

$$\tilde{B}_{0,n}b - \tilde{C}_{0,n}c = \sum_{i=1}^{n} \tilde{B}_{0,i-1}(B_i - C_i)\tilde{C}_{i,n}b + \tilde{C}_{0,n}(b - c)$$
$$= \sum_{i=1}^{n} \tilde{B}_{0,i-1}\Lambda(B_i - C_i)\tilde{C}_{i,n}b + \tilde{C}_{0,n}(b - c).$$

Then, applying (141), (142) to $\{B_n\}_{n \geq 1}$, $\{C_n\}_{n \geq 1}$, b, c, we get

$$\left\|\tilde{B}_{0,n}b - \tilde{C}_{0,n}c\right\| \leq \sum_{i=1}^{n} \left\|\tilde{B}_{0,i-1}\Lambda\right\|\|B_i - C_i\|\left\|\tilde{C}_{i,n}b\right\|$$
$$+ \left\|\tilde{C}_{0,n}(b - c)\right\|$$
$$\leq 4N\beta^{n-1}\|b\| \sum_{i=1}^{n} \|B_i - C_i\|$$
$$+ 2N^{1/2}\beta^n\|b - c\|$$
$$\leq K\beta^n(\|b\| + \|c\|) \sum_{i=1}^{n} \|B_i - C_i\|$$
$$+ K\beta^n\|b - c\|.$$

Hence, we have

$$\|B_1 \cdots B_n b - C_1 \cdots C_n c\| = \left\|\tilde{B}_{0,n}b - \tilde{C}_{0,n}c\right\|$$
$$\leq K\beta^n(\|b\| + \|c\|) \sum_{i=1}^{n} \|B_i - C_i\|$$
$$+ K\beta^n\|b - c\|.$$

$\square$

APPENDIX B

In this section, we prove Lemma 4.1. We rely on the following notation. $\tilde{r}_\theta(y, x'|x)$ is the function defined for $\theta \in \Theta$, $x, x' \in \mathcal{X}$, $y \in \mathcal{Y}$ by

$$\tilde{r}_\theta(y, x'|x) = q_\theta(y|x')p_\theta(x'|x). \quad (143)$$

$\tilde{h}_{\theta,y}(x|\xi, \zeta)$ and $\tilde{H}_{\theta,y}(\xi, \zeta)$ are the functions defined for $\xi \in \mathcal{P}(\mathcal{X})$, $\zeta \in \mathcal{M}_s^d(\mathcal{X})$ by

$$\tilde{h}_{\theta,y}(x|\xi, \zeta) = \frac{\int \tilde{r}_\theta(y, x|x')\zeta(dx') + \int \nabla_\theta \tilde{r}_\theta(y, x|x')\xi(dx')}{\iint \tilde{r}_\theta(y, x''|x')\mu(dx'')\xi(dx')}, \quad (144)$$

$$\tilde{H}_{\theta,y}(\xi, \zeta) = \int \tilde{h}_{\theta,y}(x|\xi, \zeta)\mu(dx), \quad (145)$$

while $\tilde{f}_{\theta,y}(x|\xi)$ and $\tilde{g}_{\theta,y}(x|\xi, \zeta)$ are defined as

$$\tilde{f}_{\theta,y}(x|\xi) = \frac{\int \tilde{r}_\theta(y, x|x')\xi(dx')}{\iint \tilde{r}_\theta(y, x''|x)\mu(dx'')\xi(dx')},$$
$$\tilde{g}_{\theta,y}(x|\xi, \zeta) = \tilde{h}_{\theta,y}(x|\xi, \zeta) - \tilde{f}_{\theta,y}(x|\xi)\tilde{H}_{\theta,y}(\xi, \zeta).$$

$\tilde{F}_{\theta,y}(dx|\xi)$ and $\tilde{G}_{\theta,y}(dx|\xi, \zeta)$ are the measures defined for $B \in \mathcal{B}(\mathcal{X})$ by

$$\tilde{F}_{\theta,y}(B|\xi) = \int_B \tilde{f}_{\theta,y}(x|\xi)\mu(dx),$$
$$\tilde{G}_{\theta,y}(B|\xi, \zeta) = \int_B \tilde{g}_{\theta,y}(x|\xi, \zeta)\mu(dx).$$

Measures $\tilde{F}_{\theta,y}(dx|\xi)$ and $\tilde{G}_{\theta,y}(dx|\xi, \zeta)$ are also denoted by $\tilde{F}_{\theta,y}(\xi)$ and $\tilde{G}_{\theta,y}(\xi, \zeta)$ (short-hand notation). $\tilde{r}_{\theta,\boldsymbol{y}}^{m:n}(x'|x)$ is the function recursively defined by

$$\tilde{r}_{\theta,\boldsymbol{y}}^{m:m+1}(x'|x) = \tilde{r}_\theta(y_{m+1}, x'|x),$$
$$\tilde{r}_{\theta,\boldsymbol{y}}^{m:n+1}(x'|x) = \int \tilde{r}_{\theta,\boldsymbol{y}}^{n:n+1}(x'|x'')\tilde{r}_{\theta,\boldsymbol{y}}^{m:n}(x''|x)\mu(dx'')$$

for $n > m \geq 0$ and a sequence $\boldsymbol{y} = \{y_n\}_{n \geq 0}$ in $\mathcal{Y}$. $\tilde{h}_{\theta,\boldsymbol{y}}^{m:n}(x|\xi, \zeta)$ and $\tilde{H}_{\theta,\boldsymbol{y}}^{m:n}(\xi, \zeta)$ are the functions defined by

$$h_{\theta,\boldsymbol{y}}^{m:n}(x|\xi, \zeta) = \frac{\int r_{\theta,\boldsymbol{y}}^{m:n}(x|x')\zeta(dx') + \int \nabla_\theta r_{\theta,\boldsymbol{y}}^{m:n}(x|x')\xi(dx')}{\iint r_{\theta,\boldsymbol{y}}^{m:n}(x''|x')\xi(dx')\mu(dx'')},$$

$$\tilde{H}_{\theta,\boldsymbol{y}}^{m:n}(\xi, \zeta) = \int \tilde{h}_{\theta,\boldsymbol{y}}^{m:n}(x|\xi, \zeta)\mu(dx), \quad (146)$$

while $\tilde{f}_{\theta,\boldsymbol{y}}^{m:n}(x|\xi)$ and $\tilde{g}_{\theta,\boldsymbol{y}}^{m:n}(x|\xi, \zeta)$ are defined as

$$\tilde{f}_{\theta,\boldsymbol{y}}^{m:n}(x|\xi) = \frac{\int \tilde{r}_{\theta,\boldsymbol{y}}^{m:n}(x|x')\xi(dx')}{\iint \tilde{r}_{\theta,\boldsymbol{y}}^{m:n}(x''|x')\xi(dx')\mu(dx'')},$$
$$\tilde{g}_{\theta,\boldsymbol{y}}^{m:n}(x|\xi, \zeta) = \tilde{h}_{\theta,\boldsymbol{y}}^{m:n}(x|\xi, \zeta) - \tilde{f}_{\theta,\boldsymbol{y}}^{m:n}(x|\xi)\tilde{H}_{\theta,\boldsymbol{y}}^{m:n}(\xi, \zeta).$$

$\tilde{F}_{\theta,\boldsymbol{y}}^{m:m}(dx|\xi)$, $\tilde{F}_{\theta,\boldsymbol{y}}^{m:n}(dx|\xi)$ and $\tilde{G}_{\theta,\boldsymbol{y}}^{m:m}(dx|\xi,\zeta)$, $\tilde{G}_{\theta,\boldsymbol{y}}^{m:n}(dx|\xi,\zeta)$ are the measures defined by $\tilde{F}_{\theta,\boldsymbol{y}}^{m:m}(B|\xi) = \xi(B)$, $\tilde{G}_{\theta,\boldsymbol{y}}^{m:m}(B|\xi,\zeta) = \zeta(B)$ and

$$\tilde{F}_{\theta,\boldsymbol{y}}^{m:n}(B|\xi) = \int_B \tilde{f}_{\theta,\boldsymbol{y}}^{m:n}(x|\xi)\mu(dx),$$

$$\tilde{G}_{\theta,\boldsymbol{y}}^{m:n}(B|\xi,\zeta) = \int_B \tilde{g}_{\theta,\boldsymbol{y}}^{m:n}(x|\xi,\zeta)\mu(dx).$$

Measures $\tilde{F}_{\theta,\boldsymbol{y}}^{m:n}(dx|\xi)$ and $\tilde{G}_{\theta,\boldsymbol{y}}^{m:n}(dx|\xi,\zeta)$ are also denoted by $\tilde{F}_{\theta,\boldsymbol{y}}^{m:n}(\xi)$ and $\tilde{G}_{\theta,\boldsymbol{y}}^{m:n}(\xi,\zeta)$ (short-hand notation). Then, it is easy to show that $\tilde{F}_{\theta,\boldsymbol{y}}^{m:n}(\xi)$ and $\tilde{G}_{\theta,\boldsymbol{y}}^{m:n}(\xi,\zeta)$ are the optimal filter and its gradient, i.e.,

$$\tilde{F}_{\theta,\boldsymbol{y}}^{0:n}(B|\lambda) = P\left(X_n^{\theta,\lambda} \in B \middle| Y_{1:n}^{\theta,\lambda} = y_{1:n}\right),$$

$$\tilde{G}_{\theta,\boldsymbol{y}}^{0:n}(B|\lambda,\boldsymbol{0}) = \nabla_\theta \tilde{F}_{\theta,\boldsymbol{y}}^{0:n}(B|\lambda)$$

for each $\lambda \in \mathcal{P}(\mathcal{X})$, $n \geq 1$. Moreover, it is straightforward to verify

$$\tilde{F}_{\theta,\boldsymbol{y}}^{m:n+1}(\xi) = \tilde{F}_{\theta,y_{n+1}}\left(\tilde{F}_{\theta,\boldsymbol{y}}^{m:n}(\xi)\right),$$

$$\tilde{G}_{\theta,\boldsymbol{y}}^{m:n+1}(\xi,\zeta) = \tilde{G}_{\theta,y_{n+1}}\left(\tilde{F}_{\theta,\boldsymbol{y}}^{m:n}(\xi), \tilde{G}_{\theta,\boldsymbol{y}}^{m:n}(\xi,\zeta)\right)$$

for each $\xi \in \mathcal{P}(\mathcal{X})$, $\zeta \in \mathcal{M}_s^d(\mathcal{X})$, $n \geq m \geq 0$.

Remark: We recall here that Q stands for any compact set satisfying $Q \subset \Theta$.

Lemma A2.1: Let Assumptions 2.3 and 2.4 hold. Then, there exists a real number $C_{7,Q} \in [1,\infty)$ (independent of N and depending only on $p_\theta(x'|x)$, $q_\theta(y|x)$) such that

$$\left\|\tilde{F}_{\theta,y}(\xi) - \tilde{F}_{\theta',y}(\xi)\right\| \leq C_{7,Q}\|\theta - \theta'\|,$$

$$\left\|\tilde{G}_{\theta,y}(\xi,\zeta) - \tilde{G}_{\theta',y}(\xi,\zeta)\right\| \leq C_{7,Q}\|\theta - \theta'\|(1 + \|\zeta\|),$$

$$\left\|\tilde{H}_{\theta,y}(\xi,\zeta) - \tilde{H}_{\theta',y}(\xi',\zeta')\right\|$$
$$\leq C_{7,Q}(\|\theta - \theta'\| + \|\xi - \xi'\|)(1 + \|\zeta\|) + C_{7,Q}\|\zeta - \zeta'\|$$

for all $\theta, \theta' \in Q$, $\xi, \xi' \in \mathcal{P}(\mathcal{X})$, $\zeta, \zeta' \in \mathcal{M}_s^d(\mathcal{X})$.

Proof: Throughout the proof, the following notation is used. θ, θ' are any elements of Q. x, x' are any elements of $\mathcal{X}$, while y is any element of $\mathcal{Y}$. ξ, ξ' are any elements of $\mathcal{P}(\mathcal{X})$, while ζ, ζ' are any elements of $\mathcal{M}_s^d(\mathcal{X})$.

Let $\tilde{C}_{1,Q} = 2\varepsilon_Q^{-3}K_{1,Q}(1 + \|\mu\|)$ (ε_Q, $K_{1,Q}$ are specified in Assumptions 2.3, 2.4, while $\mu(dx)$ is defined in Subsection II-A). Owing to Assumption 2.3, we have

$$\varepsilon_Q^2 \leq \varepsilon_Q p_\theta(x'|x) \leq \tilde{r}_\theta(y,x'|x) \leq \frac{1}{\varepsilon_Q}p_\theta(x'|x) \leq \frac{1}{\varepsilon_Q^2}. \quad (147)$$

Consequently, we get

$$\varepsilon_Q \leq \int \tilde{r}_\theta(y,x'|x)\mu(dx') \leq \frac{1}{\varepsilon_Q}, \quad (148)$$

$$\varepsilon_Q^3 \leq \tilde{f}_{\theta,y}(x|\xi) \leq \frac{1}{\varepsilon_Q^3} \leq \tilde{C}_{1,Q}. \quad (149)$$

Moreover, due to Assumptions 2.3, 2.4, we have

$$\|\nabla_\theta \tilde{r}_\theta(x'|y,x)\| \leq \|\nabla_\theta q_\theta(y|x')\|p_\theta(x'|x)$$
$$+ q_\theta(y|x')\|\nabla_\theta p_\theta(x'|x)\|$$
$$\leq \frac{2K_{1,Q}}{\varepsilon_Q}. \quad (150)$$

Therefore, we get

$$\|\tilde{h}_{\theta,y}(x|\xi,\zeta)\| \leq \frac{\int \|\nabla_\theta \tilde{r}_\theta(y,x|x')\|\xi(dx')}{\iint \tilde{r}_\theta(y,x''|x')\mu(dx'')\xi(dx')}$$
$$+ \frac{\int \tilde{r}_\theta(y,x|x')|\zeta|(dx')}{\iint \tilde{r}_\theta(y,x''|x')\mu(dx'')\xi(dx')}$$
$$\leq \frac{2K_{1,Q}}{\varepsilon_Q^3}(1 + \|\zeta\|) \leq \tilde{C}_{1,Q}(1 + \|\zeta\|). \quad (151)$$

Hence, we have

$$\|\tilde{H}_{\theta,y}(\xi,\zeta)\| \leq \int \|\tilde{h}_{\theta,y}(x|\xi,\zeta)\|\mu(dx)$$
$$\leq \frac{2K_{1,Q}\|\mu\|}{\varepsilon_Q^3}(1 + \|\zeta\|) \leq \tilde{C}_{1,Q}(1 + \|\zeta\|). \quad (152)$$

Let $\tilde{C}_{2,Q} = 6\varepsilon_Q^{-3}\tilde{C}_{1,Q}K_{1,Q}^2(1 + \|\mu\|)$. Due to Assumptions 2.3, 2.4, we have

$$|\tilde{r}_\theta(x'|y,x) - \tilde{r}_{\theta'}(x'|y,x)| \leq |q_\theta(y|x') - q_{\theta'}(y|x')|p_\theta(x'|x)$$
$$+ q_{\theta'}(y|x')|p_\theta(x'|x) - p_{\theta'}(x'|x)|$$
$$\leq \frac{2K_{1,Q}}{\varepsilon_Q}\|\theta - \theta'\|. \quad (153)$$

Owing to Assumptions 2.3, 2.4, we also have

$$\|\nabla_\theta \tilde{r}_\theta(x'|y,x) - \nabla_\theta \tilde{r}_{\theta'}(x'|y,x)\|$$
$$\leq \|\nabla_\theta q_\theta(y|x') - \nabla_\theta q_{\theta'}(y|x')\|p_\theta(x'|x)$$
$$+ \|\nabla_\theta q_{\theta'}(y|x')\||p_\theta(x'|x) - p_{\theta'}(x'|x)|$$
$$+ |q_\theta(y|x') - q_{\theta'}(y|x')|\|\nabla_\theta p_\theta(x'|x)\|$$
$$+ q_{\theta'}(y|x')\|\nabla_\theta p_\theta(x'|x) - \nabla_\theta p_{\theta'}(x'|x)\|$$
$$\leq \frac{4K_{1,Q}^2}{\varepsilon_Q}\|\theta - \theta'\|. \quad (154)$$

Then, using (148), (149), (153), we conclude

$$\left|\tilde{f}_{\theta,y}(x|\xi) - \tilde{f}_{\theta',y}(x|\xi)\right|$$
$$\leq \frac{\int |\tilde{r}_\theta(y,x|x') - \tilde{r}_{\theta'}(y,x|x')|\xi(dx')}{\iint \tilde{r}_\theta(y,x''|x')\mu(dx'')\xi(dx')}$$
$$+ \frac{\iint |\tilde{r}_\theta(y,x''|x') - \tilde{r}_{\theta'}(y,x''|x')|\mu(dx'')\xi(dx')}{\iint \tilde{r}_\theta(y,x''|x')\mu(dx'')\xi(dx')}$$
$$\cdot \tilde{f}_{\theta',y}(x|\xi)$$
$$\leq \left(\frac{2K_{1,Q}}{\varepsilon_Q^2} + \frac{2\tilde{C}_{1,Q}K_{1,Q}\|\mu\|}{\varepsilon_Q^2}\right)\|\theta - \theta'\|$$
$$\leq \tilde{C}_{2,Q}\|\theta - \theta'\|. \quad (155)$$

Similarly, relying on (148), (151), (153), (154), we deduce

$$\left\|\tilde{h}_{\theta,y}(x|\xi,\zeta) - \tilde{h}_{\theta',y}(x|\xi,\zeta)\right\|$$
$$\leq \frac{\int |\tilde{r}_\theta(y,x|x') - \tilde{r}_{\theta'}(y,x|x')| |\zeta|(dx')}{\iint \tilde{r}_\theta(y,x''|x')\mu(dx'')\xi(dx')}$$
$$+ \frac{\int \|\nabla_\theta \tilde{r}_\theta(y,x|x') - \nabla_\theta \tilde{r}_{\theta'}(y,x|x')\|\xi(dx')}{\iint \tilde{r}_\theta(y,x''|x')\mu(dx'')\xi(dx')}$$
$$+ \frac{\iint |\tilde{r}_\theta(y,x''|x') - \tilde{r}_{\theta'}(y,x''|x')|\mu(dx'')\xi(dx')}{\iint \tilde{r}_\theta(y,x''|x')\mu(dx'')\xi(dx')}$$
$$\cdot \|\tilde{h}_{\theta',y}(x|\xi,\zeta)\|$$

$$\leq \frac{2K_{1,Q}}{\varepsilon_Q^2}\|\theta - \theta'\|\|\zeta\| + \frac{4K_{1,Q}^2}{\varepsilon_Q^2}\|\theta - \theta'\|$$
$$+ \frac{2\tilde{C}_{1,Q}K_{1,Q}\|\mu\|}{\varepsilon_Q^2}\|\theta - \theta'\|(1 + \|\zeta\|)$$
$$\leq \tilde{C}_{2,Q}\|\theta - \theta'\|(1 + \|\zeta\|). \tag{156}$$

Moreover, (147) – (151) imply

$$\|\tilde{h}_{\theta,y}(\xi,\zeta) - \tilde{h}_{\theta,y}(\xi',\zeta')\|$$
$$\leq \frac{\int \|\nabla_\theta \tilde{r}_\theta(y,x|x')\|\|\xi - \xi'|(dx')}{\iint \tilde{r}_\theta(y,x''|x')\mu(dx'')\xi'(dx')}$$
$$+ \frac{\int \tilde{r}_\theta(y,x|x')|\zeta - \zeta'|(dx')}{\iint \tilde{r}_\theta(y,x''|x')\mu(dx'')\xi'(dx')}$$
$$+ \|\tilde{h}_{\theta,y}(\xi,\zeta)\|\frac{\iint \tilde{r}_\theta(y,x''|x')\mu(dx'')|\xi - \xi'|(dx')}{\iint \tilde{r}_\theta(y,x''|x')\mu(dx'')\xi'(dx')}$$
$$\leq \frac{2K_{1,Q}}{\varepsilon_Q^2}\|\xi - \xi'\| + \frac{1}{\varepsilon_Q^3}\|\zeta - \zeta'\| + \frac{\tilde{C}_{1,Q}}{\varepsilon_Q^3}\|\xi - \xi'\|(1 + \|\zeta\|)$$
$$\leq \tilde{C}_{2,Q}\|\xi - \xi'\|(1 + \|\zeta\|) + \tilde{C}_{2,Q}\|\zeta - \zeta'\|. \tag{157}$$

Let $\tilde{C}_{3,Q} = 2\tilde{C}_{1,Q}\tilde{C}_{2,Q}(1 + \|\mu\|)$. Then, (148), (149), (151), (155), (156) imply

$$\|\tilde{g}_{\theta,y}(x|\xi,\zeta) - \tilde{g}_{\theta',y}(x|\xi,\zeta)\|$$
$$\leq \|\tilde{h}_{\theta,y}(x|\xi,\zeta) - \tilde{h}_{\theta',y}(x|\xi,\zeta)\|$$
$$+ |\tilde{f}_{\theta,y}(x|\xi) - \tilde{f}_{\theta',y}(x|\xi)| \int \|\tilde{h}_{\theta,y}(x'|\xi,\zeta)\|\mu(dx')$$
$$+ \tilde{f}_{\theta',y}(x|\xi) \int \|\tilde{h}_{\theta,y}(x'|\xi,\zeta) - \tilde{h}_{\theta',y}(x'|\xi,\zeta)\|\mu(dx')$$
$$\leq (\tilde{C}_{2,Q} + 2\tilde{C}_{1,Q}\tilde{C}_{2,Q}\|\mu\|)\|\theta - \theta'\|(1 + \|\zeta\|)$$
$$\leq \tilde{C}_{3,Q}\|\theta - \theta'\|(1 + \|\zeta\|). \tag{158}$$

Moreover, (156) yields

$$\|\tilde{H}_{\theta,y}(\xi,\zeta) - \tilde{H}_{\theta',y}(\xi,\zeta)\|$$
$$\leq \int \|\tilde{h}_{\theta,y}(x|\xi,\zeta) - \tilde{h}_{\theta',y}(x|\xi,\zeta)\|\mu(dx)$$
$$\leq \tilde{C}_{2,Q}\|\mu\|\|\theta - \theta'\|(1 + \|\zeta\|) \leq \tilde{C}_{3,Q}\|\theta - \theta'\|(1 + \|\zeta\|). \tag{159}$$

Similarly, (157) implies

$$\|\tilde{H}_{\theta,y}(\xi,\zeta) - \tilde{H}_{\theta,y}(\xi',\zeta')\|$$
$$\leq \int \|\tilde{h}_{\theta,y}(x|\xi,\zeta) - \tilde{h}_{\theta,y}(x|\xi',\zeta')\|\mu(dx)$$
$$\leq \tilde{C}_{2,Q}\|\mu\|\|\xi - \xi'\|(1 + \|\zeta\|) + \tilde{C}_{2,Q}\|\mu\|\|\zeta - \zeta'\|$$
$$\leq \tilde{C}_{3,Q}\|\xi - \xi'\|(1 + \|\zeta\|) + \tilde{C}_{3,Q}\|\zeta - \zeta'\|. \tag{160}$$

Let $C_{7,Q} = \tilde{C}_{3,Q}(1 + \|\mu\|)$. Then, using (155), we conclude

$$\|\tilde{F}_{\theta,y}(\xi) - \tilde{F}_{\theta',y}(\xi)\|$$
$$\leq \int |\tilde{f}_{\theta,y}(x|\xi) - \tilde{f}_{\theta',y}(x|\xi)|\mu(dx)$$
$$\leq \tilde{C}_{1,Q}\|\mu\|\|\theta - \theta'\| \leq C_{7,Q}\|\theta - \theta'\|$$

(notice that $\tilde{C}_{1,Q} \leq \tilde{C}_{3,Q}$). Similarly, relying on (158), we deduce

$$\|\tilde{G}_{\theta,y}(\xi,\zeta) - \tilde{G}_{\theta',y}(\xi,\zeta)\|$$
$$\leq \int |\tilde{g}_{\theta,y}(x|\xi,\zeta) - \tilde{g}_{\theta',y}(x|\xi,\zeta)|\mu(dx)$$
$$\leq \tilde{C}_{3,Q}\|\mu\|\|\theta - \theta'\|(1 + \|\zeta\|) \leq C_{7,Q}\|\theta - \theta'\|(1 + \|\zeta\|).$$

Moreover, combining (159), (160), we get

$$\|\tilde{H}_{\theta,y}(\xi,\zeta) - \tilde{H}_{\theta',y}(\xi',\zeta')\|$$
$$\leq \|\tilde{H}_{\theta,y}(\xi,\zeta) - \tilde{H}_{\theta',y}(\xi,\zeta)\| + \|\tilde{H}_{\theta',y}(\xi,\zeta) - \tilde{H}_{\theta',y}(\xi',\zeta')\|$$
$$\leq \tilde{C}_{3,Q}(\|\theta - \theta'\| + \|\xi - \xi'\|)(1 + \|\zeta\|) + \tilde{C}_{3,Q}\|\zeta - \zeta'\|$$
$$\leq C_{7,Q}(\|\theta - \theta'\| + \|\xi - \xi'\|)(1 + \|\zeta\|) + C_{7,Q}\|\zeta - \zeta'\|.$$

$\square$

Lemma A2.2: Let Assumptions 2.3 and 2.4 hold. Then, the following is true:

(i) There exist real numbers $\rho_{4,Q} \in (0,1)$, $C_{8,Q} \in [1,\infty)$ (independent of N and depending only on $p_\theta(x'|x)$, $q_\theta(y|x)$) such that

$$\|\tilde{G}_{\theta,\boldsymbol{y}}^{m:n}(\xi,\zeta)\| \leq C_{8,Q}(1 + \|\zeta\|), \tag{161}$$
$$\|\tilde{F}_{\theta,\boldsymbol{y}}^{m:n}(\xi) - \tilde{F}_{\theta,\boldsymbol{y}}^{m:n}(\xi')\| \leq C_{8,Q}\rho_{4,Q}^{n-m}\|\xi - \xi'\|, \tag{162}$$
$$\|\tilde{G}_{\theta,\boldsymbol{y}}^{m:n}(\xi,\zeta) - \tilde{G}_{\theta,\boldsymbol{y}}^{m:n}(\xi',\zeta')\|$$
$$\leq C_{8,Q}\rho_{4,Q}^{n-m}\|\xi - \xi'\|(1 + \|\zeta\|) + C_{8,Q}\rho_{4,Q}^{n-m}\|\zeta - \zeta'\| \tag{163}$$

for all $\theta \in Q$, $\xi,\xi' \in \mathcal{P}(\mathcal{X})$, $\zeta,\zeta' \in \mathcal{M}_s^d(\mathcal{X})$, $n \geq m \geq 0$ and any sequence $\boldsymbol{y} = \{y_n\}_{n\geq 0}$ in $\mathcal{Y}$.

(ii) There exists a real number $C_{9,Q} \in [1,\infty)$ (independent of N and depending only on $p_\theta(x'|x)$, $q_\theta(y|x)$) such that

$$\|\tilde{F}_{\theta,\boldsymbol{y}}^{m:n}(\xi) - \tilde{F}_{\theta',\boldsymbol{y}}^{m:n}(\xi)\| \leq C_{9,Q}\|\theta - \theta'\|,$$
$$\|\tilde{G}_{\theta,\boldsymbol{y}}^{m:n}(\xi,\zeta) - \tilde{G}_{\theta',\boldsymbol{y}}^{m:n}(\xi,\zeta)\| \leq C_{9,Q}\|\theta - \theta'\|(1 + \|\zeta\|)$$

for all $\theta, \theta' \in Q$, $\xi \in \mathcal{P}(\mathcal{X})$, $\zeta \in \mathcal{M}_s^d(\mathcal{X})$, $n \geq m \geq 0$ and any sequence $\boldsymbol{y} = \{y_n\}_{n\geq 0}$ in $\mathcal{Y}$.

Proof: (i) See [29, Theorems 3.1, 3.2] (or [32, Theorem 2.2]).

(ii) Throughout this part of the proof, the following notation is used. θ, θ' are any elements of Q. ξ, ζ are any elements of $\mathcal{P}(\mathcal{X})$, $\mathcal{M}_s^d(\mathcal{X})$ (respectively). $\boldsymbol{y} = \{y_n\}_{n\geq 0}$ is any sequence in $\mathcal{Y}$. n,m are any integers satisfying $n > m \geq 0$.

It is straightforward to verify

$$\tilde{F}_{\theta,\boldsymbol{y}}^{m:n}(\xi) - \tilde{F}_{\theta',\boldsymbol{y}}^{m:n}(\xi)$$
$$= \sum_{i=m}^{n-1} \left(\tilde{F}_{\theta,\boldsymbol{y}}^{i:n}\big(\tilde{F}_{\theta',\boldsymbol{y}}^{m:i}(\xi)\big) - \tilde{F}_{\theta,\boldsymbol{y}}^{i+1:n}\big(\tilde{F}_{\theta',\boldsymbol{y}}^{m:i+1}(\xi)\big) \right)$$
$$= \sum_{i=m}^{n-1} \left(\tilde{F}_{\theta,\boldsymbol{y}}^{i+1:n}\big(\tilde{F}_{\theta,y_i}\big(\tilde{F}_{\theta',\boldsymbol{y}}^{m:i}(\xi)\big)\big) - \tilde{F}_{\theta,\boldsymbol{y}}^{i+1:n}\big(\tilde{F}_{\theta',\boldsymbol{y}}^{m:i+1}(\xi)\big) \right). \tag{164}$$

It is also easy to show

$$
\tilde{G}_{\theta,\boldsymbol{y}}^{m:n}(\xi,\zeta) - \tilde{G}_{\theta',\boldsymbol{y}}^{m:n}(\xi,\zeta)
$$
$$
= \sum_{i=m}^{n-1} \Big(\tilde{G}_{\theta,\boldsymbol{y}}^{i:n} \big(\tilde{F}_{\theta',\boldsymbol{y}}^{m:i}(\xi), \tilde{G}_{\theta',\boldsymbol{y}}^{m:i}(\xi,\zeta) \big)
$$
$$
- \tilde{G}_{\theta,\boldsymbol{y}}^{i+1:n} \big(\tilde{F}_{\theta',\boldsymbol{y}}^{m:i+1}(\xi), \tilde{G}_{\theta',\boldsymbol{y}}^{m:i+1}(\xi,\zeta) \big) \Big)
$$
$$
= \sum_{i=m}^{n-1} \Big(\tilde{G}_{\theta,\boldsymbol{y}}^{i+1:n} \big(\tilde{F}_{\theta,y_i} \big(\tilde{F}_{\theta',\boldsymbol{y}}^{m:i}(\xi) \big), \tilde{G}_{\theta,y_i} \big(\tilde{F}_{\theta',\boldsymbol{y}}^{m:i}(\xi), \tilde{G}_{\theta',\boldsymbol{y}}^{m:i}(\xi,\zeta) \big) \big)
$$
$$
- \tilde{G}_{\theta,\boldsymbol{y}}^{i+1:n} \big(\tilde{F}_{\theta',\boldsymbol{y}}^{m:i+1}(\xi), \tilde{G}_{\theta',\boldsymbol{y}}^{m:i+1}(\xi,\zeta) \big) \Big). \tag{165}
$$

Let $C_{9,Q} = 4C_{7,Q}C_{8,Q}^2 (1-\rho_{4,Q})^{-1}$ ($C_{7,Q}$ is specified in Lemma A2.1). Relying on Lemma A2.1 and (162), (164), we deduce

$$
\big\| \tilde{F}_{\theta,\boldsymbol{y}}^{m:n}(\xi) - \tilde{F}_{\theta',\boldsymbol{y}}^{m:n}(\xi) \big\|
$$
$$
\leq \sum_{i=m}^{n-1} \big\| \tilde{F}_{\theta,\boldsymbol{y}}^{i+1:n} \big(\tilde{F}_{\theta,y_i} \big(\tilde{F}_{\theta',\boldsymbol{y}}^{m:i}(\xi) \big) \big) - \tilde{F}_{\theta,\boldsymbol{y}}^{i+1:n} \big(\tilde{F}_{\theta',\boldsymbol{y}}^{m:i+1}(\xi) \big) \big\|
$$
$$
\leq C_{8,Q} \sum_{i=m}^{n-1} \rho_{4,Q}^{n-i-1} \big\| \tilde{F}_{\theta,y_i} \big(\tilde{F}_{\theta',\boldsymbol{y}}^{m:i}(\xi) \big) - \tilde{F}_{\theta',y_i} \big(\tilde{F}_{\theta',\boldsymbol{y}}^{m:i}(\xi) \big) \big\|
$$
$$
\leq C_{7,Q}C_{8,Q} \|\theta-\theta'\| \sum_{i=m}^{n-1} \rho_{4,Q}^{n-i-1}
$$
$$
\leq C_{7,Q}C_{8,Q} (1-\rho_{4,Q})^{-1} \|\theta-\theta'\| \leq C_{9,Q} \|\theta-\theta'\|.
$$

Similarly, using (163), (165), we conclude

$$
\big\| \tilde{G}_{\theta,\boldsymbol{y}}^{m:n}(\xi,\zeta) - \tilde{G}_{\theta',\boldsymbol{y}}^{m:n}(\xi,\zeta) \big\|
$$
$$
\leq \sum_{i=m}^{n-1} \big\| \tilde{G}_{\theta,\boldsymbol{y}}^{i+1:n} \big(\tilde{F}_{\theta,y_i} \big(\tilde{F}_{\theta',\boldsymbol{y}}^{m:i}(\xi) \big), \tilde{G}_{\theta,y_i} \big(\tilde{F}_{\theta',\boldsymbol{y}}^{m:i}(\xi), \tilde{G}_{\theta',\boldsymbol{y}}^{m:i}(\xi,\zeta) \big) \big)
$$
$$
- \tilde{G}_{\theta,\boldsymbol{y}}^{i+1:n} \big(\tilde{F}_{\theta',\boldsymbol{y}}^{m:i+1}(\xi), \tilde{G}_{\theta',\boldsymbol{y}}^{m:i+1}(\xi,\zeta) \big) \big\|
$$
$$
\leq C_{8,Q} \sum_{i=m}^{n-1} \rho_{4,Q}^{n-i-1} \big\| \tilde{F}_{\theta,y_i} \big(\tilde{F}_{\theta',\boldsymbol{y}}^{m:i}(\xi) \big) - \tilde{F}_{\theta',y_i} \big(\tilde{F}_{\theta',\boldsymbol{y}}^{m:i}(\xi) \big) \big\|
$$
$$
\cdot \Big(1 + \big\| \tilde{G}_{\theta',\boldsymbol{y}}^{m:i+1}(\xi,\zeta) \big\| \Big)
$$
$$
+ C_{8,Q} \sum_{i=m}^{n-1} \rho_{4,Q}^{n-i-1} \big\| \tilde{G}_{\theta,y_i} \big(\tilde{F}_{\theta',\boldsymbol{y}}^{m:i}(\xi), \tilde{G}_{\theta',\boldsymbol{y}}^{m:i}(\xi,\zeta) \big)
$$
$$
- \tilde{G}_{\theta',y_i} \big(\tilde{F}_{\theta',\boldsymbol{y}}^{m:i}(\xi), \tilde{G}_{\theta',\boldsymbol{y}}^{m:i}(\xi,\zeta) \big) \big\|.
$$

Consequently, Lemma A2.1 and (161) imply

$$
\big\| \tilde{G}_{\theta,\boldsymbol{y}}^{m:n}(\xi,\zeta) - \tilde{G}_{\theta',\boldsymbol{y}}^{m:n}(\xi,\zeta) \big\|
$$
$$
\leq C_{7,Q}C_{8,Q} \|\theta-\theta'\| \sum_{i=m}^{n-1} \rho_{4,Q}^{n-i-1} \Big(1 + \big\| \tilde{G}_{\theta',\boldsymbol{y}}^{m:i+1}(\xi,\zeta) \big\| \Big)
$$
$$
+ C_{7,Q}C_{8,Q} \|\theta-\theta'\| \sum_{i=m}^{n-1} \rho_{4,Q}^{n-i-1} \Big(1 + \big\| \tilde{G}_{\theta',\boldsymbol{y}}^{m:i}(\xi,\zeta) \big\| \Big)
$$
$$
\leq 4C_{7,Q}C_{8,Q}^2 \|\theta-\theta'\| (1+\|\zeta\|) \sum_{i=m}^{n-1} \rho_{4,Q}^{n-i-1}
$$
$$
\leq 4C_{7,Q}C_{8,Q}^2 (1-\rho_{4,Q})^{-1} \|\theta-\theta'\| (1+\|\zeta\|)
$$
$$
= C_{9,Q} \|\theta-\theta'\| (1+\|\zeta\|).
$$

$\square$

Proof of Lemma 4.1: (i) See [32, Theorem 3.1].

(ii) and (iii) Throughout these parts of the proof, $E_{x,y}(\cdot)$ and $E_{X_0,Y_0}(\cdot)$ denote the conditional expectations $E(\cdot|X_0 = x, Y_0 = y)$ and $E(\cdot|X_0, Y_0)$ (respectively). Due to [32, Proposition 7.2], there exist real numbers $\beta_Q \in (0,1)$, $\tilde{C}_{1,Q} \in [1,\infty)$ (independent of N and depending only on $p_\theta(x'|x)$, $q_\theta(y|x)$) such that

$$
\Big\| E_{x,y} \big(\tilde{H}_{\theta,Y_n} \big(\tilde{F}_{\theta,\boldsymbol{Y}}^{0:n-1}(\xi), \tilde{G}_{\theta,\boldsymbol{Y}}^{0:n-1}(\xi,\zeta) \big) - \nabla l(\theta) \big) \Big\|
$$
$$
\leq \tilde{C}_{1,Q} \beta_Q^n (1+\|\zeta\|) \tag{166}
$$

for all $\theta \in Q$, $x \in \mathcal{X}$, $y \in \mathcal{Y}$, $\xi \in \mathcal{P}(\mathcal{X})$, $\zeta \in \mathcal{M}_s^d(\mathcal{X})$, $n \geq 1$.

Throughout the proof of (ii), (iii), the following notation is used, too. θ, θ, θ' are any elements of Q. x, y are any elements of $\mathcal{X}$, $\mathcal{Y}$ (respectively), while B, ξ, ζ are any elements of $\mathcal{B}(\mathcal{X}), \mathcal{P}(\mathcal{X}), \mathcal{M}_s^d(\mathcal{X})$ (respectively). $\boldsymbol{y} = \{y_n\}_{n\geq 0}$ is any sequence in $\mathcal{Y}$. n is any positive integer. $\tilde{\alpha}_{\theta,y}(dx|\xi)$, $\tilde{\beta}_{\theta,y}(dx|\xi,\zeta)$ are the measures defined by

$$
\tilde{\alpha}_{\theta,y}(B|\xi) = \frac{\int_B q_\theta(y|x)\xi(dx)}{\int q_\theta(y|x)\xi(dx)},
$$
$$
\tilde{\beta}_{\theta,y}(B|\xi,\zeta) = \frac{\int_B q_\theta(y|x)\zeta(dx) + \int_B \nabla_\theta q_\theta(y|x)\xi(dx)}{\int q_\theta(y|x)\xi(dx)}.
$$

It is straightforward to verify

$$
F_{\theta,\boldsymbol{y}}^{0:n}(B|\xi) = \iint I_B(x') p_\theta(x'|x) \mu(dx') \tag{167}
$$
$$
\cdot \tilde{F}_{\theta,\boldsymbol{y}}^{0:n-1} \big(dx|\tilde{\alpha}_{\theta,y_0}(\xi) \big),
$$
$$
G_{\theta,\boldsymbol{y}}^{0:n}(B|\xi,\zeta) = \iint I_B(x') \nabla_\theta p_\theta(x'|x) \mu(dx')
$$
$$
\cdot \tilde{F}_{\theta,\boldsymbol{y}}^{0:n-1} \big(dx|\tilde{\alpha}_{\theta,y_0}(\xi) \big)
$$
$$
+ \iint I_B(x') p_\theta(x'|x) \mu(dx')
$$
$$
\cdot \tilde{G}_{\theta,\boldsymbol{y}}^{0:n-1} \big(dx|\tilde{\alpha}_{\theta,y_0}(\xi), \tilde{\beta}_{\theta,y_0}(\xi,\zeta) \big). \tag{168}
$$

(for a detailed derivation of (167), (168), see [33, Lemma SM2.1]). Then, using (26), (146), we conclude

$$
H_{\theta,y_n} \big(F_{\theta,\boldsymbol{y}}^{0:n}(\zeta), G_{\theta,\boldsymbol{y}}^{0:n}(\xi,\zeta) \big)
$$
$$
= \tilde{H}_{\theta,y_n} \big(\tilde{F}_{\theta,\boldsymbol{y}}^{0:n-1} \big(\tilde{\alpha}_{\theta,y_0}(\xi) \big), \tilde{G}_{\theta,\boldsymbol{y}}^{0:n-1} \big(\tilde{\alpha}_{\theta,y_0}(\xi), \tilde{\beta}_{\theta,y_0}(\xi,\zeta) \big) \big).
$$

Hence, we have

$$
E_{x,y} \big(H_{\theta,Y_n} \big(F_{\theta,\boldsymbol{Y}}^{0:n}(\zeta), G_{\theta,\boldsymbol{Y}}^{0:n}(\xi,\zeta) \big) \big)
$$
$$
= E_{x,y} \Big(\tilde{H}_{\theta,Y_n} \big(\tilde{F}_{\theta,\boldsymbol{Y}}^{0:n-1} \big(\tilde{\alpha}_{\theta,y}(\xi) \big),
$$
$$
\tilde{G}_{\theta,\boldsymbol{Y}}^{0:n-1} \big(\tilde{\alpha}_{\theta,y}(\xi), \tilde{\beta}_{\theta,y}(\xi,\zeta) \big) \big) \Big). \tag{169}
$$

Let $\tilde{C}_{2,Q} = \varepsilon_Q^{-2} K_{1,Q}$, $\tilde{C}_{3,Q} = 5C_{7,Q}C_{8,Q}C_{9,Q}$, while $C_{1,Q} = 3C_{8,Q}\tilde{C}_{2,Q}(1+\|\mu\|)$ ($\varepsilon_Q, K_{1,Q}, C_{7,Q}, C_{8,Q}$ are specified in Assumption 2.3, 2.4 and Lemmas A2.1, A2.2, while $\mu(dx)$ is defined in Subsection II-A). Owing to Assumptions 2.3, 2.4, we have

$$
\big\| \tilde{\beta}_{\theta,y}(\xi,\zeta) \big\| \leq \frac{\int q_\theta(y|x)|\zeta|(dx) + \int \|\nabla_\theta q_\theta(y|x)\|\xi(dx)}{\int q_\theta(y|x)\xi(dx)}
$$
$$
\leq \frac{1}{\varepsilon_Q^2} \|\zeta\| + \frac{K_{1,Q}}{\varepsilon_Q} \leq \tilde{C}_{2,Q}(1+\|\zeta\|). \tag{170}
$$

Consequently, Assumption 2.4, Lemma A2.2 and (168) yield

$$\left\|G_{\theta,\boldsymbol{y}}^{0:n}(\xi,\zeta)\right\|$$
$$\leq \iint \|\nabla_\theta p_\theta(x'|x)\|\mu(dx')\tilde{F}_{\theta,\boldsymbol{y}}^{0:n-1}\big(dx|\tilde{\alpha}_{\theta,y_0}(\xi)\big)$$
$$+ \iint p_\theta(x'|x)\mu(dx')\big|\tilde{G}_{\theta,\boldsymbol{y}}^{0:n-1}\big|\big(dx|\tilde{\alpha}_{\theta,y_0}(\xi),\tilde{\beta}_{\theta,y_0}(\xi,\zeta)\big)$$
$$\leq K_{1,Q}\|\mu\| + \left\|\tilde{G}_{\theta,\boldsymbol{y}}^{0:n-1}\big(\tilde{\alpha}_{\theta,y_0}(\xi),\tilde{\beta}_{\theta,y_0}(\xi,\zeta)\big)\right\|$$
$$\leq K_{1,Q}\|\mu\| + C_{8,Q}\left(1+\|\tilde{\beta}_{\theta,y_0}(\xi,\zeta)\|\right)$$
$$\leq K_{1,Q}\|\mu\| + 2C_{8,Q}\tilde{C}_{2,Q}(1+\|\zeta\|) \leq C_{1,Q}(1+\|\zeta\|).$$

Hence, (iii) holds.

Combining (166), (169), (170), we get

$$\left\|E_{x,y}\big(H_{\theta,Y_n}\big(F_{\theta,\boldsymbol{Y}}^{0:n}(\xi),G_{\theta,\boldsymbol{Y}}^{0:n}(\xi,\zeta)\big)-\nabla l(\theta)\big)\right\|$$
$$\leq \tilde{C}_{1,Q}\beta_Q^n\left(1+\|\tilde{\beta}_{\theta,y}(\xi,\zeta)\|\right) \leq 2\tilde{C}_{1,Q}\tilde{C}_{2,Q}\beta_Q^n(1+\|\zeta\|).$$

Therefore, we have

$$\left\|E\big(H_{\theta,Y_n}\big(F_{\theta,\boldsymbol{Y}}^{0:n}(\xi),G_{\theta,\boldsymbol{Y}}^{0:n}(\xi,\zeta)\big)\big)-\nabla l(\theta)\right\|$$
$$\leq E\big(\big\|E_{X_0,Y_0}\big(H_{\theta,Y_n}\big(F_{\theta,\boldsymbol{Y}}^{0:n}(\xi),G_{\theta,\boldsymbol{Y}}^{0:n}(\xi,\zeta)\big)-\nabla l(\theta)\big)\big\|\big)$$
$$\leq 2\tilde{C}_{1,Q}\tilde{C}_{2,Q}\beta_Q^n(1+\|\zeta\|).$$

Thus, (29) holds.

Owing to Lemmas A2.1, A2.2, we have

$$\left\|\tilde{H}_{\theta,Y_n}\big(\tilde{F}_{\theta,\boldsymbol{Y}}^{0:n-1}(\xi),\tilde{G}_{\theta,\boldsymbol{Y}}^{0:n-1}(\xi,\zeta)\big)\right.$$
$$\left.-\tilde{H}_{\theta',Y_n}\big(\tilde{F}_{\theta',\boldsymbol{Y}}^{0:n-1}(\xi),\tilde{G}_{\theta',\boldsymbol{Y}}^{0:n-1}(\xi,\zeta)\big)\right\|$$
$$\leq C_{7,Q}\left(\|\theta-\theta'\|+\|\tilde{F}_{\theta,\boldsymbol{Y}}^{0:n-1}(\xi)-\tilde{F}_{\theta',\boldsymbol{Y}}^{0:n-1}(\xi)\|\right)$$
$$\cdot\left(1+\|\tilde{G}_{\theta,\boldsymbol{Y}}^{0:n-1}(\xi,\zeta)\|\right)$$
$$+ C_{7,Q}\|\tilde{G}_{\theta,\boldsymbol{Y}}^{0:n-1}(\xi,\zeta)-\tilde{G}_{\theta',\boldsymbol{Y}}^{0:n-1}(\xi,\zeta)\|$$
$$\leq 5C_{7,Q}C_{8,Q}C_{9,Q}\|\theta-\theta'\|(1+\|\zeta\|)$$
$$\leq \tilde{C}_{3,Q}\|\theta-\theta'\|(1+\|\zeta\|).$$

Moreover, due to (166), we have

$$\left\|E\left(\tilde{H}_{\theta,Y_n}\big(\tilde{F}_{\theta,\boldsymbol{Y}}^{0:n-1}(\xi),\tilde{G}_{\theta,\boldsymbol{Y}}^{0:n-1}(\xi,\zeta)\big)\right)-\nabla l(\theta)\right\|$$
$$\leq E\Big(\big\|E_{X_0,Y_0}\Big(\tilde{H}_{\theta,Y_n}\big(\tilde{F}_{\theta,\boldsymbol{Y}}^{0:n-1}(\xi),\tilde{G}_{\theta,\boldsymbol{Y}}^{0:n-1}(\xi,\zeta)\big)-\nabla l(\theta)\Big)\big\|\Big)$$
$$\leq \tilde{C}_{1,Q}\beta_Q^n(1+\|\zeta\|).$$

Therefore, we get

$$\|\nabla l(\theta)-\nabla l(\theta')\|$$
$$\leq E\Big(\big\|\tilde{H}_{\theta,Y_n}\big(\tilde{F}_{\theta,\boldsymbol{Y}}^{0:n-1}(\xi),\tilde{G}_{\theta,\boldsymbol{Y}}^{0:n-1}(\xi,\zeta)\big)$$
$$-\tilde{H}_{\theta',Y_n}\big(\tilde{F}_{\theta',\boldsymbol{Y}}^{0:n-1}(\xi),\tilde{G}_{\theta',\boldsymbol{Y}}^{0:n-1}(\xi,\zeta)\big)\big\|\Big)$$
$$+\left\|E\left(\tilde{H}_{\theta,Y_n}\big(\tilde{F}_{\theta,\boldsymbol{Y}}^{0:n-1}(\xi),\tilde{G}_{\theta,\boldsymbol{Y}}^{0:n-1}(\xi,\zeta)\big)\right)-\nabla l(\theta)\right\|$$
$$+\left\|E\left(\tilde{H}_{\theta',Y_n}\big(\tilde{F}_{\theta',\boldsymbol{Y}}^{0:n-1}(\xi),\tilde{G}_{\theta',\boldsymbol{Y}}^{0:n-1}(\xi,\zeta)\big)\right)-\nabla l(\theta')\right\|$$
$$\leq \tilde{C}_{3,Q}\|\theta-\theta'\|(1+\|\zeta\|)+2\tilde{C}_{1,Q}\beta_Q^n(1+\|\zeta\|).$$

Letting $n\to\infty$, we deduce

$$\|\nabla l(\theta)-\nabla l(\theta')\| \leq \tilde{C}_{3,Q}\|\theta-\theta'\|.$$

Since Q is any compact set in Θ, we deduce that (ii) holds. $\square$

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